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# A MONTE CARLO STUDY ON THE PERFORMANCE OF EMPIRICAL THRESHOLD AUTOREGRESSIVE MODELS UNDER VIOLATION OF STATIONARITY ASSUMPTIONS

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**ABSTRACT**

One of the major importance of modeling in time series is to forecast the future values of that series. And this requires the use of appropriate method to fit the time series data which are dependent on the nature of the data. We are aware that most financial and economic data are mostly non-stationary. . The study is an extension of the work of Romsen et al (2020) which dealt with forecasting of nonlinear data that are stationary with only two threshold regimes. The study recommendations that In further research, the above models can be extended to other regimes (such as the 3 – regimes Threshold models) as well as comparing them with other regimes to understand the behaviors of the other regimes in selecting a suitable model for a data. STAR (2,1) and SETAR (2,2) are recommended to fit and forecast nonlinear data of trigonometric, exponential and polynomial forms respectively that are non-stationary.

**Keywords**: Stationarity, Exponential, Regimes, Auto regressive

## INTRODUCTION

A time series forecasting is the use of model to predict the future values based on the past values. Predictions were made when the actual outcome of event(s) may not be known until in some future time. The goal of time series is to forecast and identify meaningful characteristics in data that can be used in making statements about the future outcomes. Time series are generally classified into stationary and non-stationary. A stationary time series has its statistical properties such as the mean, variance, auto covariance's, auto correlation etc. are all constant over time. Since its characteristics are constant, then a stationary time series can be easily predicted because all its statistical properties that were constant in the past will for likely be constant at the future. On the other hand, a non-stationary time series is the one whose statistical properties change over time. And this need to be further converted into stationary data because using non-stationary time series especially in financial models produces unreliable result and leads to poor understanding and forecasting.

### Related literatures

The most commonly used linear time series models are Auto regressive (AR), moving average (MA), Auto regressive moving average (ARMA) and the Auto regressive integrated moving average (ARIMA) models. Autoregressive (AR) process is a model in which future values are forecast purely on the basis of past values of the time series. Moving average (MA) process is a model in which future values are forecast purely on the basis of past shocks (or noise or random disturbances). A model that uses both past values of the time series and past shocks is called an autoregressive-moving average (ARMA) process. And lastly, the ARMA model of a differenced series is called an ARIMA model. The approach proposed by Box and Jenkins came to be known as the Box-Jenkins methodology to ARIMA models, where the letter "I", between AR and MA, stood for the word "Integrated". The three models (AR, MA and ARMA) assumes that the time series is stationary, that is its statistical properties are all constant over time. As stated earlier that most time series problems are known to be non-stationary and there is need to be transformed to achieve stationarity. Differencing is usually needed to be employed, hence an ARMA model of a differenced series is called an ARIMA model. Where the word Integrated was included in order to obtain an output needs to be anti-differenced or integrated, to forecast the original series.

Makridaki&Insead (1997) aims to apply the Box-Jenkins methodology to ARIMA models and determine the reasons why in empirical tests in the post-sample forecasting the accuracy of such models is generally worse than much simpler time series methods. The paper concludes that the major problem is the way of making the series stationary in its mean (i.e. the method of differencing) that has been proposed by Box and Jenkins. The result also shows that using ARMA models to seasonally adjusted data slightly improves post-sample accuracies while simplifying the use of ARMA models. The result also confirmed that transformations slightly improve post-sample forecasting accuracy, particularly for long forecasting horizons. And finally the result demonstrated that AR(1), AR(2) and ARMA(1,1) models can produce more accurate post-sample forecasts than those found through the application of Box-Jenkins methodology.

Guidolin et. al (2019) examines the comparative predictive performances of a number of linear and non-linear models for stock and bond returns in the G7 economies. Non-linear models appear to forecast better in the case of US and UK asset returns, whereas simple linear models (such as the random walk and univariate autoregressions) appear to forecast better in the case of French, Italian and German asset returns.

## MATERIALS AND METHODS

### Estimation of Parameter to be Fixed for Simulation Model

From the p^th^ order of autoregressive [AR (p)], the first order AR (1) and second order AR (2) were deduced

AR (1): $X_t = \propto X_{t-1} + e_t$- (1)

$$\sigma_t^2 = \frac{\sigma_e^2}{1-\propto^2} \quad (2)$$

AR (2): $X_t = \propto_1 X_{t-1} + \propto_2 X_{t-2} + e_t$ -- (3)

$$\sigma_t^2 = \frac{\sigma_e^2}{1-\alpha_1^2-\propto_2^2} \quad (4)$$

The specified parameter values were estimated by fixing $\sigma_e^2 = 2,\ \sigma_t^2 = 4\ and\ \alpha_1 = 0.7,$ to be $\alpha_2 = 0.6$. This imply that the simulation was simulated using these parameters estimated will have a unit root and hence non-stationary

**Models Selected for Simulation**

The simulation Data is generated from several linear and nonlinear second orders of general classes of autoregressive functions given as follows:

**Model 1.** AR(2): $X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t$, (5)
$t = 1,2,\ldots,20,40,60,80,100,120,140,160,180, and\ 200.$
$i = 1,2,\ldots,1000$

The following codes were written to simulate data of sample size 20 from model 1 above

```
e1 <- rnorm (20,1,2)
x1 <- rnorm (20,2,4)
for (t in 3:20)x[t] <- 0.7 * x[t - 1] - 0.6 * x[t - 2] + e[t]  -   -    (6)
```

Up to the model 4 which is

Model 4: PL(2):$X_{ti} = 0.7X_{ti-1}^2 - 0.6X_{ti-2} + e_t$, (7)

**Test of Linearity/ Nonlinearity**

Two tests of nonlinearity were used to confirm the nonlinear data generated, these are Keenan and Tsay F tests. Both statistics test null hypothesis that data series is nonlinear. The procedure for each statistic are stated as follows

**Keenan's One-Degree Test for Nonlinearity**

The Keenan Test examines and tests the Quadratic Nonlinearity Hypothesis and provides information on threshold nonlinearity. This situation refers to the F test in the following model:

$$X_t = \mu + \sum_{u=-\infty}^{\infty} a_u \varepsilon_{t-u} + \sum_{u,v=-\infty}^{\infty} a_{u,v}\varepsilon_{t-u}\varepsilon_{t-v} \quad (8)$$

**Tsay's Test for Nonlinearity**

Tsay's (Tsay, 1986) linearity test is based on recursive auto regression and destructive term estimators, and firstly, the recursive auto regressions are established starting from ***b*.** is expressed as *b*=(n/10) + p observation value in return for the *p* and the relevant *d* values with *AR* level, and then the model is established between $\hat{e}_t$values and $(1, X_{t-1}, X_{t-2}, \cdots, X_{t-p})$. Then the following test is obtained among the inclusions of the model formed with $\hat{e}_t$

$$\hat{F}(p,d) = \frac{[\sum \hat{e}_t^2 - \sum \hat{\varepsilon}_t^2]/(p+1)}{\sum \hat{\varepsilon}_t^2/(n-d-b-p-h)} \quad - \quad (9)$$

**Estimation of SETAR Models**

Estimation of SETAR models is often carried out by the nonlinear least squares method. Properties of the resulting estimators are hard to establish, however. As a matter of fact, limiting properties of the least squares estimates are only available for the two-regime SETAR(*p*) model. It is believed that similar results can be shown for multi-regime SETAR models. Chan (1993) considered the estimation of the two-regime SETAR(*p*) model of Equation (9). Let $\emptyset = (\emptyset_0, \emptyset_1, \cdots, \emptyset_p)'$ and $\theta = (\theta_0, \theta_1, \cdots, \theta_p)'$ be the parameter vectors of the two regimes, respectively. Let $\mathbf{\Theta} = (\emptyset', \theta, r, d)'$ be the coefficient vector of the model in Equation (9) and $\Theta_0$ be the true coefficient vector. Suppose that the realizations $\{X_1, \cdots X_T\}$ are available, where $T$ denotes the sample size.

**Smooth transition Autoregressive (STAR) model**

Another class of nonlinear time series models is smooth transition autoregressive (STAR) models. The STAR model is similar to the self-exciting threshold autoregressive model. The main difference between these two models is the mechanism governing the transition between regimes. A two-regime model will be considered here. The concept can be extended to the case with more than two regimes.

**Evaluation and Comparison of the Models**

Simulation studies were conducted to investigate the performances of Autoregressive (AR), Self- Exciting Threshold autoregressive (SETAR), Smooth Transition Autoregressive Models (STAR) in fitting and forecasting linear, trigonometric, exponential and polynomial forms of autoregressive time series under study (model 1-4). Effect of sample size and the stationarity of the models were examined on each of the general linear and nonlinear data simulated.

**Criteria for Assessment of the Study**

The goodness of fit for each model was assessed using three common information criteria (AIC, MAPE and MSE) in time series. The model with lowest criteria is the best among the models.

**Akaike Information Criteria**

There are several information criteria available to determine the best model of autoregressive process. All of them are likelihood based; the well-known is Akaike information criterion (AIC).

## RESULTS AND DISCUSSION

**Table 1: MSE of SETAR and STAR Models across Sample Sizes Fitted on AR (2):** $\mathbf{X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t}$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 3.4916 | 3.3893 | 4.8638 | 4.1091 | 4.4153 | 3.5648 |
| 40 | 3.9660 | 3.7852 | 5.8296 | 4.1983 | 4.1402 | 4.1849 |
| 60 | 4.0522 | 4.0157 | 5.3391 | 4.9072 | 4.9424 | 4.8346 |
| 80 | 3.9358 | 4.1073 | 5.0705 | 4.6135 | 4.6693 | 4.6003 |
| 100 | 3.9329 | 4.0630 | 5.1909 | 4.1576 | 4.2277 | 4.1639 |
| 120 | 3.7256 | 3.8715 | 5.0806 | 4.0375 | 4.0628 | 4.1595 |
| 140 | 3.7219 | 3.7614 | 4.5859 | 3.9492 | 3.9252 | 4.0130 |
| 160 | 3.7069 | 3.6009 | 4.0380 | 3.9196 | 3.9352 | 3.9730 |
| 180 | 3.7286 | 3.5142 | 3.9421 | 3.9159 | 3.9050 | 3.9380 |
| 200 | 3.5581 | 3.1785 | 3.9928 | 3.7854 | 3.7638 | 3.7683 |

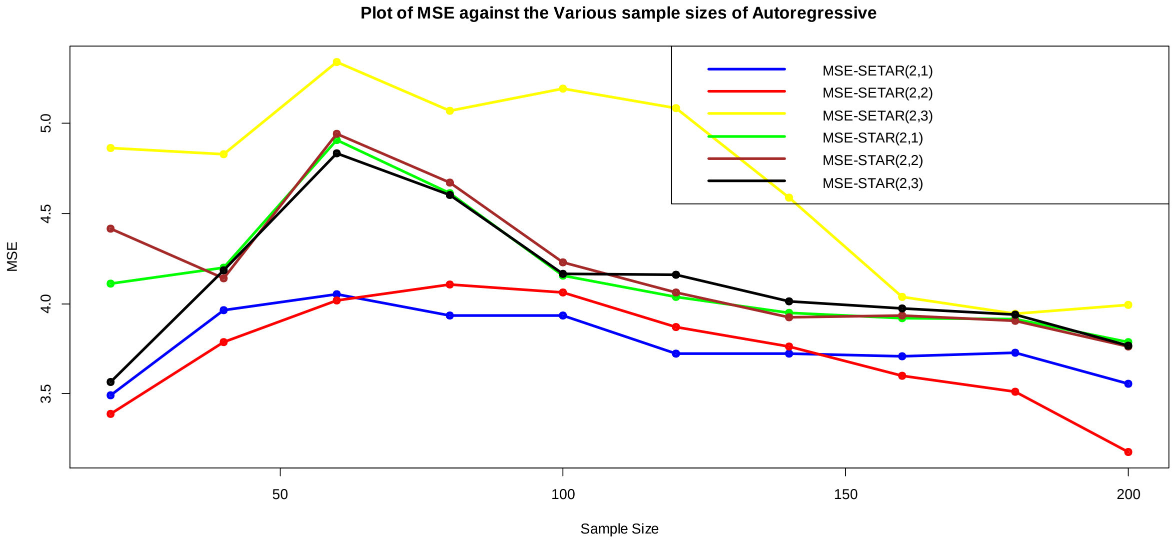



Figure 1: MSE of the fitted models on linear AR

**Table 2: MAPE of SETAR and STAR Models across the Sample Sizes Fitted on AR:** $X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 0.1191 | 0.1198 | 0.1196 | 0.1195 | 0.1197 | 0.1192 |
| 40 | 0.1187 | 0.1198 | 0.1196 | 0.1195 | 0.1198 | 0.1190 |
| 60 | 0.1188 | 0.1199 | 0.1193 | 0.1195 | 0.1199 | 0.1190 |
| 80 | 0.1186 | 0.1196 | 0.1190 | 0.1192 | 0.1199 | 0.1187 |
| 100 | 0.1187 | 0.1195 | 0.1193 | 0.1193 | 0.1197 | 0.1189 |
| 120 | 0.1179 | 0.1196 | 0.1194 | 0.1192 | 0.1195 | 0.1184 |
| 140 | 0.1184 | 0.1195 | 0.1192 | 0.1193 | 0.1196 | 0.1182 |
| 160 | 0.1180 | 0.1195 | 0.1189 | 0.1188 | 0.1194 | 0.1181 |
| 180 | 0.1178 | 0.1194 | 0.1186 | 0.1183 | 0.1193 | 0.1180 |
| 200 | 0.1179 | 0.1195 | 0.1187 | 0.1187 | 0.1193 | 0.1182 |

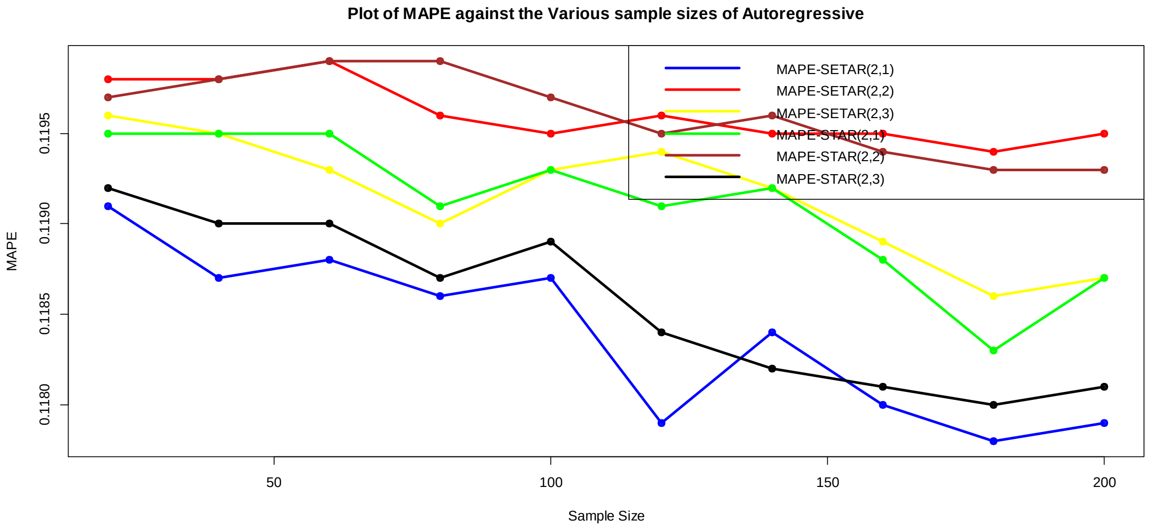



Figure 2: MASE of the fitted models on linear AR

**Table 3: AIC of SETAR and STAR Models across the Sample Sizes Fitted on AR:** $X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 39.0071 | 31.4197 | 38.8472 | 34.2640 | 35.7014 | 30.3998 |
| 40 | 63.1104 | 64.2440 | 99.3844 | 69.3870 | 67.8300 | 65.2588 |
| 60 | 101.9361 | 102.0535 | 145.4402 | 101.4427 | 101.8713 | 102.5476 |
| 80 | 129.4853 | 112.8418 | 213.8843 | 128.3188 | 129.2798 | 130.0890 |
| 100 | 150.9370 | 148.1906 | 247.2784 | 148.4934 | 150.1648 | 150.6457 |
| 120 | 171.8259 | 170.4354 | 254.7929 | 173.4744 | 174.2252 | 179.0462 |
| 140 | 197.9930 | 197.4720 | 255.5188 | 198.2904 | 197.4382 | 202.5348 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 160 | 223.6301 | 225.6387 | 268.5576 | 224.5566 | 225.1940 | 228.7234 |
| 180 | 250.8862 | 247.9722 | 275.4175 | 251.7081 | 251.2052 | 254.7187 |
| 200 | 267.8437 | 264.5026 | 288.6473 | 272.2327 | 271.0831 | 273.3267 |

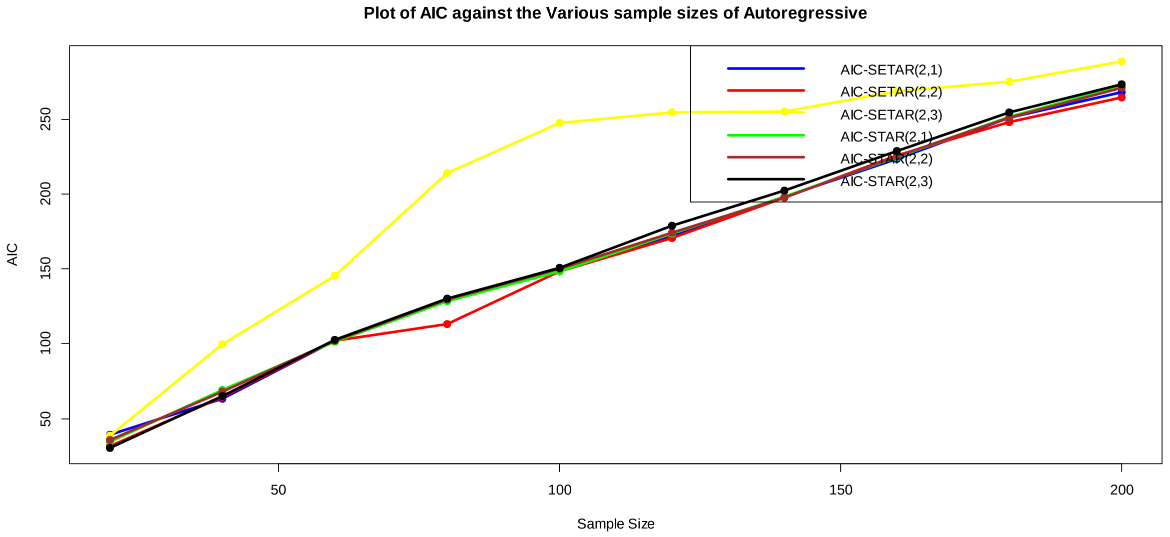



Figure 3: AIC of the fitted models on linear AR

**Table 4: MSE of SETAR and STAR Models across the Sample Sizes Fitted on Trigonometric:** $X_{ti} = 0.7sin(X_{ti-1}) - 0.6cosX_{ti-2} + e_t$

| Sample Size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 3.1317 | 3.9799 | 5.1181 | 4.9133 | 6.2620 | 2.9152 |
| 40 | 3.8272 | 4.0402 | 9.2536 | 4.6781 | 5.2848 | 4.3708 |
| 60 | 4.7649 | 5.2886 | 9.2200 | 5.0171 | 6.0036 | 5.0686 |
| 80 | 4.3264 | 5.0239 | 10.5929 | 5.0076 | 5.5915 | 5.1949 |
| 100 | 4.0247 | 4.4264 | 9.2359 | 4.0140 | 4.9518 | 4.4327 |
| 120 | 4.1275 | 4.3633 | 10.8565 | 4.1048 | 4.8799 | 4.5714 |
| 140 | 3.9586 | 4.1253 | 14.9776 | 3.9784 | 4.6057 | 4.2851 |
| 160 | 3.9487 | 4.0205 | 18.3391 | 3.8492 | 4.6082 | 4.3944 |
| 180 | 3.9404 | 3.9853 | 12.3866 | 3.6468 | 4.5457 | 3.9673 |
| 200 | 3.8162 | 3.9065 | 15.6097 | 3.5532 | 4.3554 | 4.4240 |

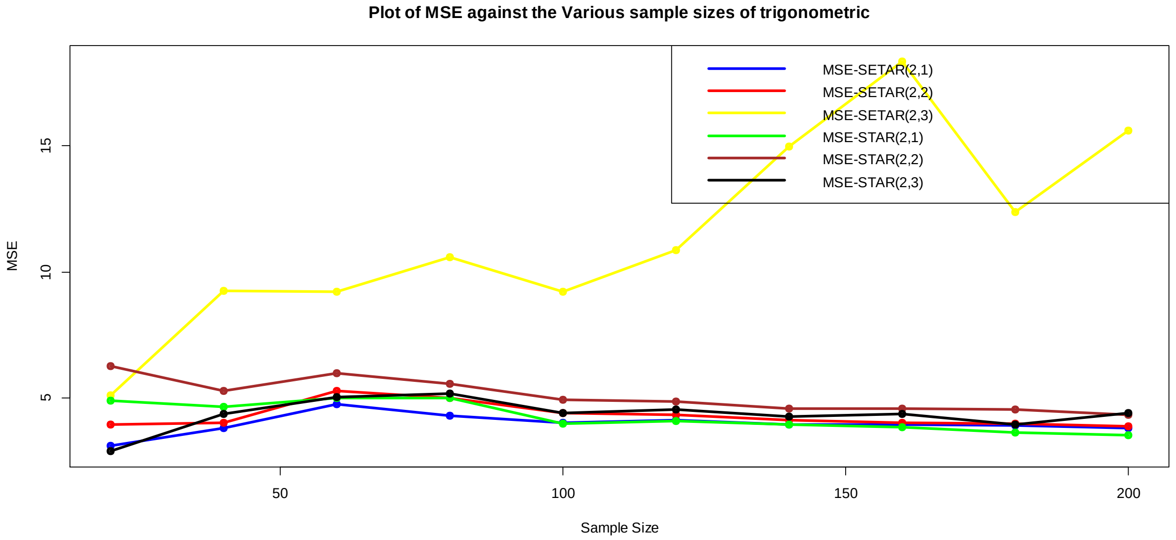
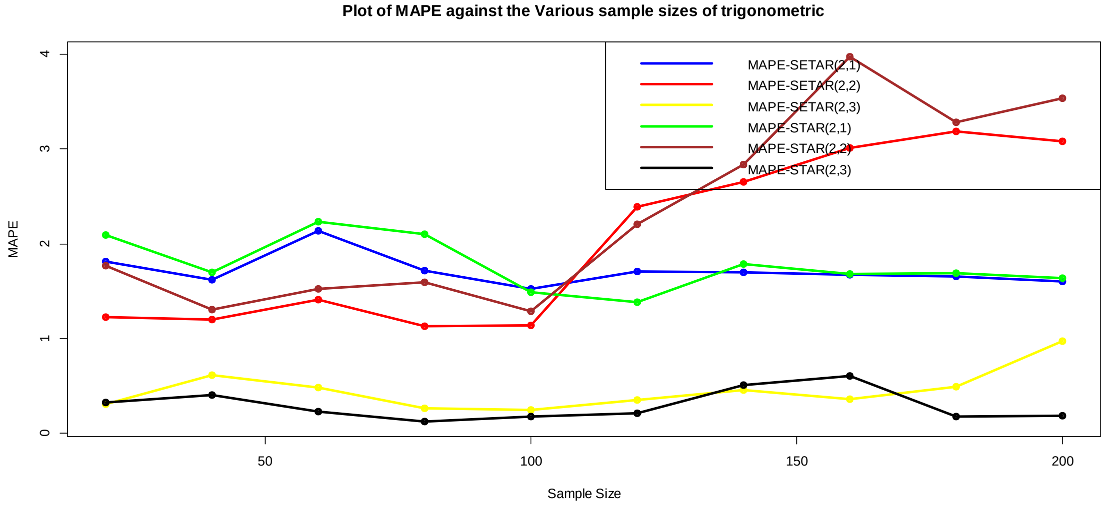



Figure 4: MSE of the fitted models on linear AR

**Table 5: MAPE of SETAR and STAR Models across the Sample Sizes Fitted on Trigonometric:**
$X_{ti} = 0.7sin(X_{ti-1}) - 0.6cosX_{ti-2} + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 1.8096 | 1.2232 | 0.3113 | 2.0948 | 1.7716 | 0.3272 |
| 40 | 1.6180 | 1.1999 | 0.6146 | 1.7030 | 1.3015 | 0.4018 |
| 60 | 2.1354 | 1.4137 | 0.7799 | 2.2351 | 1.5232 | 0.2263 |
| 80 | 1.7134 | 1.1274 | 0.2609 | 2.1014 | 1.5958 | 0.1241 |
| 100 | 1.5243 | 1.1416 | 0.2443 | 1.4898 | 1.2845 | 0.1790 |
| 120 | 1.7059 | 2.3918 | 0.9492 | 1.3879 | 2.2063 | 0.2119 |
| 140 | 1.6949 | 2.6559 | 0.4590 | 1.7847 | 2.8363 | 0.5093 |
| 160 | 1.6691 | 3.0106 | 0.3566 | 1.6834 | 3.9757 | 0.6079 |
| 180 | 1.6578 | 3.1846 | 0.4930 | 1.6911 | 3.2817 | 0.1740 |
| 200 | 1.5990 | 2.0819 | 0.9717 | 1.6339 | 3.5406 | 0.1868 |




. Figure 5: MAPE of the fitted models on linear AR

**Table 6: AIC of SETAR and STAR Models across the Sample Sizes Fitted on Trigonometric:**
$X_{ti} = 0.7sin(X_{ti-1}) - 0.6cosX_{ti-2} + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 36.8313 | 41.6249 | 29.0102 | 37.8389 | 42.6899 | 29.3989 |
| 40 | 67.6850 | 69.85207 | 132.3078 | 67.7157 | 72.5929 | 66.9982 |
| 60 | 107.6771 | 113.9337 | 216.4913 | 108.4712 | 113.5413 | 105.3834 |
| 80 | 131.1796 | 143.1360 | 247.8224 | 138.4694 | 143.7004 | 139.8138 |
| 100 | 153.2460 | 162.7592 | 333.0052 | 154.9776 | 165.9744 | 156.8997 |
| 120 | 184.1218 | 190.7881 | 363.8278 | 185.4591 | 196.2145 | 190.3778 |
| 140 | 206.6239 | 212.3983 | 399.2453 | 209.3212 | 219.8211 | 211.7200 |
| 160 | 233.7404 | 236.6254 | 402.7533 | 231.6594 | 250.4544 | 244.8519 |
| 180 | 260.8318 | 262.8701 | 479.2933 | 258.8910 | 278.5537 | 256.0564 |
| 200 | 281.8490 | 286.5255 | 523.4700 | 279.5675 | 300.2820 | 305.4102 |

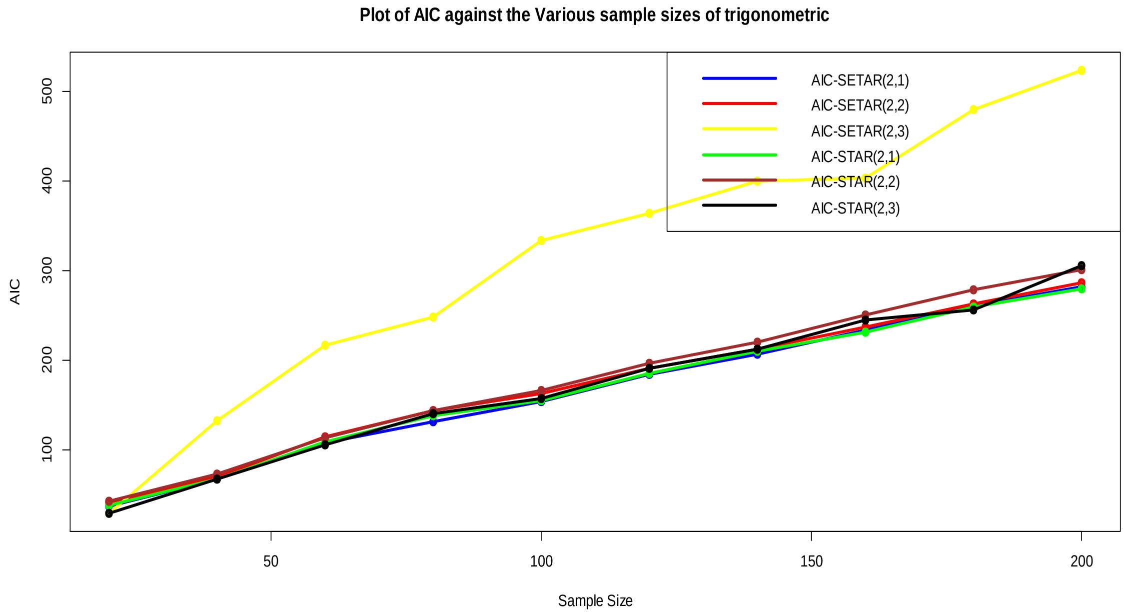



Figure 6: AIC of the fitted models on linear AR

**Table 7: MSE of SETAR and STAR Models across the Sample Sizes Fitted on exponential:** $X_{ti} = 0.7(X_{ti-1}) - exp(0.6X_{ti-2}) + e_t,$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 3.02894 | 2.8190 | 3.6895 | 3.3292 | 4.2523 | 3.8184 |
| 40 | 3.6768 | 3.1874 | 5.9413 | 4.3560 | 4.1094 | 3.5792 |
| 60 | 4.3777 | 4.1330 | 6.5362 | 4.7981 | 4.8647 | 4.2307 |
| 80 | 4.3474 | 4.1861 | 6.5450 | 4.6497 | 4.6758 | 4.3435 |
| 100 | 4.1241 | 3.9152 | 6.2770 | 4.5205 | 4.2475 | 4.1224 |
| 120 | 4.0419 | 3.9094 | 5.4367 | 4.1803 | 4.1205 | 4.0421 |
| 140 | 3.9211 | 3.8008 | 5.7090 | 4.1060 | 4.0306 | 3.9237 |
| 160 | 3.9863 | 4.1000 | 4.6101 | 4.0739 | 4.3352 | 3.9861 |
| 180 | 4.0032 | 4.1030 | 4.7183 | 4.1407 | 4.1641 | 3.8738 |
| 200 | 3.8448 | 3.8091 | 4.8949 | 4.1562 | 3.9389 | 3.7661 |

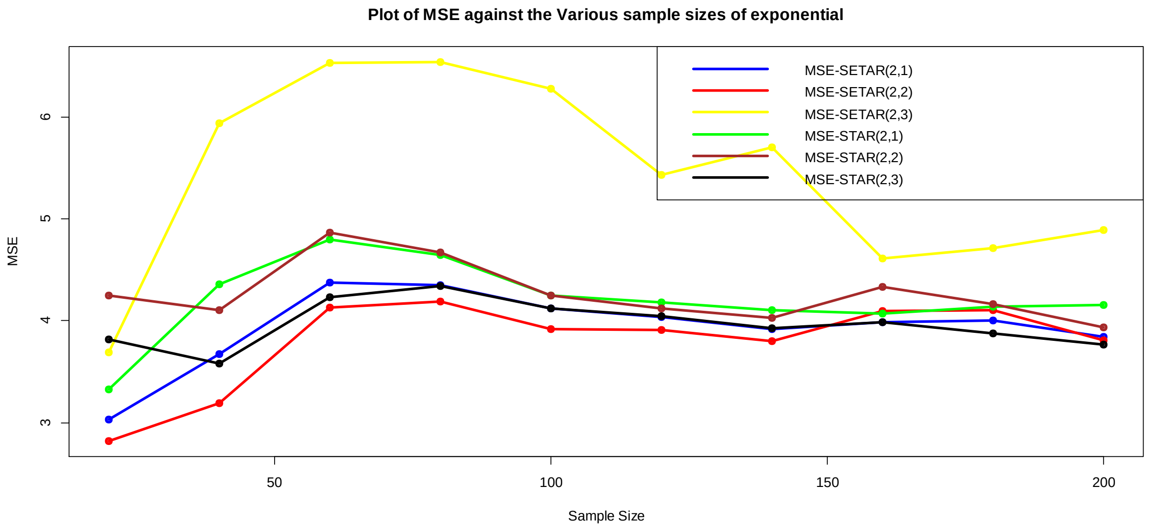



Figure 7: MSE of the fitted models on linear exponential

**Table 8: MAPE of SETAR and STAR Models across the Sample Sizes Fitted on exponential:** $X_{ti} = 0.7(X_{ti-1}) - exp(0.6X_{ti-2}) + e_t$,

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 1.3342 | 0.4047 | 0.1393 | 2.2055 | 1.3890 | 0.0856 |
| 40 | 1.6161 | 0.9140 | 0.1453 | 1.5373 | 1.0737 | 0.0353 |
| 60 | 1.8865 | 1.1834 | 0.1516 | 2.0111 | 1.3406 | 0.0166 |
| 80 | 1.5033 | 1.1021 | 0.0433 | 1.4952 | 1.4610 | 0.0251 |
| 100 | 1.7352 | 1.0882 | 0.0691 | 1.7313 | 1.2953 | 0.0164 |
| 120 | 1.6202 | 1.1602 | 0.0795 | 1.6204 | 1.1695 | 0.0255 |
| 140 | 1.5126 | 1.0245 | 0.1535 | 1.5145 | 0.9938 | 0.0239 |
| 160 | 2.7140 | 0.9093 | 0.1059 | 2.7156 | 0.9457 | 0.0128 |
| 180 | 2.5708 | 1.2089 | 0.1491 | 2.7088 | 1.0777 | 0.0205 |
| 200 | 2.9151 | 1.1651 | 0.2868 | 3.0500 | 1.2230 | 0.0284 |

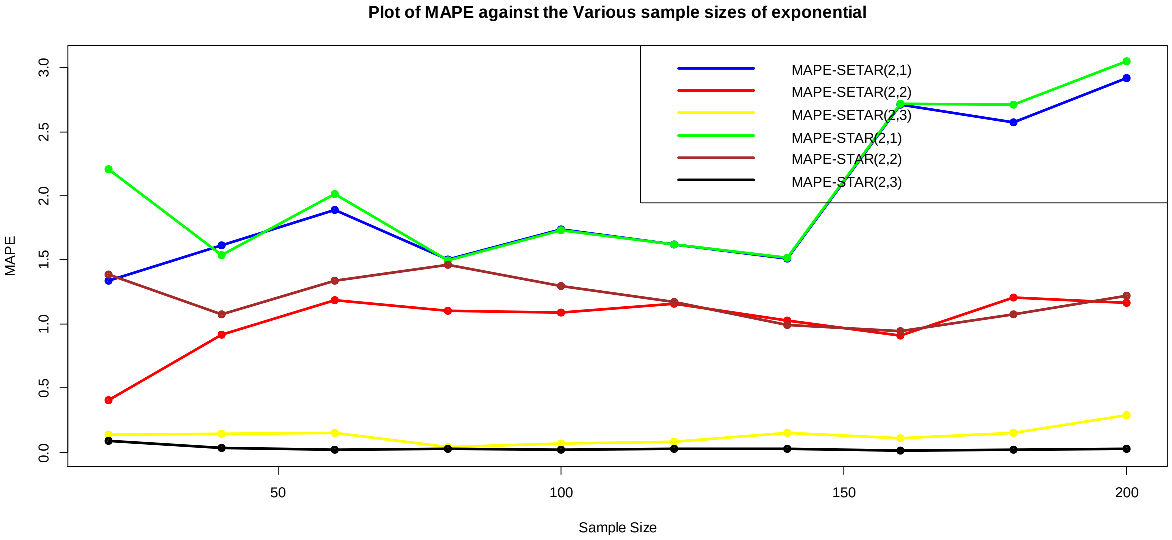



Figure 8: MAPE of the fitted models on linear exponential

**Table 9: AIC of SETAR and STAR Models across the Sample Sizes Fitted on exponential:** $X_{ti} = 0.7(X_{ti-1}) - exp(0.6X_{ti-2}) + e_t$,

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 28.1502 | 34.7274 | 33.7870 | 37.4488 | 34.9490 | 32.0544 |
| 40 | 66.0818 | 60.3679 | 99.7018 | 66.8613 | 62.5307 | 66.8624 |
| 60 | 102.5909 | 99.1399 | 122.6586 | 108.6190 | 100.9202 | 102.0925 |
| 80 | 131.5660 | 128.5413 | 181.0831 | 133.4939 | 129.3918 | 130.9438 |
| 100 | 155.6841 | 150.4854 | 225.2943 | 157.6435 | 150.6339 | 152.6946 |
| 120 | 181.6056 | 177.6071 | 298.4070 | 183.6111 | 175.9158 | 179.6469 |
| 140 | 205.2917 | 200.9311 | 334.8540 | 207.3851 | 201.1480 | 192.3197 |
| 160 | 235.2588 | 231.6578 | 376.7380 | 237.2519 | 240.6836 | 228.7670 |
| 180 | 263.6756 | 258.8677 | 397.9810 | 259.7636 | 262.7697 | 254.1874 |
| 200 | 283.3450 | 271.4776 | 400.4040 | 281.2083 | 280.1821 | 266.0767 |

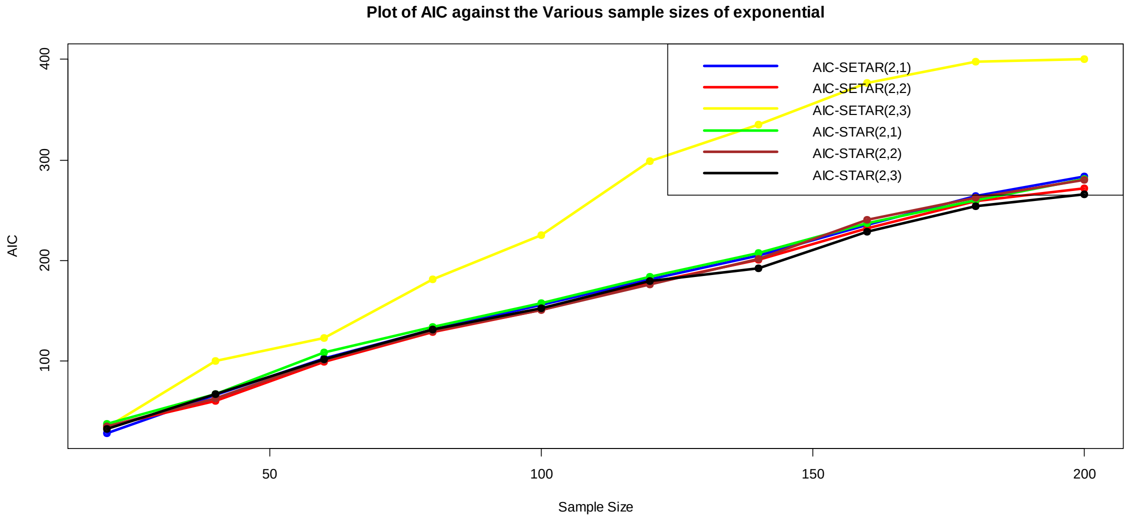



Figure 9: AIC of the fitted models on linear exponential

**Table 10: MSE of SETAR and STAR Models across the Sample Sizes Fitted on polynomial:**
$\boldsymbol{X_{ti} = 0.7(X^2{}_{ti-1}) - 0.6(X_{ti-2}) + +e_t}$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 853.4312 | 587.3325 | 757.1245 | 645.3714 | 395.0563 | 745.3714 |
| 40 | 793.5645 | 477.3434 | 703.3665 | 585.3477 | 337.4387 | 725.3426 |
| 60 | 777.1982 | 398.0101 | 694.7014 | 503.1258 | 320.3332 | 703.1235 |
| 80 | 741.2009 | 387.4829 | 659.2508 | 567.5661 | 304.9870 | 687.5608 |
| 100 | 721.0786 | 218.8977 | 646.2354 | 517.8725 | 299.4432 | 657.7815 |
| 120 | 693.3134 | 197.4535 | 612.0578 | 414.1761 | 282.8970 | 624.1766 |
| 140 | 656.0975 | 193.7908 | 585.3008 | 460.1515 | 270.0070 | 616.1530 |
| 160 | 614.7413 | 190.0952 | 548.6817 | 415.3444 | 190.0170 | 595.3423 |
| 180 | 583.3406 | 190.0163 | 517.3045 | 387.2332 | 189.3435 | 587.2315 |
| 200 | 551.9715 | 189.8818 | 492.1313 | 377.1748 | 179.6745 | 567.1702 |

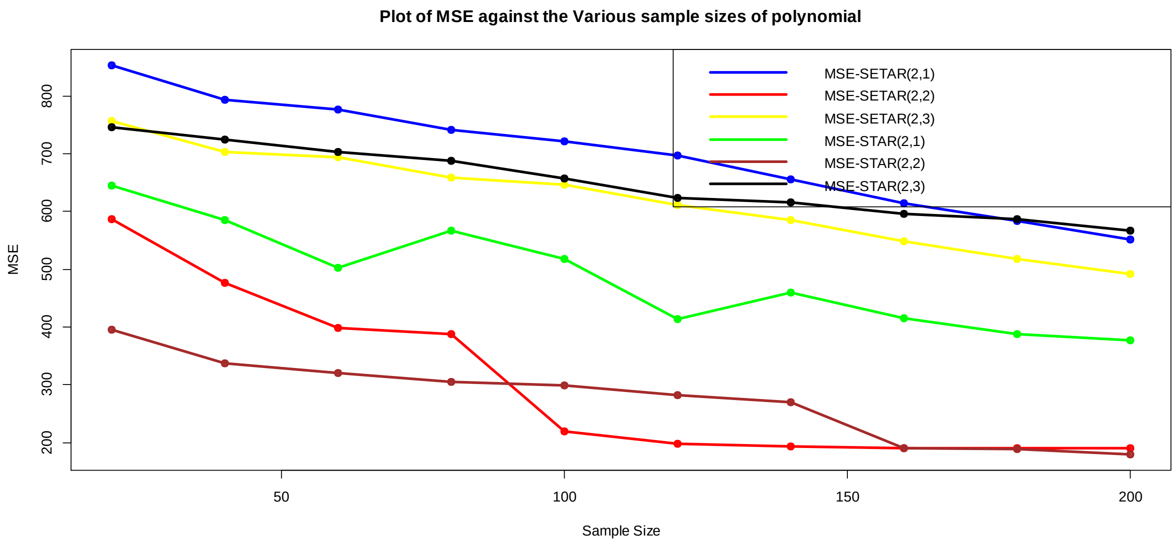



Figure 10: MSE of the fitted models on linear polynomial

**Table 11: MAPE of SETAR and STAR Models across the Sample Sizes Fitted on polynomial:**
$X_{ti} = 0.7(X^2{}_{ti-1}) - 0.6(X_{ti-2}) + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 1340.7926 | 980.6726 | 1995.2816 | 1897.5615 | 18.0715 | 1255.9872 |
| 40 | 1320.4634 | 943.7916 | 1875.5932 | 1858.2432 | 17.6511 | 1243.0681 |
| 60 | 1318.8533 | 898.0138 | 1846.5414 | 1787.5018 | 17.2596 | 1223.0994 |
| 80 | 1298.3176 | 552.9945 | 1787.3855 | 1508.1065 | 16.7551 | 1204.0231 |
| 100 | 1257.6158 | 521.8885 | 1654.4346 | 1210.8366 | 16.5065 | 1169.0772 |
| 120 | 1244.9438 | 396.4515 | 1574.4774 | 1198.8048 | 16.1365 | 1172.0032 |
| 140 | 1216.2613 | 389.9954 | 1327.0113 | 1186.8262 | 16.0938 | 1156.7621 |
| 160 | 1183.6763 | 382.0065 | 1225.5437 | 1156.9183 | 15.9967 | 1135.0931 |
| 180 | 1165.5827 | 367.8916 | 1203.2573 | 1123.6213 | 15.6539 | 1124.8775 |
| 200 | 1153.0547 | 367.8731 | 1162.7307 | 1020.4163 | 15.3027 | 1090.0752 |

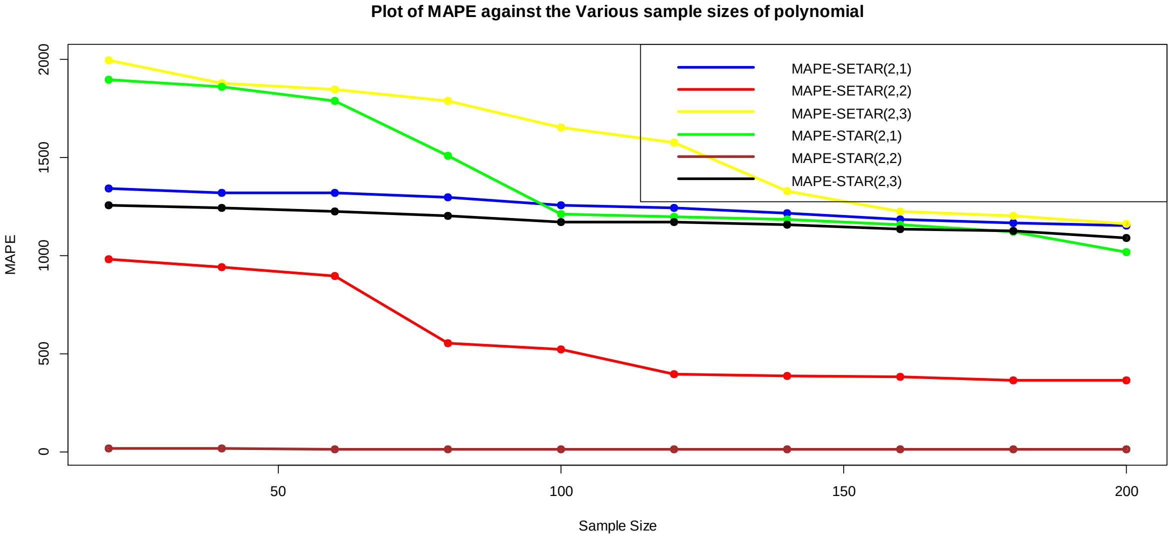



Figure 11: MAPE of the fitted models on linear polynomial

**Table 12: AIC of SETAR and STAR Models across the Sample Sizes Fitted on polynomial:**
$X_{ti} = 0.7(X^2{}_{ti-1}) - 0.6(X_{ti-2}) + e_t$

| Sample size | setar(2,1) | setar(2,2) | setar(2,3) | star(2,1) | star(2,2) | star(2,3) |
|---|---|---|---|---|---|---|
| 20 | 1340.7912 | 980.6720 | 1995.2842 | 1997.5645 | 118.0715 | 1010.0173 |
| 40 | 1150.4624 | 943.7914 | 1875.5967 | 1858.2445 | 117.6511 | 1009.1632 |
| 60 | 1188.8505 | 898.0165 | 1786.5432 | 1787.5050 | 117.2596 | 918.7866 |
| 80 | 1108.3111 | 552.9954 | 1687.3822 | 1508.1068 | 116.7551 | 918.7016 |
| 100 | 1097.6128 | 521.8885 | 1664.4313 | 1210.8318 | 116.5065 | 908.6115 |
| 120 | 1074.9462 | 396.4519 | 1574.4708 | 998.8074 | 116.1365 | 908.5015 |
| 140 | 1046.2612 | 389.9903 | 1527.0154 | 886.8266 | 116.0938 | 908.4335 |
| 160 | 1033.6735 | 382.0054 | 1485.5414 | 860.9141 | 115.9967 | 908.2214 |
| 180 | 1027.5814 | 367.8913 | 1453.2584 | 753.6258 | 115.6539 | 872.1825 |
| 200 | 1013.0575 | 367.8754 | 1362.7303 | 720.4166 | 115.3027 | 821.1074 |

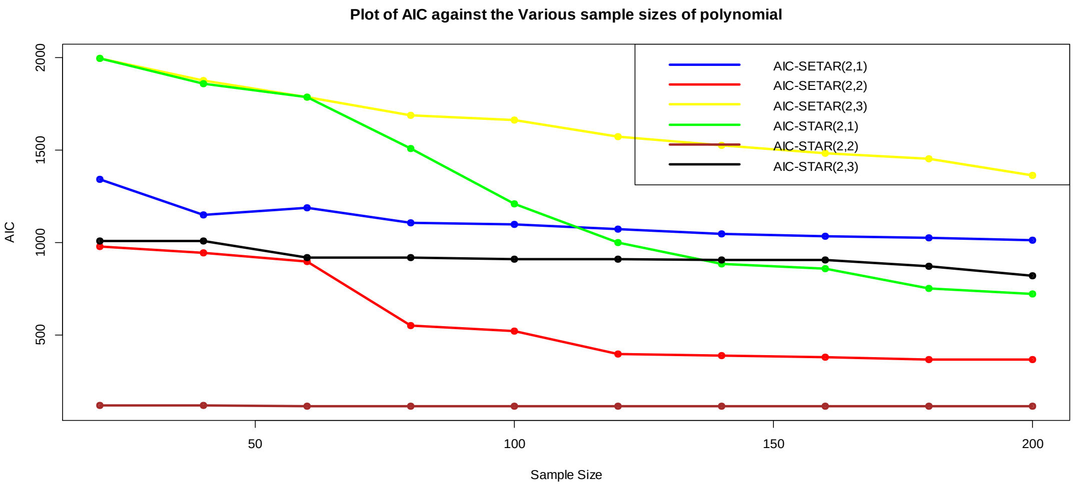



Figure 12: AIC of the fitted models on linear polynomial

**Table 13: MSE of the Forecast of the Best Models from AR:$X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t$**

| **Sample Size** | **n=20** | | **n=100** | | **n=200** | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,1)** | **setar (2,2)** | **setar (2,1)** | **setar (2,2)** | **setar (2,1)** | **setar (2,2)** |
| 5 | 1.18e-33 | 1.28e-33 | 1.11e-32 | 6.23e-34 | 1.20e-33 | 1.03e-32 |
| 10 | 1.16e-33 | 1.34e-32 | 1.28e-32 | 6.16e-34 | 1.28e-33 | 1.52e-32 |
| 15 | 1.99e-32 | 1.09e-32 | 9.42e-33 | 2.79e-33 | 4.57e-32 | 8.12e-33 |
| 20 | 2.67e-32 | 1.20e-32 | 7.61e-32 | 1.47e-32 | 4.61e-32 | 8.50e-33 |
| 25 | 2.07e-32 | 1.82e-32 | 3.65e-32 | 4.16e-33 | 5.21e-32 | 2.80e-32 |
| 30 | 2.48e-32 | 1.91e-32 | 2.48e-32 | 1.88e-32 | 5.58e-32 | 1.55e-32 |
| 35 | 1.09e-31 | 2.01e-33 | 4.52e-32 | 2.10e-34 | 7.76e-32 | 1.75e-32 |
| 40 | 4.11e-32 | 3.77e-34 | 5.69e-32 | 1.23e-32 | 7.56e-32 | 2.47e-32 |
| 45 | 5.24e-32 | 7.94e-33 | 8.19e-32 | 1.21e-32 | 8.27e-32 | 2.69e-32 |
| 50 | 4.73e-31 | 3.85e-31 | 8.65e-32 | 8.69e-32 | 9.24e-32 | 1.55e-32 |

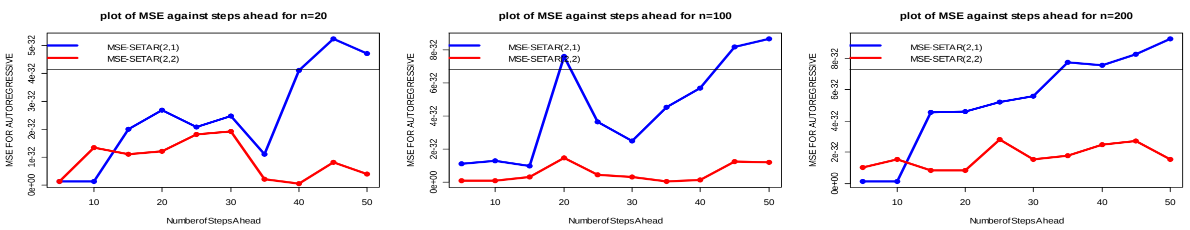



**Table 14: AIC of the Forecast of the Best Models from AR:$X_{ti} = 0.7X_{ti-1} - 0.6X_{ti-2} + e_t$**

| **Sample Size** | **n=20** | | **n=100** | | **n=200** | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,1)** | **Setar (2,2)** | **setar (2,1)** | **Setar (2,2)** | **setar (2,1)** | **Setar (2,2)** |
| 5 | -1334.92 | -1248.27 | -1845.86 | -2533.65 | -1765.36 | -2623.65 |
| 10 | -963.56 | -691.83 | -1012.66 | -1784.85 | -1008.51 | -2209.88 |
| 15 | -1058.40 | -1165.88 | -1054.62 | -2053.01 | -1063.69 | -2375.66 |
| 20 | -1416.88 | -1483.91 | -1126.80 | -2242.51 | -1415.46 | -2486.88 |
| 25 | -1820.95 | -1885.34 | -1276.71 | -2358.38 | -1509.41 | -2499.80 |
| 30 | -1973.35 | -2594.84 | -1553.26 | -2559.43 | -1692.35 | -2560.91 |
| 35 | -1982.94 | -2615.02 | -1694.32 | -2658.07 | -1720.35 | -2586.49 |
| 40 | -2077.78 | -2751.49 | -1858.98 | -2744.65 | -1812.59 | 2605.20 |
| 45 | -2290.65 | -2945.85 | -2089.80 | -2807.97 | -1878.93 | -2752.65 |
| 50 | -2221.36 | -2962.23 | -2158.43 | -2911.28 | -1944.77 | -2813.29 |

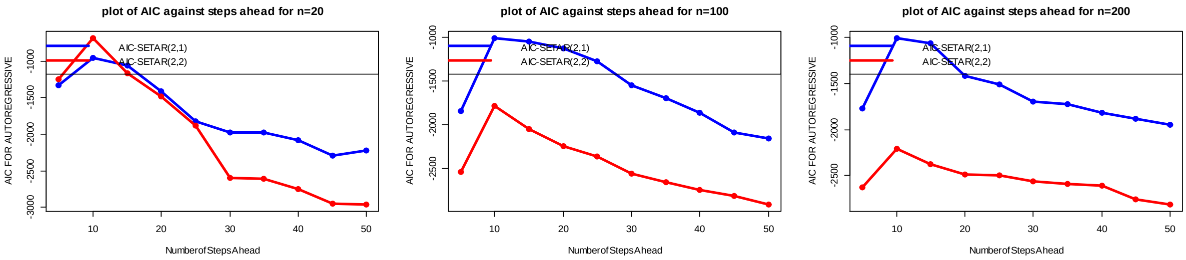



**Table 15: MSE of the Forecast of the Best Models from Trigonometric AR (2): $X_{ti} = 0.7sin(X_{ti-1}) - 0.6cos(X_{ti-2}) + et$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,1)** | **star (2,1)** | **setar (2,1)** | **star (2,1)** | **setar (2,1)** | **star (2,1)** |
| 5 | 3.96e-32 | 1.19e-32 | 6.15e-33 | 2.37e-33 | 5.32e-33 | 2.34e-33 |
| 10 | 4.66e-32 | 3.24e-33 | 6.59e-33 | 2.85e-33 | 4.66e-33 | 2.09e-33 |
| 15 | 3.51e-32 | 7.79e-34 | 5.58e-33 | 2.04e-33 | 9.50e-33 | 2.95e-33 |
| 20 | 4.44e-32 | 5.64e-33 | 6.90e-33 | 2.27e-33 | 8.58e-33 | 2.30e-33 |
| 25 | 5.10e-32 | 5.68e-33 | 5.10e-33 | 2.08e-33 | 9.48e-33 | 2.49e-33 |
| 30 | 3.15e-32 | 5.03e-34 | 5.83e-33 | 2.35e-33 | 8.13e-33 | 2.21e-33 |
| 35 | 9.04e-32 | 1.05e-32 | 5.82e-33 | 2.35e-33 | 9.05e-33 | 2.03e-33 |
| 40 | 4.47e-32 | 1.45e-32 | 5.68e-33 | 2.73e-33 | 8.39e-33 | 2.42e-33 |
| 45 | 4.29e-32 | 7.37e-33 | 5.50e-33 | 2.10e-33 | 8.84e-33 | 2.27e-33 |
| 50 | 4.24e-32 | 1.84e-33 | 6.11e-33 | 2.34e-33 | 8.75e-33 | 2.03e-33 |

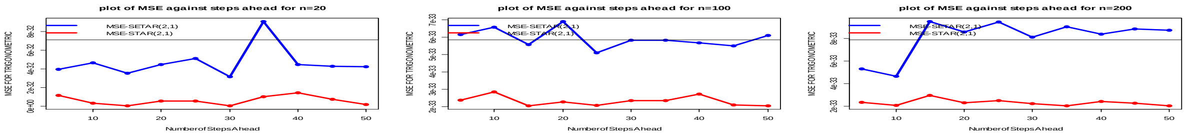



**Table 16: AIC of the forecast of the best models from trigonometric: $X_{ti} = 0.7sin(X_{ti-1}) - 0.6cos(X_{ti-2}) + et$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,1)** | **star (2,1)** | **setar (2,1)** | **star (2,1)** | **setar (2,1)** | **star (2,1)** |
| 5 | -1328.93 | -3328.54 | -998.94 | -1228.08 | -1379.34 | -2328.64 |
| 10 | -750.29 | -1749.28 | -1019.96 | -1752.73 | -1036.60 | -1750.27 |
| 15 | -1117.91 | -2002.77 | -1013.73 | -1825.30 | -1147.93 | -1905.52 |
| 20 | -1376.63 | -2187.14 | -1083.49 | -1924.29 | -1269.26 | -2583.81 |
| 25 | -1444.05 | -2035.41 | -1027.15 | -1999.58 | -1273.40 | -2895.70 |
| 30 | -1553.53 | -2267.84 | -1085.12 | -2239.38 | -1304.14 | -2828.10 |
| 35 | -2002.23 | -2590.20 | -1193.86 | -2233.04 | -1577.55 | -2905.76 |
| 40 | -2287.60 | -2990.80 | -1411.71 | -2263.58 | -1975.73 | -3137.97 |
| 45 | -2344.87 | -3081.43 | -1458.42 | -2358.52 | -2126.52 | -3064.65 |
| 50 | -2356.05 | -3023.79 | -1512.53 | -2408.25 | -2091.44 | -3198.36 |

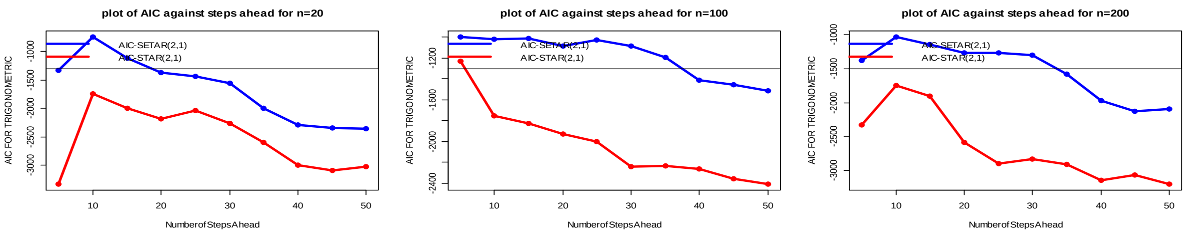


**Table 17: MSE of the forecast of the best models from exponential: Xti = $0.7X_{ti-2} - exp(0.6X_{ti-2}) + et$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,2)** | **Star(2,3)** | **setar (2,2)** | **Star(2,3)** | **setar (2,2)** | **Star(2,3)** |
| 5 | 1.67e-32 | 3.11e-31 | 4.15e-34 | 1.08e-33 | 2.98e-34 | 2.08e-32 |
| 10 | 1.31e-32 | 3.71e-31 | 5.82e-34 | 1.28e-32 | 3.63e-34 | 2.58e-32 |
| 15 | 1.10e-32 | 1.68e-31 | 4.54e-33 | 1.47e-32 | 5.92e-33 | 7.70e-32 |
| 20 | 1.07e-32 | 2.79e-31 | 3.59e-33 | 1.79e-32 | 7.08e-33 | 9.15e-32 |
| 25 | 1.14e-32 | 2.92e-31 | 5.15e-33 | 5.76e-32 | 6.37e-33 | 9.18e-32 |
| 30 | 1.14e-32 | 2.40e-31 | 5.83e-33 | 5.60e-32 | 4.37e-33 | 8.04e-32 |
| 35 | 1.26e-32 | 2.60e-31 | 8.32e-33 | 8.84e-32 | 4.48e-33 | 3.11e-31 |
| 40 | 1.37e-32 | 3.04e-31 | 1.11e-32 | 1.54e-31 | 3.29e-33 | 6.43e-32 |
| 45 | 1.48e-32 | 3.34e-31 | 1.94e-32 | 1.45e-31 | 4.54e-33 | 4.20e-32 |
| 50 | 1.62e-32 | 4.42e-31 | 1.13e-32 | 1.37e-31 | 4.37e-33 | 4.65e-31 |

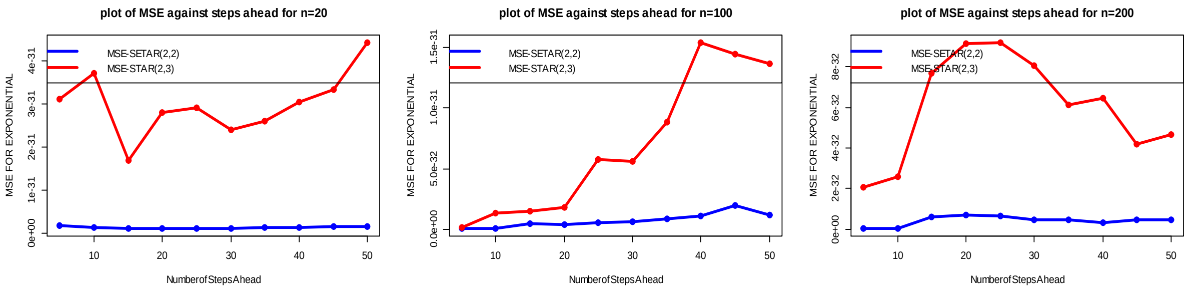



**Table 18: AIC of the forecast of the best models from exponential: Xti = $0.7X_{ti-1} - exp(0.6X_{ti-2}) + et$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **setar (2,2)** | **star (2,3)** | **setar (2,2)** | **star (2,3)** | **setar (2,2)** | **star (2,3)** |
| 5 | -1523.79 | -978.44 | -1713.79 | -981.443 | -1981.43 | -1000.79 |
| 10 | -1995.29 | -995.29 | -1812.34 | -1009.2899 | -2105.0211 | -1119.78 |
| 15 | -2129.71 | -1034.82 | -1829.71 | -1110.82 | -2451.975 | -1189.71 |
| 20 | -2364.74 | -1375.50 | -1911.83 | -1217.504 | -2441.269 | -1211.834 |
| 25 | -2424.09 | -1445.65 | -2153.74 | -1280.654 | -2680.202 | -1253.744 |
| 30 | -2493.67 | -1511.80 | -2366.46 | -1235.80 | -2716.947 | -1356.56 |
| 35 | -2768.97 | -1668.80 | -2579.92 | -1295.82 | -2763.372 | -1389.78 |
| 40 | -2893.46 | -1773.30 | -2663.03 | -1315.34 | -2844.455 | -1497.58 |
| 45 | -3108.93 | -1980.92 | -2778.56 | -1468.93 | -3220.099 | -1587.33 |
| 50 | -3136.40 | -2106.79 | -2800.67 | -1459.63 | -3220.974 | -1645.43 |

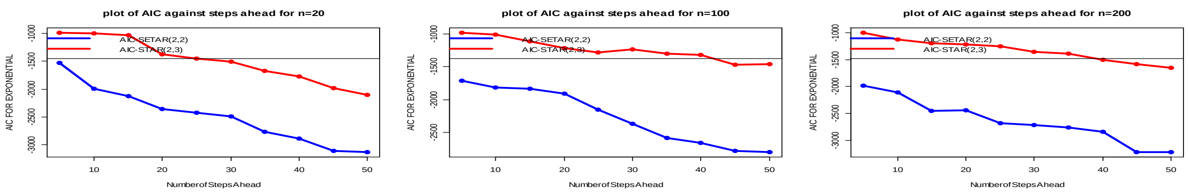



**Table 19: MSE of the forecast of the best models from polynomial: Xti = $0.7X_{ti-1}^2 - 0.6X_{ti-2} + e_t$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **Setar(2,2)** | **Star(2,2)** | **Setar(2,2)** | **Star(2,2)** | **Setar(2,2)** | **Star(2,2)** |
| 5 | 2.91e-06 | 0.0040 | 2.33e-04 | 6.22e-04 | 2.91e-06 | 0.0040 |
| 10 | 9.10e-07 | 0.0038 | 9.57e-05 | 5.81e-04 | 9.10e-07 | 0.0038 |
| 15 | 6.05e-07 | 0.0038 | 7.45e-05 | 5.14e-04 | 6.05e-07 | 0.0038 |
| 20 | 4.87e-07 | 0.0038 | 5.00e-05 | 4.97e-04 | 4.87e-07 | 0.0038 |
| 25 | 3.54e-07 | 0.0035 | 9.57e-06 | 4.83e-04 | 3.54e-07 | 0.0035 |
| 30 | 9.43e-08 | 0.0035 | 9.03e-06 | 4.77e-04 | 9.43e-08 | 0.0035 |
| 35 | 8.13e-08 | 0.0035 | 8.00e-06 | 3.43e-04 | 8.13e-08 | 0.0035 |
| 40 | 7.06e-08 | 0.0031 | 7.30e-06 | 2.32e-04 | 7.06e-08 | 0.0031 |
| 45 | 4.53e-08 | 0.0022 | 4.55e-06 | 1.92e-04 | 4.53e-08 | 0.0022 |
| 50 | 1.01e-08 | 0.0013 | 2.10e-06 | 1.77e-04 | 1.01e-08 | 0.0013 |

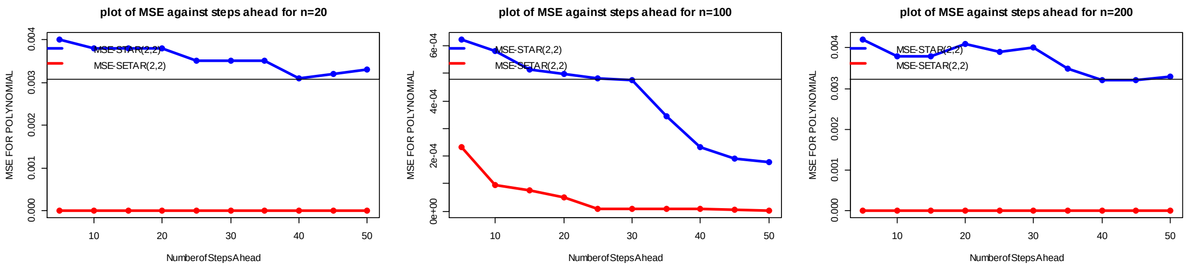
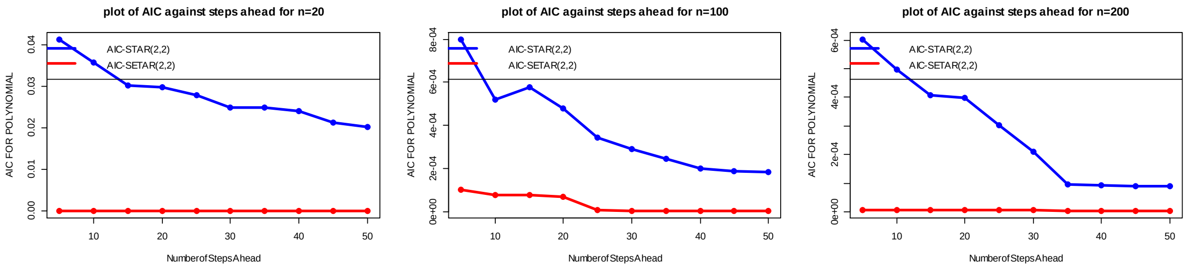



**Table 20: AIC of the forecast of the best models from polynomial: -$X_{ti} = 0.7X_{ti-1}^2 - 0.6X_{ti-2} + e_t$**

| Sample Size | n=20 | | n=100 | | n=200 | |
|---|---|---|---|---|---|---|
| **Steps ahead (h)** | **Setar(2,2)** | **Star(2,2)** | **Setar(2,2)** | **Star(2,2)** | **Setar(2,2)** | **Star(2,2)** |
| 5 | 9.15e-06 | 0.0412 | 9.89e-05 | 7.98e-04 | 5.22e-06 | 6.01e-04 |
| 10 | 7.33e-06 | 0.0358 | 7.75e-05 | 5.20e-04 | 6.02e-06 | 4.98e-04 |
| 15 | 6.45e-06 | 0.0302 | 7.56e-05 | 5.77e-04 | 5.51e-06 | 4.07e-04 |
| 20 | 4.44e-06 | 0.0297 | 6.88e-05 | 4.78e-04 | 5.88e-06 | 3.98e-04 |
| 25 | 3.16e-06 | 0.0279 | 5.07e-06 | 3.42e-04 | 5.07e-06 | 3.02e-04 |
| 30 | 1.14e-06 | 0.0248 | 4.06e-06 | 2.89e-04 | 5.06e-06 | 2.09e-04 |
| 35 | 7.11e-07 | 0.0249 | 3.77e-06 | 2.43e-04 | 4.07e-06 | 9.43e-05 |
| 40 | 5.58e-07 | 0.0181 | 3.11e-06 | 2.00e-04 | 3.87e-06 | 9.12e-05 |
| 45 | 5.01e-07 | 0.0172 | 1.78e-06 | 1.88e-04 | 2.92e-06 | 8.88e-05 |
| 50 | 4.89e-07 | 0.0143 | 1.59e-06 | 1.81e-04 | 2.88e-06 | 7.21e-05 |




**Summary of Findings**

This study focuses on investigating the relative performance of two nonlinear thresholds autoregressive with two regimes namely SETAR and STAR models. Their forecasting performance were studied on three common forms of nonlinear functions, these are polynomial, exponential and trigonometric. Simulated data with features of nonlinearity and non-stationarity was used to compare the performance of the model. The relative performance of each model was examined with a view to identifying the best models using the following criteria, mean square error (MSE), Akaike Information Criteria (AIC) and Mean Absolute Percentage Error (MAPE). The results for the 2- regime SETAR and STAR models of order 1, 2 and 3 were discussed as follows:

For the linear auto regressive models, the SETAR (2, 1) performs better than the other models followed by SETAR (2, 2) on the basis of the MSE and AIC criteria. Whereas the MAPE criteria shows STAR (2, 3) is better than others in relative performance of the model. Hence, since two among the three criteria shows that the SETAR (2, 1) and SETAR (2, 2) were selected to be the best in forecasting the linear auto regressive models.

Nevertheless, for the trigonometric nonlinear functions, it can be seen that SETAR (2, 1) and STAR (2, 1) are known to be the best based on the MSE and the AIC criteria and we therefore conclude that the two mentioned models were used for the best forecasting performance of the fitted models across the steps ahead. Moreover, the SETAR (2, 2) performs better for the MSE and the AIC criteria and STAR (2, 3) was considered to be the best based on all of the three criteria in an exponential nonlinear function. The SETAR (2, 2) and the STAR (2, 2) were known to be the best models for the relative performance of the models.

Finally, in determining the forecasting ability of the fitted models, the SETAR (2, 1) and SETAR (2, 2) are taken as the best models and therefore used to forecast for future values at different steps ahead for an Auto regressive form of linear function. The SETAR (2, 1) and STAR (2, 1) for the trigonometric form, the SETAR (2, 2) and STAR (2, 3) for an exponential and the SETAR (2, 2) and STAR (2, 2) for a polynomial nonlinear function. The MSE and AIC of the values of the forecasted models are recorded to compare the relative forecast performance of the models at lower, moderate and large sample sizes respectively.

It was recorded that, in linear form of autoregressive, SETAR (2, 2) forecasted better than SETAR (2, 1) from the low to the high steps ahead for all of the sample sizes. However, the SETAR (2, 1) superseded SETAR (2, 2) at the lowest steps when the sample is 20 and 200 in forecasting the 5 step ahead while SETAR (2, 2) took the lead as the step ahead increases based on the criteria. The STAR (2, 1) outperforms the SETAR (2, 1) in trigonometric function from small, moderate and large sample sizes. As the steps ahead increases, the self-exciting threshold auto regressive model tends to come closer to the smooth transition auto regressive model., the results of exponential and polynomial functions show that SETAR (2, 2) was known to be the best model that can be used for forecasting.

## CONCLUSION

In this study, comparative performances of the nonlinear models with non-stationarity features were carried out. SETAR and STAR models at order 1, 2 and 3 and regime 2 respectively was applied to AR, trigonometric, exponential and polynomial functions. It was concluded that SETAR (2, 2) forecasted better at different steps ahead on both AR and polynomial functions which is in line with the findings of Akeyede et al (2016). Whereas in forecasting trigonometric nonlinear form of data, it can be seen that STAR (2, 1) outperforms the other models. However, SETAR (2, 2) has shown to have the best forecasting performance for an exponential function.

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