CONSTRUCTION AND ANALYSIS OF BALANCED INCOMPLETE SUDOKU SQUARE DESIGN

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ABSTRACT

Sudoku squares have been widely used to design an experiment where each treatment occurs exactly once in each row, column or sub-block. For some experiments, the size of row (or column or sub-block) may be less than the number of treatments. Since not all the treatments can be compared within each block, a new class of designs called balanced incomplete Sudoku squares design (BISSD) is proposed. A general method for constructing BISSD is proposed by an intelligent selection of certain cells from a complete Latin square via orthogonal Sudoku designs. The relative efficiencies of a delete-one-transversal balance incomplete Latin Square (BILS) design with respect to Sudoku design are derived. In addition, linear model, numerical examples and procedure for the analysis of data for BISSD are proposed.

Keywords: Balanced incomplete Latin square, Sudoku design, NP-complete.

INTRODUCTION

A Sudoku design is a block matrix filled with different Latin letters or numbers, such that each letter or number appears in a row (column or sub-block) only once. Hui-Dong and Ru-Gen (2008) stated that all non-prime number can construct a Sudoku square while prime number k cannot, k is the number of treatments (or rows or columns) of the square. It was presented by Subramani and Ponnuswamy (2009) for the construction of Sudoku designs of order $k = m^2$ only and also proposed linear models for analyzing the data obtained from their design, and applied the designs to agricultural experiments. It has been shown that Sudoku design may be incomplete as reported by Bejar *et al* (1960) and Mahdian Mahmoodian (2015). In addition Donovan *et al (2015)* and Kumar *et al* (2015) studied the Sudoku based space filling designs.

Researchers address a classical problem of experimental designs, in the construction of balanced incomplete Latin squares (BILS) design. However the construction of balanced incomplete Sudoku squares design (BISSD) are popularly known as experimental designs for several reasons; like the randomized complete block designs, balanced incomplete Latin square designs and Latin square designs, they are variance balanced. This problem of constructing balanced incomplete Latin squares has drawn the attention of many leading mathematicians and statisticians for a long period. Recent contribution in this area includes Das and Dey (1990), and Subramani and Ponnuswamy (2009). Where Das and Dey (1990) have presented a simple method of constructing a pair of orthogonal Latin squares of odd order by developing two initial rows. A Graeco Latin square can be obtained by superimposing any two orthogonal Latin squares. By extending the method of

Das and Dey (1990), a more generalized method of constructing several orthogonal (Graeco) Latin squares of odd order to give Latin square was presented by Subramani and Ponnuswamy (2009) . Recently Subramani (2013), extended the work of Das and Dey (1990) and Subramani and Ponnuswamy (2009), to construct and analysis of orthogonal (Graeco) Sudoku square designs. A sequential method of constructing and analyzing orthogonal Sudoku square designs of odd order was presented in the paper.

Mahdian and Mahmoodian (2015) report that a balanced incomplete Sudoku design is an incomplete filled block boxes with some empty cells which satisfied that each letter appears only once in (row, column or sub-block). This balanced incomplete Sudoku design has been shown to be non deterministic polynomial time complete for the particular case of sub-blocks square. In general, Sudoku square is said to be non complete 9complete if a balanced incomplete Sudoku square is completable see Colbourn (1984), and Mahdian and Mahmoodian (2015).

A general method for constructing BILS was discussed by Mingyao *et al* (2013) for selection of certain cells from a complete Latin square via orthogonal Latin squares. Mingyao *et al* (2013), investigated the optimality for BILS designs and concluded that the transversal BILS designs are asymptotically optimal for all the row, column and treatment effects. Recently Danbaba *et al* (2018) presented a paper on joint analysis of several experiments conducted via orthogonal Sudoku square design of odd order and conclude .that joint analysis of multi-environment experiments increases the accuracy of evaluation, instead of analyzing each experiment separately. Mingyao *et al* (2013) also discussed that balanced incomplete Latin square designs occurs only once in

each row and column of the design. However the analysis in Mingyao *et al* (2013) is not Sudoku design in nature. This paper proposed construction and analysis of balanced incomplete Sudoku square designs that is to select certain cells from a complete Sudoku square such that the remaining cells satisfy the condition of balanced occurrence of numbers. The method of constructing balanced incomplete Sudoku square designs is presented in section 2 with other related results and numerical examples. Furthermore the computational formulae of the error sum of squares together with the error degree of freedom of four different types of the balanced incomplete Sudoku square designs in line with the four different Sudoku square designs are given in section 3. Finally the numerical analysis and results are given in section 4.

Construction of balanced incomplete Sudoku square designs.

A natural way of constructing a BISSD is to select certain cells from a complete Latin square such that the remaining cells satisfy the condition of balanced occurrence of symbols. It can be done by removing one or more "transversal". For a given Latin square (k), a transversal is a set of cells such that only one cell is followed in each row and column, and further, each symbol appears in each cell exactly once. It is known that any

two Latin squares of the same order are said to be orthogonal, if these two squares when superimposed have the property that each pair of symbols appears exactly once Bose *et al,* Subramani and Ponnuswamy (2009), Mingyao *et al* (2013), Subramani (2013), and Danbaba *et al* (2018).

Example 1. Consider the experiment from Subramani (2013), for testing the abrasion resistance of rubber-covered fabric in a Martindale wear tester. The original design is the complete Latin square in Table 1(a), where we have LS(4) for four material A,B,C and D respectively. The response is the loss in weight in 0.1(mg) over a period of time. Two blocking variables "application" and "position" are assigned to the rows and columns, respectively. The weight data is given below. Now we consider the BILS (4,3) design obtained in example 1, the data along the diagonal in Table 1(a) are removed, as shown in Table 1(b). The underlying linear model for BILS (k, r) design is:

$$Y_{ijl} = \mu + \alpha_i + \beta_j + \tau_l + e_{ijl}$$

Where i, j takes vales in $(1, 2, \ldots, k)$ and l is the symbol in the (i, j)-th cell of the BILS(k, r), μ, is the overall mean, α_i is the ith row effect, β_j is the jth column effect, τ_l is the effect of the lth treatment, and the errors e_{ijl} are the independent N(0, σ^2).

Table 1: Weight loss data for LS(4) and BILS(3).

235	236	218	268
251	**241**	227	229
234	273	**274**	226
195	270	230	**225**

	236	218	268
251		227	229
234	273		226
195	270	230	

(a) Data of LS(4)   (b) Data of BILS(4,3)

Table 2: Anova table for the BILS(4,3) wear experiment.

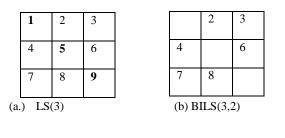| Source	degree of freedom	Sum of squares	Mean squares	F-value	P-value (> F)
Application	3	278.2	92.75	3.66	0.22192
Position	3	2243.5	747.83	29.52	0.03294
Material	3	3424.5	1141.50	45.06	0.02179
Residual	2	50.7	25.33		

Balanced incomplete Sudoku squares design.

In this study, four models of orthogonal Sudoku square designs discussed by Subramani (2013) were modified and balanced incomplete Sudoku square from each were discussed under the construction. For all the models considered in this study, it is assumed that all effects are fixed.

Consider the construction of an incomplete Latin square ILS(k,r) of order k in which each row and each column has r empty cells. If ILS(k,r) satisfies the condition that each number appears exactly (k) times in the whole square, then the ILS(k,r) is called a balanced incomplete Sudoku square design.

Example 2. For $k = 9 \; and \; r = 3, \; let \; k - r = x,$ suppose we have Latin square of order (3), if we remove the transversal number of Table 3(a) below the cells will be a BILS(3,2) as in Table 3(b) then the balanced incomplete Sudoku square design BISSD(9,6) is obtained as displayed in table 3(c) below.

Table 3: LS(3), BILS(3,2) and BISS(9,6).

1	2	3
4	5	6
7	8	9

(a.)   LS(3)

	2	3
4		6
7	8	

(b) BILS(3,2)

	2	3	4		6	7	8	
4		6	7	8			2	3
7	8			2	3	4		6
	6	4	8		7	2	3	
8		7	2	3			6	4
2	3			6	4	8		7
	7	8	3		2	6	4	
3		2	6	4			7	8
6	4			7	8	3		2

(c) BISS(9, 6)

 CONSTRUCTION AND ANALYSIS OF BALANCED INCOMPLETE SUDOKU SQUARE DESIGN
Consider the four models of orthogonal Sudoku square designs discussed by Subramani (2013) below with the error sum of squares together with the corresponding degrees of freedom.
Balanced Incomplete Sudoku square designs – Type 1

In this model we assume row, column and treatment effects as in the Latin square designs.
In addition to these we assume row block, column block and square as fixed effects model. The Sudoku design of order (x) and its Analysis of Variance (ANOVA) model is given below:

$$Y_{ij(l,n,p,q)} = \mu + \alpha_{ix} + \beta_{jx} + \tau_n + \gamma_{lx} + C_{px} + S_{qx} + e_{ij(l,n,p,q)}$$

$$\text{where} \quad i,j = 1,2,\dots,m, \quad l,n,p,q = 1,2,\dots,x$$

$\mu = $ General Mean effect

$\alpha_{i,x} = i^{th}$ Row block effect in x^{th} empty cells,

$\beta_{j,x} = j^{th}$ Column block effect in x^{th} empty cells,

$\tau_n = n^{th}$ Treatment effect

$\gamma_{l,x} = l^{th}$ Row effect in x^{th} empty cells,

$C_{p,x} = p^{th}$ Column effect in x^{th} empty cells,

$S_{q,x} = q^{th}$ Square effect in x^{th} empty cells,

$e_{ij(l,n,p,q)} = $ Error component effect with mean zero and variance σ^2

$$G = \sum_{i=1}^{x}\sum_{j=1}^{x} Y_{ij} \; and \; N = m^2 x$$

$$TSS = \sum_{i=1}^{x}\sum_{j=1}^{x} Y_{ij}^2 - \frac{G^2}{N}$$

$$SST = \sum_{n=1}^{x} \frac{\tau_n^2}{x} - \frac{G^2}{N}$$

$$SSRB = \sum_{i=1}^{m} \frac{RB_i^2}{mx} - \frac{G^2}{N}$$

$$SSCB = \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \frac{G^2}{N}$$

$$SSR = \sum_{l=1}^{x} \frac{\gamma_l^2}{x} - \frac{G^2}{N}$$

$$SSC = \sum_{p=1}^{x} \frac{C_p^2}{x} - \frac{G^2}{N}$$

$$SSS = \sum_{q=1}^{x} \frac{S_q^2}{x} - \frac{G^2}{N}$$

$$ESS = \sum_{i=1}^{x} \sum_{j=1}^{x} Y_{ij}^2 - \sum_{i=1}^{m} \frac{RB_i^2}{mx} - \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \sum_{n=1}^{x} \frac{\tau_n^2}{x} - \sum_{l=1}^{x} \frac{\gamma_l^2}{x} - \sum_{p=1}^{x} \frac{C_p^2}{x} - \sum_{q=1}^{x} \frac{S_q^2}{x} + \frac{5G^2}{N}$$

Error degrees of freedom $= (m + 1)(x - 1)$

Table 4. ANOVA table for BISSD Model Type I

Source	Sum of Squares	Degree of Freedom	Mean Square	F-Ratio
Treatments	SST	$x - 1$	$MST = SST/df$	MST/MSE
Row	SSR	$x - 1$	$MSR = SSR/df$	MSR/MSE
Column	SSC	$x - 1$	$MSC = SSC/df$	MSC/MSE
Row-block	SSRB	$m - 1$	$MSRB = SSRB/df$	$MSRB/MSE$
Column-block	SSCB	$m - 1$	$MSCB = SSCB/df$	$MSCB/MSE$
Sub-Square	SSS	$x - 1$	$MSS = SSS/df$	MSS/MSE
Error	SSE	$(m + 1)(x - 1)$	$MSE = SSE/df$	
Total	TSS	$m^2x - 1$		

Balanced Incomplete Sudoku square designs – Type II

aIn this model it is assumed that the row effects are nested in the row block effects and the column effects are nested in the column block effects. The Sudoku design of order (x) and its Analysis of Variance (ANOVA) model is given below:

$$Y_{ij(l,n,p,q)} = \mu + \alpha_{ix} + \beta_{jx} + \tau_n + \gamma(\alpha)_{l(i)x} + C(\beta)_{p(j)x} + S_{qx} + e_{ij(l,n,p,q)}$$

where $i, j, l, p = 1,2, …, m$ and $n, q = 1,2, …, x$

$\mu =$ General Mean effect

$\alpha_{ix} = i^{th}$ Block (row) effect in x^{th} empty cells,

$\gamma_{jx} = j^{th}$ Block (column) effect in x^{th} empty cells,

$\tau_n = n^{th}$ Treatment effect

$\gamma(\alpha)_{l(i)x} = l^{th}$ Row effect nested in ith block (row) in x^{th} empty cells,

$C(\beta)_{p(j)x} = p^{th}$ Column effect nested in jth block (column) in x^{th} empty cells,

$S_{qx} = q^{th}$ Square effect in x^{th} empty cells,

$e_{ij(l,n,p,q)} =$ Error component with mean zero and variance σ^2

$$TSS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 - \frac{G^2}{N}$$

$$SST = \sum_{n=1}^{x} \frac{T_n^2}{x} - \frac{G^2}{N}$$

$$SSRB = \sum_{i=1}^{m} \frac{RB_i^2}{mx} - \frac{G^2}{N}$$

$$SSCB = \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \frac{G^2}{N}$$

$$SSR = \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{R_{l(i)}^2}{x} - \sum_{i=1}^{m} \frac{RB_i^2}{mx}$$

$$SSC = \sum_{j=1}^{m} \sum_{p=1}^{m} \frac{C_{p(j)}^2}{x} - \sum_{j=1}^{m} \frac{CB_i^2}{mx}$$

$$SSS = \sum_{q=1}^{m^2} \frac{S_q^2}{x} - \frac{G^2}{N}$$

$$ESS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 - \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{R_{l(i)}^2}{x} - \sum_{j=1}^{m} \sum_{p=1}^{m} \frac{C_{p(j)}^2}{x} - \sum_{n=1}^{x} \frac{T_n^2}{x} - \sum_{q=1}^{m^2} \frac{S_q^2}{x} + \frac{3G^2}{N}$$

Error degrees of freedom $= m(x + 2)$

Table 5: ANOVA table for BISSD Model Type II

Source	Sum of Squares	Degree of Freedom	Mean Square	F-Ratio
Treatments	SST	$x - 1$	$MST = SST/df$	MST/MSE
Row-block	SSRB	$m - 1$	$MSRB = SSRB/df$	$MSRB/MSE$
Column-block	SSCB	$m - 1$	$MSCB = SSCB/df$	$MSCB/MSE$
Row within RB	SSR(RB)	$m(m - 1)$	$MSR = SSR/df$	MSR/MSE
Column within CB	SSC(CB)	$m(m - 1)$	$MSC = SSC/df$	MSC/MSE
Sub-Square	SSS	$x - 1$	$MSS = SSS/df$	MSS/MSE
Error	SSE	$m(x + 2)$	$MSE = SSE/df$	
Total	TSS	$m^2 x - 1$		

Balanced Incomplete Sudoku square designs – Type 1II
In this model it is assumed that the horizontal square effects are nested in the row block effects and the vertical square effects are nested in the column block effects.

$$Y_{ij(l,n,p,q,w)} = \mu + \alpha_{ix} + \beta_{jx} + \tau_n + \gamma_{lx} + C_{px} + S(\alpha)_{q(i)x} + \pi(\beta)_{w(j)x} + e_{ij(l,n,p,q,w)}$$
$$\text{where} \quad i, j, n, q, w = 1, \dots, m \text{ and } l, p = 1, 2, \dots, x$$

$\mu =$ General mean effect
$\alpha_{ix} = i^{th}$ Row block effect in x^{th} empty cells,
$\beta_{jx} = j^{th}$ Column block effect in x^{th} empty cells,
$\tau_n = n^{th}$ Treatment effect,
$\gamma_{lx} = l^{th}$ Row effect in x^{th} empty cells,

$C_{px} = p^{th}$ Column effectin x^{th} empty cells,

$S(\alpha)_{q(i)x} = q^{th}$ Horizontal Square effect nested in i^{th} Row block effect in x^{th} empty cells,

$\pi(\beta)_{w(j)x} = w^{th}$ Vertical Square effect nested in j^{th} Column block effect in x^{th} empty cells,

$e_{ij(l,n,p,q,w)} =$ Error component with mean zero and variance σ^2

$$TSS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 - \frac{G^2}{N}$$

$$SST = \sum_{n=1}^{x} \frac{T_n^2}{x} - \frac{G^2}{N}$$

$$SSRB = \sum_{i=1}^{m} \frac{RB_i^2}{mx} - \frac{G^2}{N}$$

$$SSCB = \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \frac{G^2}{N}$$

$$SSR = \sum_{l=1}^{x} \frac{R_l^2}{x} - \frac{G^2}{N}$$

$$SSC = \sum_{p=1}^{x} \frac{C_p^2}{x} - \frac{G^2}{N}$$

$$SSHS = \sum_{i=1}^{m} \sum_{p=1}^{m} \frac{S(\alpha)_{q(i)}}{x} - \sum_{i=1}^{m} \frac{RB_i^2}{mx}$$

$$SSVS = \sum_{j=1}^{m} \sum_{w=1}^{m} \frac{\pi(\beta)_{w(j)}}{x} - \sum_{j=1}^{m} \frac{CB_j^2}{mx}$$

$$ESS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 - \sum_{l=1}^{x} \frac{R_l^2}{x} - \sum_{p=1}^{x} \frac{C_p^2}{x} - \sum_{n=1}^{x} \frac{T_n^2}{x} - \sum_{i=1}^{m} \sum_{p=1}^{m} \frac{S(\alpha)_{q(i)}}{x} - \sum_{j=1}^{m} \sum_{w=1}^{m} \frac{\pi(\beta)_{w(j)}}{x} + \frac{4G^2}{N}$$

Error degrees of freedom $= (m-1)(x+2)$

Table 6: ANOVA table for BISSD Model Type III

Source	Sum of squares	Degree of freedom	Mean Square	F-Ratio
Treatments	SST	$x-1$	$MST = SST/df$	$MST/_{MSE}$
Row	SSR	$x-1$	$MSR = SSR/df$	$MSR/_{MSE}$
Column	SSC	$x-1$	$MSC = SSC/df$	$MSC/_{MSE}$
Row-block	SSRB	$m-1$	$MSRB = SSRB/df$	$MSRB/_{MSE}$
Column-block	SSCB	$m-1$	$MSCB = SSCB/df$	$MSCB/_{MSE}$
Square within RB	SSHS	$m(m-1)$	$MSHS = SSHS/df$	$MSHRB/_{MSE}$
Square within CB	SSVS	$m(m-1)$	$MSVS = SSVS/df$	$MSVCB/_{MSE}$
Error	SSE	$(m-1)(x+2)$	$MSE = SSRB/df$	
Total	TSS	$m^2x - 1$		

Balanced Incomplete Sudoku square designs – Type 1V

In this model it is assumed that the row effects and the horizontal square effects are nested in the row block effects and the column effects and the vertical square effects are nested in the column block effects.

$$Y_{ij(l,n,p,q,w)} = \mu + \alpha_{ix} + \beta_{jx} + \tau_n + \gamma(\alpha)_{l(i)x} + C(\beta)_{p(j)x} + S(\alpha)_{q(i)x} + \pi(\beta)_{w(j)x} + e_{ij(ln,p,q,w)}$$

$$\text{where } i, j, l, p, q, w = 1,2, \dots m, \ \ n = 1,2, \dots, x$$

$\mu =$ General Mean effect

$\alpha_{ix} = i^{th}$ Block (row) effect in x^{th} empty cells,

$\beta_{jx} = j^{th}$ Block (column) effect in x^{th} empty cells,

$T_n = n^{th}$ Treatment effect

$\gamma(\alpha)_{l(i)x} = l^{th}$ Row effect nested in ith row block effect in x^{th} empty cells,

$C(\beta)_{p(j)x} = p^{th}$ Column effect nested in jth column block effect in x^{th} empty cells,

$S(\alpha)_{q(i)x} = q^{th}$ Horizontal square effect nested in ith row block effect in x^{th} empty cells,

$\pi(\beta)_{w(j)x} = w^{th}$ Vertical square effect nested in jth column block effect in x^{th} empty cells,

$e_{ij(l,n,p,q,w)} =$ Error component with mean zero and variance σ^2

$$TSS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 - \frac{G^2}{N}$$

$$SST = \sum_{n=1}^{x} \frac{T_n^2}{x} - \frac{G^2}{N}$$

$$SSRB = \sum_{i=1}^{m} \frac{RB_i^2}{mx} - \frac{G^2}{N}$$

$$SSCB = \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \frac{G^2}{N}$$

$$SSR = \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{R_{l(i)}^2}{x} - \sum_{i=1}^{m} \frac{RB_i^2}{mx}$$

$$SSC = \sum_{j=1}^{m} \sum_{p=1}^{m} \frac{C_{p(j)}^2}{x} - \sum_{j=1}^{m} \frac{CB_i^2}{mx}$$

$$SSHS = \sum_{i=1}^{m} \sum_{p=1}^{m} \frac{S(\alpha)_{q(i)}}{x} - \sum_{i=1}^{m} \frac{RB_i^2}{mx}$$

$$SSVS = \sum_{j=1}^{m} \sum_{w=1}^{m} \frac{\pi(\beta)_{w(j)}}{x} - \sum_{j=1}^{m} \frac{CB_j^2}{mx}$$

$$ESS = \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} Y_{ij}^2 + \sum_{i=1}^{m} \frac{RB_i^2}{mx} + \sum_{j=1}^{m} \frac{CB_i^2}{mx} - \sum_{n=1}^{x} \frac{T_n^2}{x} - \sum_{i=1}^{m} \sum_{l=1}^{m} \frac{R_{l(i)}^2}{x} - \sum_{j=1}^{m} \sum_{p=1}^{m} \frac{C_{p(j)}^2}{x} - \sum_{i=1}^{m} \sum_{p=1}^{m} \frac{S(\alpha)_{q(i)}}{x} - \sum_{j=1}^{m} \sum_{w=1}^{m} \frac{\pi(\beta)_{w(j)}}{x} + \frac{2G^2}{N}$$

Error degrees of freedom= $(m + 1)(x - 1)$.

Table 7: ANOVA table for BISSD Model Type IV

Source	Sum of squares	Degree of freedom	Mean Square	F-Ratio
Treatments	SST	$x-1$	$MST = SST/df$	$MST/_{MSE}$
Row blocks	SSRB	$x-1$	$MSR = SSR/df$	$MSR/_{MSE}$
Column blocks	SSCB	$x-1$	$MSC = SSC/df$	$MSC/_{MSE}$
Row within RB	SSR	$m(m-1)$	$MSRB = SSRB/df$	$MSRB/_{MSE}$
Column within CB	SSC	$m(m-1)$	$MSCB = SSCB/df$	$MSCB/_{MSE}$
Square within RB	SSHS	$m(m-1)$	$MSHS = SSHS/df$	$MSHRB/_{MSE}$
Square within CB	SSVS	$m(m-1)$	$MSVS = SSVS/df$	$MSVCB/_{MSE}$
Error	SSE	$(m+1)(x-1)$	$MSE = SSRB/df$	
Total	TSS	$m^2x - 1$		

Illustration.

Table 8 give hypothetical data of a balanced incomplete Sudoku square design. The BISSD ANOVA tables obtained using R software for models 1 to 4 are presented in tables 9 to 12, respectively.

Table 8: Hypothetical data of a BISSD

	35	35	39		28	31	42	
28		26	27	33			31	37
24	25			29	43	41		35
	31	29	36		44	34	27	
23		35	24	27			28	43
28	40			30	25	36		37
	33	37	24		35	37	40	
25		26	27	23			30	25
27	32			34	39	28		32

TABLE 9. ANOVA RESULT FOR BISSD MODEL I

Source	Sum of square	degree of freedom	Mean squares	F-Ratio	Table-value* 5% level
Treatments	89.7037	5	17.9407	0.7957	2.71
Rows	362.4815	8	45.3102	2.0097	2.45
Columns	484.8148	8	60.6019	2.6879	2.45
Row blocks	35.1481	2	17.5741	0.7795	3.49
Column blocks	159.5926	2	79.7963	3.5392	3.49
Sub-square	252.1481	8	31.5185	1.3979	2.45
Error	450.9260	20	22.5463	____	____
Total	1834.8148	53	_____	____	

TABLE 10: ANOVA RESULT FOR BISSD MODEL II

Source	Sum of square	degree of freedom	Mean squares	F-Ratio	Table-value* 5% level
Treatments	89.7037	5	17.9407	0.6676	2.62
Row blocks	35.1481	2	17.5741	0.6539	3.40
Column blocks	159.5926	2	79.7963	2.9692	3.40
Rows within RB	327.3334	6	54.5556	2.0230	2.51
Columns within CB	325.2222	6	54.2037	2.0169	2.51
Sub-square	252.1481	8	31.5185	1.1728	2.36
Error	645.0000	24	26.8750	____	____
Total	1834.8148	53	_____	____	

TABLE 11. ANOVA RESULT FOR BISSD MODEL III

Source	Sum of square	degree of freedom	Mean squares	F-Ratio	Table-value* 5% level
Treatments	89.7037	5	17.9407	0.7294	2.85
Rows	362.4815	8	45.3102	1.8423	2.59
Columns	484.8148	8	60.6019	2.4640	2.59
Row blocks	35.1481	2	17.5741	0.7145	3.63
Column blocks	159.5926	2	79.7963	3.2444	3.63
Squares within RB	217.0000	6	36.1667	1.4705	2.74
Squares within CB	92.5555	6	15.4259	0.6272	2.74
Error	393.5186	16	24.5949	____	____
Total	1834.8148	53	_____	____	

TABLE 12: ANOVA RESULT FOR BISSD MODEL IV

Source	Sum of square	degree of freedom	Mean squares	F-Ratio	Table-value* 5% level
Treatments	89.7037	5	17.9407	0.6070	2.71
Row blocks	35.1481	2	17.5741	0.5975	3.49
Column blocks	159.5926	2	79.7963	2.7130	3.49
Rows within BR	327.3334	6	54.5556	1.8548	2.60
Columns within BC	325.2222	6	54.2037	1.8428	2.60
Squares within RB	217.0000	6	36.1667	1.2296	2.60
Squares within CB	92.5555	6	15.4259	0.5245	2.60
Error	588.2593	20	29.4130	____	____
Total	1834.8148	53	_____	____	

CONCLUSION

In this paper we introduce a new class of design called balanced incomplete Sudoku squares (BISS) design. This design allows experimenter to investigate the blocks and treatments variables, where the size of blocks may be less than the number of treatments. A general construction method of (BISS) design is proposed via orthogonal Sudoku squares. In addition, the models of the orthogonal Sudoku designs where modified to contain balanced incomplete Sudoku squares design.

REFERENCES

Bejar R., Fernandez C. Mateu C. Megda Valls M. (2012). The Sudoku Completion Problem With Rectangular Hole

Pattern Is NP-Complete. *Discrete Mathematics* 312, 3306-3315.

Bose R.C., Shrikhande, S.S., Parker, E.T., (1960). Further results on the construction of mutually orthogonal Latin

squares and the falsify of Euler's conjecture. *Canadian Journal of Mathematics* 12, 189-203.

Colbourn C., The complexity of completing partial latin squares, *Discrete Appl. Math*. 8 (1984), 151–158.

Danbaba, A., Odeyale. A.B., Musa. Y., (2018). Joint Analysis of Several Experiments Conducted via Orthogonal

Sudoku Design of Odd Order, *International Journal of Statistics and Applications,* 8(6), 323-331

Das, A., Dey, A. (1990). A note on construction of Graeco Latin square of order $2n + 1$,*Journal of India Soc Agric Statist 42, 247-249*.D.,

Donovan, D., Haaland B. and Nott D.J (2015). A Simple Approach to Constructing Quasi-Sudoku-based Sliced Space-Filling Designs. arxiv.1502.05522v1

Hui-Dong M. and Ru-Gen, X. (2008). Sudoku Square- a New Design in field Experiment, Acta Agron Sin, 34(9), 1489–1493.

Kanaana, I. and Ravikumar B. (2010). Row-filled completion problem for Sudoku. *Util. Math.*, 81, 65–84

Kumar, A., Varghese C. Varghese, E. and Jaggi S. (2015). On the construction of designs with three-way blocking. *Model;Assisted Statistics and Applications* 10, 43–52 43

Mahdian, M., E.S. Mahmoodian (2015) , Sudoku Rectangle Completion, *Electronic Notes in Discrete Mathematics 49, 747–755*

Mingyao, Ai., Kang, Li., Sanmao, Liu., Dennis, and KJ Lin., (2013). Balanced Incomplete Latin Square Designs. *Journal of Statistical and Inference* 143 (2013) 1575-1582.

Subramani, J. and K.N. Ponnuswamy (2009). Construction and Analysis of Sudoku designs. Model Assisted Statistics and Applications, 4(4), 287-301.

Subramani, J. (2013). Construction of Graeco Sudoku square Designs of Odd Orders, Bonfring International Journal of Data Mining, 2 (2), 37-41.

Wu C.F.J, and Hamada, M. (2000). Experiments: Planning, Analysis, and Parameter Design Optimization. *Wiley, New York.*