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LOCATIONS OF TRIANGULAR EQUILIBRIUM POINTS OF THE RESTRICTED THREE-BODY PROBLEM WITH POYNTING-ROBERTSON DRAG AND VARIABLE MASSES

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ABSTRACT

The restricted three-body problem (R3BP) is a fascinating problem that has been receiving attentions of astronomers and scientists because of its vast implications in diverse scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics. In this paper, we examine the locations of the triangular equilibrium points of the R3BP with Poynting-Robertson (P-R) drag forces and variable masses. The primaries are assumed to vary under the unified Mestschersky law and their dynamics defined by the Gylden-Mestschersky equation, while the smaller primary is assumed to be a radiation emitter with P-R drag. The dynamical equations are obtained for both the non-autonomous with variable coefficients and autonomized system with constant coefficients. Further, the locations of the triangular points of the autonomized systems are obtained using perturbation method. It is seen that the positions are defined by the mass parameter, radiation pressure and P-R drag of the smaller primary. The triangular points of the non-autonomous equations are obtained with help of the Mestschersky transformation, and differ from those of the autonomized system due to a function of time. The equilibrium points have several applications in space missions, satellites constellations and station-keeping.

Keywords: R3BP, Satellite, Triangular Equilibrium Points, P-R Drag, Variable Masses

INTRODUCTION

The restricted three-body problem (R3BP) constitutes one of the most important problems in dynamical astronomy. This formulation is the most searched and interesting problem for astrophysicists and the problem defines the motion of an infinitesimal mass moving under the gravitational influence of two massive bodies called primaries which move in circular orbits around their common center of mass on the account of their mutual attraction. Because no general solution in the R3BP is available, particular solutions are sought to obtain insight into the problem, these particular solutions referred to as the equilibrium points (EPs) are five for the classical R3BP; two triangular equilibrium points (TEPs) and three collinear EPs (Szebehely 1967). The R3BP has been investigated by several researchers and scientists under different modifications, (for example, see Singh and Leke (2014), Ansari et. al (2019a), Taura and Leke (2022), and Amuda and Singh (2022).

The formulation of the classical R3BP did not consider the case when the primaries as sources of radiation pressure. Radiation pressure acts as an orbital perturbation and can displace a dust grain from its position. Radzievskii (1950,1953) discussed the introduction of radiation pressure of one and both primaries and observed that their presence allows for the existence of additional EPs. In view of this, several investigations have been carried out when one or both primaries are sources of radiation pressure. Notable among these are Singh and Ishwar (1999), Singh and Leke (2010, 2012, 2013a) and Singh and Sunusi (2020).

Further, characterization of the primaries which involves radiation force is the perturbing effect of Poynting–Robertson (P–R) drag. This force is a component of the radiation force and can sweep small particles of the solar system into the Sun at a cosmically rapid rate. Several authors have conducted researches on the R3BP with P-R drag, amongst them are Ragos and Zafiropoulos (1995), Kushvah (2008), Das et. al (2009) Singh and Abdulkarim (2014), Singh and Amuda (2019), Amuda et. al (2021), and Amuda and Singh (2022).

The formulation of the classical R3BP assumes that the masses of primaries are constant. However, the existence of absorption in stars motivated scientists to formulate the R3BP with variable mass(es). During evolution, celestial bodies change their masses, especially in a double star system were mass changes rather intensively. An interesting example of mass loss is the real physical scenario of those transiting exoplanets whose atmospheres are escaping because of the severe levels of energetic radiations, coming from their nearby parent stars, hitting them.

The R3BP with variable masses formulation is relevant in various astronomical and engineering contexts, such as, investigating the dynamics of spacecraft near asteroids or comets with variable masses due to surface outgassing. Investigating dynamics of binary stars with mass transfer between them, and also studying dynamics in the Earth-Moon system during lunar mass discharge. Several authors such as Singh and Leke (2010, 2012, 2013a, b, c), Ansari et. al (2019b), Leke and Singh (2023), Leke and Shima (2023), Leke and Mmaju (2023) and, Leke and Orum (2024) have carried out interesting investigations of the variable mass R3BP under diverse characterizations.

Inspired by the vast applications of the R3BP with variable masses, we thought it expedient to investigate in this paper, the locations of the triangular EPs when both primaries have variable masses and the smaller primary is an intense emitter of radiation force, which is a component of radiation pressure and P-R drag. The importance of the study of the R3BP with variable mass is that the model provides a more accurate representation of the real world dynamics of celestial bodies and allows for more accurate dynamical predictions of their impending behaviors.

The paper organization of the paper is as follows. In Section 2, the equations of motion are stated for the non-autonomous and autonomized systems, while Section 3 delves into the locations of the triangular EPs. The numerical and graphical illustrations of the locations of the triangular EPs are given in Section 4, while the discussion and conclusion are given in Sections 5 and 6, respectively.

MATERIALS AND METHODS

Next, let m_1 and m_2 be the masses of two radiating stars and let m_3 be the mass of the infinitesimal body. The equations of motion of the photogravitational R3BP with variable masses have the form (Bekov 1988):

$$\ddot{x} - 2\omega\dot{y} - \dot{\omega}y = \omega^2 x - \frac{\mu_1(x-x_1)}{r_1^3} - \frac{\mu_2(x-x_2)}{r_2^3}$$
$$\ddot{y} + 2\omega\dot{x} + \dot{\omega}x = y\left[\omega^2 - \frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}\right] \quad (1)$$
$$\ddot{z} = -z\left[\frac{\mu_1}{r_1^3} - \frac{\mu_2}{r_2^3}\right]$$

where $r_1^2 = (x-x_1)^2 + y^2 + z^2$ and $r_2^2 = (x-x_2)^2 + y^2 + z^2$ are the distances of m_1 and m_2 from m_3, respectively. $\mu_1(t) = Gm_1(t)$, $\mu_2(t) = Gm_2(t)$, $\mu(t) = \mu_1(t) + \mu_2(t)$. ω is the angular velocity of revolutions of the primaries.

In estimating the light radiation force of the photogravitational R3BP with variable masses, Singh and Leke (2010) took into account just one of the three components of the light pressure field, which is due to the central force: the gravitational and the radiation pressure. The other two components arising from the Doppler shift and the absorption and subsequent re-emission of the incident radiation, was not considered. The last two components constitute the Poynting-Robertson (P-R) effect. The P-R effect will operate to sweep small particles of the solar system into the Sun at a cosmically rapid rate. Therefore, the equations of motion when the P-R drag effect of the smaller primary is considered, and both primaries vary their masses in accordance with UML with their motion governed by GMP have the forms (Singh and Leke 2010; Singh and Amuda 2014):

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega}y - \frac{\mu_1(x-x_1)}{r_1^3} - \frac{\mu_2 q(x-x_2)}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{(x-x_2)}{r_2^2}\{(x-x_2)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{x} - \omega y\right],$$
$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega}x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 q y}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{(x-x_2)}{r_2^2}\{(x-x_2)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{y} + \omega(x-x_2)\right], \quad (2)$$
$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{(x-x_2)}{r_2^2}\{(x-x_2)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{z}\right],$$

where $W_2 = \frac{\mu_2(1-q)}{c_d}$ represent the P-R drag of the smaller primary; q is its radiation pressure and c_d is dimensionless velocity of light.

The Mestschersky(1952) transformation (MT) is given by:

$$x = \xi R(t),\ y = \eta R(t),\ z = \varsigma R(t),\ \frac{dt}{d\tau} = R^2(t)$$
$$r_i = \rho_i R(t),\ (i = 1, 2), r = \rho_{12} R(t) \quad (3)$$

where

m_1 ;ξ, η, ζ, τ are the new variables and ρ_{12} is constant.

The unified Mestschersky Law (UML):

$$\vec{r},\ P_1, P_2 \quad (4)$$

The particular solutions of the Gylden-Mestschersky Equation (GME):

$$\vec{C}, C^* = \rho_{12}^2\omega_0, r\mu = \kappa C^{*2}, \kappa = \frac{\beta^2 - \alpha\gamma + \omega_0^2}{\omega_0^2} \quad (5)$$

where κ is a constant of mass variation

Substituting equations (3), (4) and (5) in (2), and choosing units of measurements, we get the equations:

$$\xi'' - 2\eta' = \kappa\left[\xi - \frac{(1-\upsilon)(\xi+\upsilon)}{\rho_1^3} - \frac{q\upsilon(\xi+\upsilon-1)}{\rho_2^3} - \frac{W_2}{\rho_2^2}\left[\frac{(\xi+\upsilon-1)}{\rho_2^2}\{(\xi+\upsilon-1)\xi' + \eta\eta' + \zeta\zeta'\} + \xi' - \eta\right]\right]$$
$$\eta'' + 2\xi' = \kappa\eta\left[1 - \frac{(1-\upsilon)}{\rho_1^3} - \frac{q\upsilon}{\rho_2^3}\right] - \frac{\kappa W_2}{\rho_2^2}\left[\frac{\eta}{\rho_2^2}\{(\xi+\upsilon-1)\xi' + \eta\eta' + \zeta\zeta'\} + \eta' + (\xi+\upsilon-1)\right] \quad (6)$$
$$\zeta'' = (\kappa-1)\zeta - \frac{\kappa(1-\upsilon)\zeta}{\rho_1^3} - \frac{\kappa q\upsilon\zeta}{\rho_2^3} - \frac{\kappa W_2}{\rho_2^2}\left[\frac{\zeta}{\rho_2^2}\{(\xi+\upsilon-1)\xi' + \eta\eta' + \zeta\zeta'\} + \zeta'\right]$$

where $W_2 = \frac{\upsilon(1-q)}{c_d}$ $\ 0 < \kappa < \infty$

$$\rho_1^2 = (\xi+\upsilon)^2 + \eta^2 + \zeta^2, \rho_2^2 = (\xi+\upsilon-1)^2 + \eta^2 + \zeta^2 \text{ and } 0 < \upsilon \le 0.5 \quad (7)$$

υ is the mass parameter and defined as the ratio of the mass of the smaller primary to the sum of the masses of the primaries and is such that $0 < \upsilon \le 0.5$.

Locations of the Triangular Points

The triangular EPs are the solution of equations (6) when $\xi \ne 0, \eta \ne 0, \zeta = 0$ and $\kappa \ne 0$, That is, we solve the equation

$$\xi - \frac{(1-\upsilon)(\xi+\upsilon)}{\rho_1^3} - \frac{q\upsilon(\xi+\upsilon-1)}{\rho_2^3} - \frac{W_2\eta}{\rho_2^2} = 0 \quad (8)$$
$$\eta\left[1 - \frac{(1-\upsilon)}{\rho_1^3} - \frac{q\upsilon}{\rho_2^3}\right] - \frac{W_2(\xi+\upsilon-1)}{\rho_2^2} = 0$$

To solve equations (8), we use perturbation method by first solving these equations when the PR-drag of the smaller primary is absent. From equation (8), when $W_2 = 0$, , we get

$$\rho_1 = 1 \text{ and } \rho_2 = q^{1/3} \quad (9)$$

Therefore, the solutions of (8), in presence of P-R effect, can be assumed to change slightly by ε_1 and ε_2:

$$\rho_1 = 1 + \varepsilon_1, \rho_2 = q^{1/.3} + \varepsilon_2. \varepsilon_1, \varepsilon_2 << 1 \quad (10)$$

Following the methodology used in Singh and Amuda (2014), the exact coordinates of the TEPs are

$$\xi = \xi_0 + \varepsilon_1 - q^{1/3}\varepsilon_2 \quad (11)$$
$$\eta = \pm\sqrt{\eta_0^2 + \left(\varepsilon_1 + q^{1/3}\varepsilon_2\right) - \left(1 + q^{2/3}\right)\left(\varepsilon_1 + q^{1/3}\varepsilon_2\right)}$$

where

$$\xi_0 = \frac{1}{2} - \upsilon + \frac{1}{3}(1-q) \quad (12)$$
$$\eta_0 = \frac{\sqrt{3}}{2}\left[1 - \frac{2}{9}(1-q)\right]$$

Next, we substitute equations (10), (11) into equations (8) and simplify by neglecting higher order terms of small quantities, to get the respective equations:

$$a_1\varepsilon_1 + b_1\varepsilon_2 = c_1 \quad (13)$$
$$a_2\varepsilon_1 + b_2\varepsilon_2 = c_2$$

where

$$a_1 = 3(1-\upsilon)(\xi_0+\upsilon) + \frac{1}{2\eta_0}W_2,$$
$$b_1 = 3\upsilon(\xi_0+\upsilon-1)q^{-\frac{1}{3}} - \left[2\eta_0 q^{-1} - \frac{(2-q^{2/3})q^{-1/3}}{2\eta_0}\right]W_2, \quad (14)$$
$$c_1 = -q^{-2/3}\eta_0 W_2$$
$$a_2 = 3(1-\upsilon)\eta_0 - q^{2/3}W_2,$$
$$b_2 = 3\upsilon\eta_0 q^{-\frac{1}{3}} + \left(2\xi_0 q^{-1} + q^{-1/3} + 2\upsilon q^{-1} - 2q^{-1}\right)W_2$$
$$c_2 = (\xi_0+\upsilon-1)q^{-\frac{2}{3}}W_2$$

Equations (13) is a system of two equations in two variables $\varepsilon_1$and $\varepsilon_2$and can be determined using the Cramer's Rule:

$$\varepsilon_1 = \frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1},\ \varepsilon_2 = \frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1} \quad (15)$$

Substituting the coefficients a_i, b_i, c_i $(i=1,2)$, and applying equations (12), to get

$$\varepsilon_1 = -\frac{2W_2}{3\sqrt{3(1-\upsilon)}} \text{and } \varepsilon_2 = \frac{\rightleftarrows W_2\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows}{3\upsilon\sqrt{3}\rightleftarrows} \quad (16)$$

Substituting equations (16), in equations (14), we at once get

$$\xi = \frac{1}{2} - \upsilon + \frac{1}{3}(1-q) + \frac{(1+\upsilon)W_2}{3\sqrt{3}\upsilon(1-\upsilon)} \quad (17)$$
$$\eta = \frac{\sqrt{3}}{2}\left[1 - \frac{2}{9}(1-q) + \frac{2(1-3\upsilon)W_2}{9\sqrt{3}\upsilon(1-\upsilon)}\right]$$

The points given by equations (17) are the TEPs and are defined by the mass parameter, radiation pressure and P-R drag of the smaller primary. These points are denoted by $L_4$and L_5.

The triangular EPs $L_j\left(x_j, y_j\ 0{:}\, j=4{,}5\right)$ of the system of equations of the non-autonomous system (2) are sought using the MT (3) in the form (Luk'yanov 1989):

$$x_j(t) = \xi_j R(t),\ y_j(t) = \eta_j R(t), \quad (18)$$

where ξ_j, η_j $(j=4{,}5)$ are triangular EPs of autonomized system obtained in equation (17).

These solutions differ from those obtained in equations (17) due to the function $R(t)$.

Numerical Illustration

In this section, we devote our efforts to the numerical and graphical illustrations of the locations of the triangular EPs for the general case in which the test particle is a dust grain in the gravitational field of the primaries having variable masses with P-R drag of the smaller primary. We carry out our numerical exploration for all binaries having mass parameter in the interval $0 < \upsilon \leq 0.5$, which covers most astronomical systems, especially those containing planetary systems, while values for the radiation pressure of the smaller primary and the velocity of light are taken from Amuda ad Singh (2022):$q_2 = 0.99996$and $c_d = 46939.84$, respectively. All numerical computations have been carried out using the software package, Mathematica (Wolfram 2017).

Locations of triangular equilibrium points

The TEPs are given in equation (17) and the locations are numerically computed in Table 1 below

Table 1: Coordinates of the TEPs for $0 < \upsilon \leq 0.5$

	(Classical)		Radiation Force Effects		
Mass ratio	ξ	$\pm\eta$	W_2	ξ	$\pm\eta$
0.000000001	0.5	0.866025	4.26077×10^{-10}	0.527346	0.91336
0.00001	0.499990	0.866025	4.26077×10^{-10}	0.500006	0.866022
0.01	0.490000	0.866025	4.26077×10^{-10}	0.490013	0.866018
0.1	0.4	0.866025	4.26077×10^{-10}	0.400013	0.866018
0.2	0.3	0.866025	4.26077×10^{-10}	0.300013	0.866018
0.3	0.2	0.866025	4.26077×10^{-10}	0.200013	0.866018
0.4	0.1	0.866025	4.26077×10^{-10}	0.100013	0.866018
0.5	0	0.866025	4.26077×10^{-10}	0.0000133335	0.866018

In Table 1, we have computed the positions of the triangular EPs defined by the changing mass parameters of the binary, the radiation pressure and P-R drag of the smaller primary. For each mass parameter, we first compute the case when both primaries are non-radiating (classical case) and the case when the radiation pressure and P-R drag effects of the smaller primary is considered, is computed on the last two columns of Table 1. It is seen that, for all mass parameter, the locations of the triangular EPs under combined effects of the radiation force differ from those of the classical case. The 3D plots of the positions of the triangular EPs have been plotted in Fig 1 to 4 under different characterizations of the smaller primary.

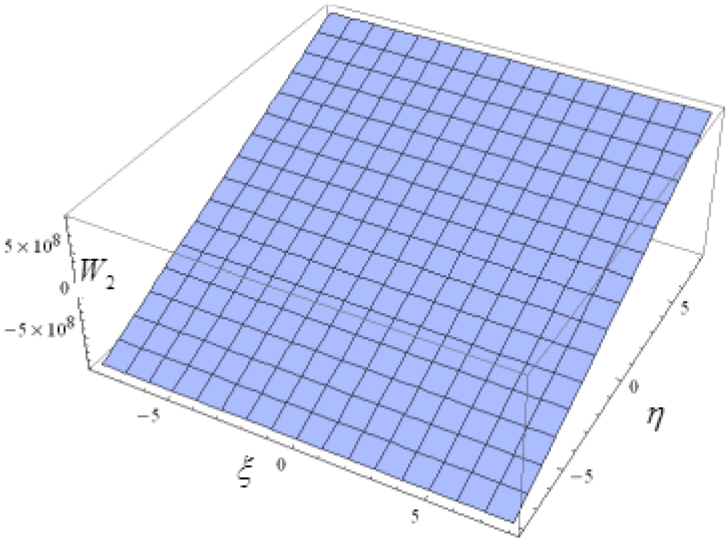

Figure 1: Locations of Triangular EPs for$\upsilon = 0.000000001$with P-R Drag

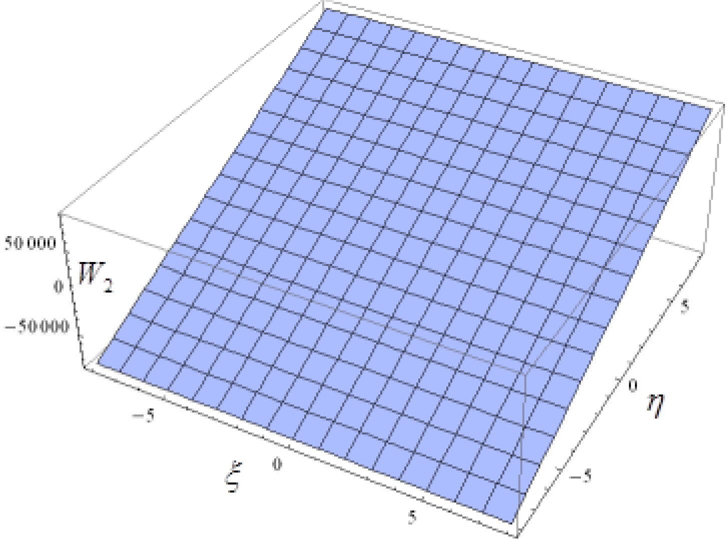

Figure 2: Locations of Triangular EPs for $\upsilon = 0.00001$, with P-R Drag

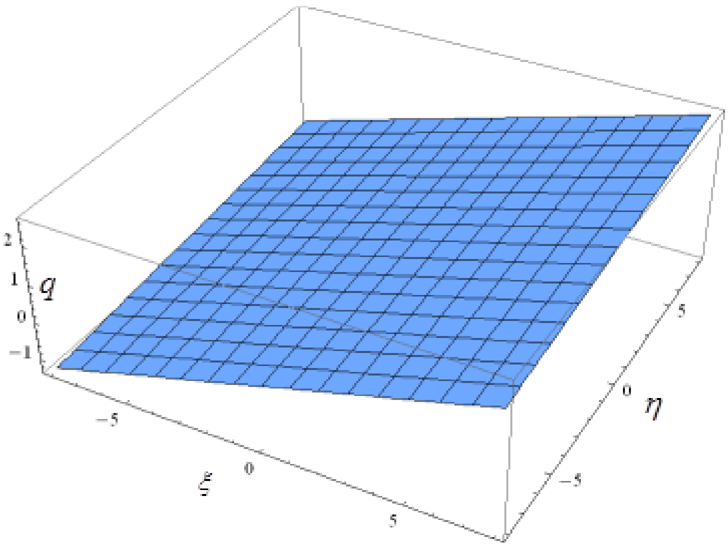

Figure 3: Locations of Triangular EPs for $\upsilon = 0.3$, with Radiation Pressure

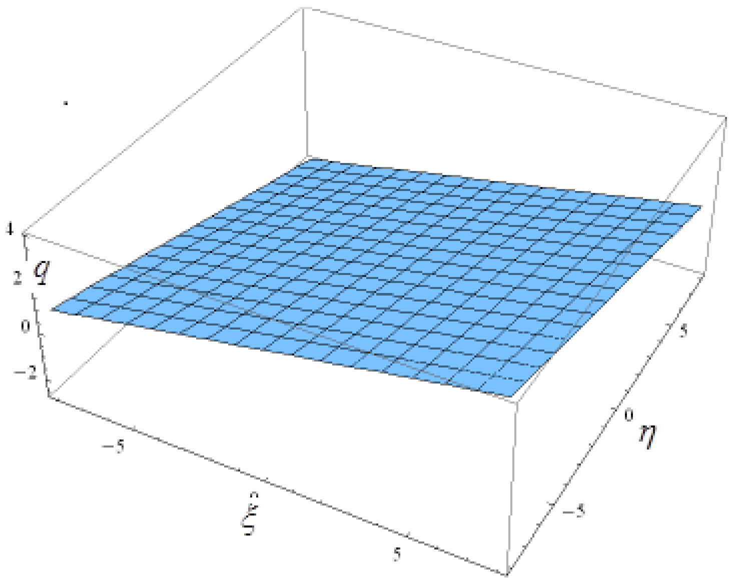

Figure 4: Locations of Triangular EPs for $\upsilon = 0.5$with Radiation Pressure.

Fig. 1 to Fig. 4 shows the effects of the changing mass parameters on deviations in the locations of the triangular EPs under radiation pressure and P-R drag effects of the smaller primary. Fig 1 and Fig 2 is drawn for $\upsilon = 0.000000001$ and $\upsilon = 0.00001$ when $W_2 = 4.26077 \times 10^{-10}$, respectively. while Fig. 3 and Fig. 4, shows the deviations in the locations of the triangular EPs when $\upsilon = 0.3$ and $\upsilon = 0.5$, under radiation pressure of the smaller primary.

Discussion

The paper investigates the locations of the triangular EPs of the R3BP with P-R drag force and variable masses of the primaries, when the smaller primary is taken as a emitter of radiation force. The masses of both primaries are assumed to vary with time under the framework of the unified Mestschersky law and their dynamics determined by the GME. The equations of motion (2) of the R3BP which depends on time have been obtained and are different from those of Bekov (1988), Singh and Leke (2010), Singh et. al (2010) due to the introduction of the P-R drag of the smaller primary. Because the equations of motion have variable coefficients, we used the MT, the UML and the particular solutions of the GME, to convert the time variable equations to one which have constant coefficients, in equations (3). These equations are analogous to those of Singh and Amuda (2014) but differs due to the parameter κ.

The particular solutions of the equations of motion of the autonomized system are solved using perturbation method, and with the aid of Crammer's rule, we were able to obtain the coordinates of the triangular EPs, we are denoted by $L_4$and L_5. These points are analogous to those obtained in Singh and Amuda (2014), Singh and Abdulkarim (2014) but differs from the triangular points obtained by Bekov (1988), Singh and Leke (2010) and Singh et. al (2010). The triangular EPs of the non-autonomous dynamical equations differ from those of the autonomized equations due to a function of time.

CONCLUSION

The restricted three-body problem (R3BP) is a fascinating problem that has been receiving attentions of astronomers and scientists because of its vast implications in diverse scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics. In this paper, we examine the locations of the triangular equilibrium points of the R3BP with variable masses when the Doppler shift and the absorption and subsequent re-emission of the incident radiation, is considered. The primaries are assumed to vary under the unified Mestschersky law and their dynamics defined in 1952 by the Gylden-Mestschersky equation, while the smaller primary is assumed to be a radiation emitter with P-R drag. The dynamical equations are obtained for both the non-autonomous with variable coefficients and autonomized system with constant coefficients. Further, the locations of the triangular points of the autonomized systems are obtained using perturbation method. It is seen that the positions are defined by the mass parameter, radiation pressure and P-R drag of the smaller primary. The triangular points of the non-autonomous equations are obtained with help of the Mestschersky transformation, and differ from those of the autonomized system due to a function of time. The equilibrium points have several applications in space missions, satellites constellations and station-keeping.

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