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MATHEMATICAL ANALYSIS OF COVID-19 INFECTION MODEL WITH DEMOGRAPHIC DYNAMICS

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ABSTRACT

This study developed a deterministic mathematical model of COVID-19 infection by incorporating asymptomatically and symptomatically infectious individuals, the vital dynamics such as birth rate and mortality rate. Face mask use, diagnosis of asymptomatic infectious individuals, and isolation of infected individuals as control strategies are also incorporated. The model is shown to have two unique equilibrium states, namely: the disease-free equilibrium points and the endemic equilibrium point. The result from the stability analysis of the critical points is shown to be local asymptotic stable and also, globally asymptotically stable provided the basic reproduction number is less than one ($R_0 < 1$), and the endemic equilibrium state is local asymptotic stable and also, globally asymptotically stable provided $R_0 > 1$. Furthermore, results of the sensitivity index on R_0 for the different parameters of the model show that the recruitment rate and the effective contact rate are the most sensitive parameters and thus critical in disease management and eradication. Thus, efforts geared at reducing the recruitment of susceptible individuals and infection transmission rate will significantly eliminate the disease burden.

Keywords: COVID-19, Positivity of Solution, Equilibrium Points, Basic Reproduction Number, Stability

INTRODUCTION

Coronaviruses are a broad family of zoonotic (transmission from animals to humans) viruses that cause illnesses ranging from the common cold to serious respiratory disorders (Vince, 2020). Coronaviruses are viral types. There are several varieties, some of which cause sickness. They are divided into three subgroups: alpha, beta, and gamma. A fourth novel category is known as delta coronaviruses (Mehmet et al., 2021).

Coronavirus disease 2019 (COVID-19), according to Indwiana and Ysrafil (2020), is an infectious illness caused by a novel coronavirus termed SARS-CoV-2, which stands for Severe Acute Respiratory Syndrome Coronavirus 2. SARS-CoV-2 is an enclosed virus with a positive-sense, single-stranded RNA of 29,891 bases that belongs to the coronaviridae family's beta subgroup. The genome encodes 29 proteins important in the processes of infection, replication, and virion assembly. The crown-like spikes on the surface of the coronavirus give it its name. The common cold, Middle East Respiratory Syndrome (MERS), and Severe Acute Respiratory Syndrome (SARS) are all examples of coronaviruses that can cause mild or severe respiratory disease (CDC, 2021).

The SARS-CoV-2 spike S protein has a receptor binding domain (RBD) that binds the human angiotensin-converting enzyme 2 (ACE2), promoting membrane fusion and viral endocytosis. The coronavirus genome's most variable section is the RBD found in the spike protein. According to structural and biochemical research, RBD from SARS-CoV-2 binds to ACE2 with more affinity than RBD from previous SARS-CoV viruses. However, the diversity of human ACE2 protein may also play a role in the high binding affinity. Because this virus is brand new, no one is immune to it. As a result, it has the potential to infect a huge number of people (WHO, 2021). The duration between being exposed to COVID-19 infection and experiencing symptoms is 5-6 days on average but can range from 1-14 days (Liu et al., 2020). According to WHO (2021), the most frequent symptoms of COVID-19 are fever, dry cough, and exhaustion. Symptoms of severe COVID-19 illness include loss of appetite, shortness of breath, disorientation, continuous discomfort or pressure in the chest, and a high fever (over). People of all ages who have a fever and/or a cough that is accompanied by trouble breathing or shortness of breath, chest discomfort or pressure, or loss of speech or movement should seek medical attention right once. Some patients may get severe sickness, which can lead to hospitalization and death (Guan et al., 2020).

The respiratory droplets produced when an infected person talks, coughs, or sneezes are the major method of transmission. These droplets can land on surrounding surfaces or directly on another person's lips, nose, or eyes, causing illness (CDC, 2022).

However, there is evidence that the virus can be spread via aerosols, which are tiny particles that can float in the air for extended periods. Indoor places that are poorly ventilated, especially those with limited air exchange or recirculation, increase the risk of airborne transmission (Zhang et al., 2021). The COVID-19 pandemic has served as a sharp reminder of how important it is to understand how infectious illnesses spread and to take the necessary safeguards to avoid their transmission.

Several models have been developed and tested to explain the behavior of COVID-19. Andrea et al. (2020) presented the difficulty of modeling and forecasting the spread of COVID-19. They stressed that modeling and forecasting the spread of COVID-19 remained difficult, but they were able to propose three macroscopic models: the exponential growth model, the self-exciting branching process, and the SIR (susceptible-infected-resistant) compartmental model. Their research revealed that dealing with the coronavirus disease 2019 (COVID-19) epidemic will be vitally dependent on the successful application of public health measures such as social distancing, shelter-in-place orders, disease surveillance, contact tracing, isolation, and quarantine.

Ghassane et al. (2020) proposed a mathematical model that described the flow of coronavirus, focusing more on asymptomatic individuals. They stated that the asymptomatic infectious individuals' contribution to the spread of the infection are largely undetected and thus can demoralize efforts to regulate the spread of the ailment. They further counseled that if social distancing is taken more seriously, then the healthcare system will not be overloaded.

The study by Masaki and Mitsuo (2020) showed a mathematical model for the COVID-19 pandemic which is characterized by presymptomatic and asymptomatic populations. They argued that infection propagation is difficult in asymptomatic persons since they are not isolated.

Veera and Prakash (2020) investigated the phase-based transmissibility of coronavirus. The generation matrix approach was recommended for calculating the basic reproduction number (). Their major goal is to expand a phase-based mathematical modeling to describe coronavirus transferability, therefore they created a reservoir-individuals spreading set of connections modeling to simulate the potential spread of a person's infectivity. They reported that for researchers to better understand and model the dynamics of a specific infection, they must consider the influence of numerous variables ranging from micro-host-pathogen interactions to host-to-host encounters, as well as global cultural, social, economic, and local customs.

Zeb et al. (2020) developed a mathematical model for coronavirus disease 2019 (COVID-19) by considering the isolation of the infected population. Their findings showed that effective contact with infected individuals is the cause of epidemics and hence, isolating sick persons can lower the risk of a pandemic.

Gnanvi et al. (2021) conducted a critical examination of modeling methodologies by investigating the reliability of predictions on COVID-19 dynamics. They performed a global systematic literature study on 242 publications to highlight trends in the modeling approach used for COVID-19 from January 1st, 2020 to November 30th, 2020. They also investigated prediction accuracy and precision by comparing anticipated and observed values for cumulative cases and fatalities, as well as the uncertainty of these forecasts. They demonstrated that compartmental and statistical growth models are the most often utilized modeling techniques for predicting COVID-19 dynamics. Artificial intelligence-based models, agent-based models, and Bayesian models were also employed, but to a lesser extent. Finally, they argued that although some forecasts were valuable in guiding policy-making, others were not.

Kumar, et al. (2021) discussed a new fractional mathematical modelling of COVID-19 in the presence of vaccine. They formulated the model with integer order and then, generalized it in the Atangana-Baleanu derivative sense. They provided the existence of a solution for the given fractional SEIRS model.

Okolo and Onoja (2021) investigated the impact of physical distance and isolation on COVID-19 transmission using a mathematical model. They calculated the fundamental reproduction number as an epidemic threshold using the next-generation technique. The study revealed that be mitigated with effective enforcement of regulations such as physical distance and isolation.

Okolo et al. (2021) constructed a deterministic mathematical model of coronavirus infection 2019 (COVID-19) transmission dynamics in the presence of social distance, face mask use, and hospitalization. The sensitivity index results suggest that the most sensitive parameter is the effective contact parameter, which was also used as the social distancing parameter. Numerical results suggest that the successful combination of social distancing, public usage of face masks, and isolation (hospitalization) of sick persons would result in a significant decrease in COVID-19 infection burden.

Though research in mathematical models of COVID-19 transmission dynamics is available and ongoing, the available models reviewed are epidemic models that did not incorporate vital/demographic dynamics. The results show that the epidemics will be mitigated with time, notwithstanding the value of the basic reproduction number of the model. We also noted that no attempt was made to incorporate the clinical diagnosis of an asymptomatic infectious population as a control measure. It is thus, instructive to formulate a mathematical model that will incorporate vital/demographic dynamics, assess the impact of face-mask use, clinical diagnosis of asymptomatic individuals, and isolation of infectious persons on the management and control of COVID-19 infection.

The present study extends some of the above-reviewed models but specifically, the Okolo et al. (2021) model by incorporating vital/demographic dynamics, a proportion of the population that uses face masks, the efficacy of face masks, the effect of isolating infected population, and the impact of clinical diagnosis of the asymptomatically-infectious individuals.

MATERIALS AND METHODS

Model Formulation

The entire population $N(t)$ is partitioned into six classes of Susceptible $S(t)$, Latent $E(t)$, Asymptomatic $I_a(t)$, Symptomatic $I(t)$, Isolated $I_s(t)$ and Recovered $R(t)$ individuals, so that,

$$N(t) = S(t) + E(t) + I_A(t) + I(t) + I_s(t) + R(t) \qquad (1a)$$

The description of the model variables and parameters are presented in Table 1

Table 1: Description of model variables and parameters

Variables / Parameters	Description
$S(t)$	Susceptible individuals at time *t.*
$E(t)$	Latent individuals at time *t.*
$I_A(t)$	Asymptomatic individuals at time *t.*
$I(t)$	Symptomatic individuals at time *t.*
$I_S(t)$	Isolated individuals at time *t.*
$R(t)$	Recovered individuals at time *t.*
Π	The influx rate of susceptible individuals.
β	Infection transmission rate.
θ	The rate of face mask compliance in public.
q	Efficiency of face masks.
η	The proportion of the asymptomatic population with symptoms as a result of clinical diagnosis.
v	The rate at which symptomatic individuals are isolated.
σ	The proportion of the latent population who show no COVID symptoms.
ω	The progression of latent individuals to the infectious class.

γ	The recovery rate for asymptomatic, symptomatic, and isolated compartments.
d	The COVID-19 fatality rate of the asymptomatically-infected, symptomatically-infected, and isolated class.
μ	Natural mortality rate.

The description and interaction in the different classes are represented by the diagram in Figure 1.

Model Description

The susceptible individuals $S(t)$ are recruited at the rate Π. It is decreased by effective contact with symptomatic individuals and effective contact with asymptomatic individuals at a rate of β. $0 < \theta \leq 1$ is the rate of face-masks compliance in public and $0 < q \leq 1$ is the efficiency of face-masks. It is decreased further by the natural mortality rate, μ. Thus,

$$\frac{dS}{dt} = \Pi - \beta(1-\theta q)S\frac{(I+I_A)}{N} - \mu S \qquad (1b)$$

The population of the latent individuals' $E(t)$ increases infection at a rate β. It is decreased by the progression of latent individuals to the infectious classes at a rate ω. A proportion, $0 < \sigma \leq 1$, of the latent population with no symptoms is moved to the asymptomatic class and $(1-\sigma)$, with symptoms are moved to the symptomatic class. It is decreased further by the natural mortality rate μ. so that,

$$\frac{dE}{dt} = \beta(1-\theta q)S\frac{(I+I_A)}{N} - (\omega+\mu)E \qquad (1c)$$

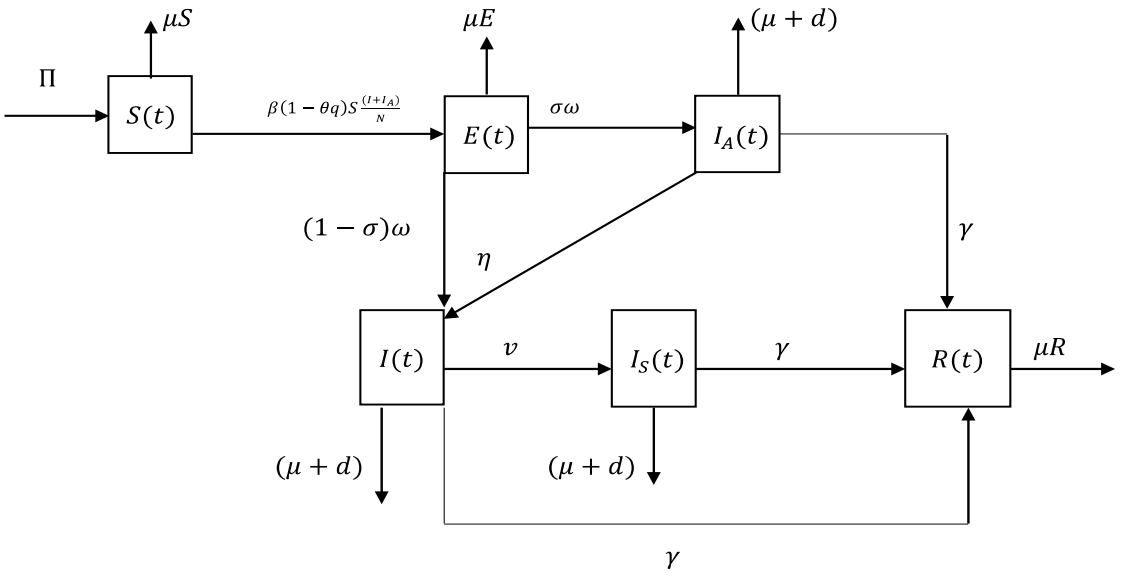

Figure 1: Schematic flow of COVID-19 transmission dynamics.

The population of the Asymptomatic individuals I_A (t) increases by the progression of the latent individuals into the asymptomatic class at a rate $\sigma\omega$, where $(0 < \sigma \leq 1)$ of the proportion of latent individuals with no symptoms. It is reduced by recovery at a rate γ, and by the proportion of individuals who develop symptoms, η. It is reduced further by COVID-19 induced-mortality rate d, and natural mortality rate μ. Therefore,

$$\frac{dI_A}{dt} = \sigma\omega E - (\gamma+\eta+\mu+d)I_A \qquad (1d)$$

The population of the Symptomatic individuals $I(t)$ increases by the progression of the latent individuals into the symptomatic class at a rate $(1-\sigma)\omega$, and the proportion η of asymptomatic individuals who developed symptoms. It is reduced by recovery at a rate γ, isolation of symptomatic infectious individuals at the rate v, COVID-19 induced-mortality rate d, and natural mortality rate μ. Hence,

$$\frac{dI}{dt} = (1-\sigma)\omega E + \eta I_A - (\gamma+v+\mu+d)I \qquad (1e)$$

The population of the Isolated individuals I_s(t) increases by progression of symptomatic individuals into the isolation class v, and decreases by recovery γ, it is reduced further by COVID-19 induced mortality rate d, and natural mortality at a rate μ. Thus,

$$\frac{dI_s}{dt} = vI - (\gamma+\mu+d)I_s \qquad (1f)$$

The Recovered population $R(t)$ is generated by the recovery of the symptomatic, asymptomatic, and isolated individuals at a rate γ. It is reduced by natural mortality, μ. Therefore,

$$\frac{dR}{dt} = \gamma I_A + \gamma I + \gamma I_s - \mu R \qquad (1g)$$

The Model Equations

For clarity, the equations from the model description are written as

$$\frac{dS}{dt} = \Pi - \beta(1-\theta q)S\frac{(I+I_A)}{N} - \mu S \qquad (2)$$

$$\frac{dE}{dt} = \beta(1-\theta q)S\frac{(I+I_A)}{N} - (\omega+\mu)E \qquad (3)$$

$$\frac{dI_A}{dt} = \sigma\omega E - (\gamma+\eta+\mu+d)I_A \qquad (4)$$

$$\frac{dI}{dt} = (1-\sigma)\omega E + \eta I_A - (\gamma+v+\mu+d)I \qquad (5)$$

$$\frac{dI_s}{dt} = vI - (\gamma+\mu+d)I_s \qquad (6)$$

$$\frac{dR}{dt} = \gamma I_A + \gamma I + \gamma I_s - \mu R \qquad (7)$$

with the non-negative initial condition

$S(0) > 0, E(0) > 0, I_A(0) > 0\ I(0) > 0, I_s(0) > 0, R(0) > 0$

RESULTS AND DISCUSSION

Positivity of Solutions

The system of equations (2) - (7) is a model that monitors the changes in the human population. It is necessary to prove that all the state variables are positive at all $t > 0$ by the theorem below.

Theorem 1: Let $N(0) = N_0, t_0 > 0$ and the initial conditions for equation (2) - (7) satisfy $S(t_0) > 0, E(t_0) > 0, I_A(t_0) > 0, I(t_0) > 0, I_S(t_0) > 0, and\ R(t_0) > 0$, then the solution $S(t), E(t), I_A(t), I(t), I_S(t), R(t)$ of system (2) – (7) are non-negative for all $t \geq 0$.

Proof: By eliminating the non-negative term Π Equation (2), it can be expressed as an inequality,

$$\frac{dS}{dt} \geq -[\beta(1-\theta q)\frac{(I+I_A)}{N} + \mu]S \quad (8)$$

Integrating equation (8) by separating the variables, gives

$$\int \frac{dS}{S} \geq \int -\left[\beta(1-\theta q)\frac{(I+I_A)}{N} + \mu\right] dt$$

That is,

$$S(t) \geq S(0)e^{-\mu t - \beta(1-\theta q)\frac{I}{N}t - \beta(1-\theta q)\frac{I_A}{N}t} \quad (9)$$

where $S(0)$ is obtained from the initial condition. Since the exponential function is always non-negative, the function $e^{-\mu t - \beta(1-\theta q)\frac{I}{N}t - \beta(1-\theta q)\frac{I_A}{N}t}$ is a positive quantity. Hence, we concluded that: $S(t) \geq 0$.

Equation (3) can be expressed as,

$$\frac{dE}{dt} \geq -(\omega + \mu)E \quad (10)$$

Integrating equation (10) by separating the variables, gives

$$E(t) \geq E(0)e^{-\omega t - \mu t} \quad (11)$$

where $E(0)$ is obtained from the initial condition. Since the exponential function is always non-negative, the function $e^{-\omega t - \mu t}$ is a positive quantity. Hence, we concluded that:

$E(t)$
≥ 0.

Similarly, following the above procedure by separating the variables and applying the initial conditions, Equations (4), (5), (6) and (7) can be solved to obtain,

$$I_A(t) \geq I_A(0)e^{-\gamma t - \eta t - \mu t - dt} \quad (12)$$

$$I(t) \geq I_A(0)e^{-\gamma t - \nu t - \mu t - dt} \quad (13)$$

$$I_S(t) \geq I_S(0)e^{-\gamma t - \mu t - dt} \quad (14)$$

$$R(t) \geq R(0)e^{-\mu t} \quad (15)$$

Therefore, $S(t), E(t), I_A(t), I(t),\ I_S(t)\ and\ R(t)$ are all non-negative for all $t \geq 0$.

Invariant Region

Consider the biologically feasible region

$$\Omega = \left\{\left(S(t), E(t), I_A(t), I(t), I_S(t), R(t)\right) \in \mathbb{R}^6 : N \leq \frac{\pi}{\mu}\right\} \quad (16)$$

Lemma 2: The closed set Ω is positively invariant with respect to the system equations (2) – (7).

Proof:

Using equation (2) – (7) we have

$$\frac{dN}{dt} = \Pi - \mu N - d(I_A + I + I_s) \quad (17)$$

It is clear from the equation (17) that

$$\frac{dN}{dt} \leq \Pi - \mu N \quad (18)$$

Thus,

$$\frac{dN}{dt} \leq 0, \text{ if } N(t) \geq \frac{\Pi}{\mu}$$

Following Lakstimikantham et al. (1989) we have that

$$N(t) = N(0)e^{-\mu t} + \frac{\Pi}{\mu}(1 - e^{-\mu t}) \quad (19)$$

Specifically,

$$N(t) \leq \frac{\Pi}{\mu} \quad \text{if} \quad N(0) \leq \frac{\Pi}{\mu}$$

Hence, $\Omega = \{(S, E, I_A\ I, I_S, R) \in \mathbb{R}^6 : N \leq \frac{\pi}{\mu}\}$ is positively and attract. However, if $N(t) \leq \frac{\Pi}{\mu}$, then either the solution enters Ω in a finite time, or $N(t)$ approaches $\frac{\Pi}{\mu}$ asymptotically.

Disease-Free Equilibrium (DFE) Point

The model equations (2) - (7) have a Disease-Free Equilibrium (DFE) state, $\mathcal{E}_0$ where

$$\mathcal{E}_0 = (S_0, 0, 0, 0, 0, 0) = \left(\frac{\Pi}{\mu}, 0, 0, 0, 0, 0\right) \quad (20)$$

The computation of the basic reproduction number R_0 is required to assess the stability of COVID-19.

Basic Reproduction Number (R_0)

The average number of secondary infections, when one sick individual is introduced into a host community where everyone is prone to the disease is the basic reproduction number. (Diekmann et al., 1990; Van Den Driessche & Watmough, 2002). To calculate the reproduction number R_0, we employ the next-generation matrix method. The maximum eigenvalue of FV^{-1} is the basic reproduction number. Thus,

$$R_0 = \sigma(FV^{-1}) \quad (21)$$

where σ denotes the maximum eigenvalue.

The non-negative matrix F, for the appearance of new infection terms, and the transition rate matrix, V, for the transfer of individuals into and out of this class by all other means given by

$$F_x(\mathcal{E}_0) = \begin{pmatrix} 0 & \beta(1-\theta q)\frac{\Pi}{\mu N} & \beta(1-\theta q)\frac{\Pi}{\mu N} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

and

$$V_x = \begin{pmatrix} (\omega+\mu) & 0 & 0 \\ -\sigma\omega & (\gamma+\eta+\mu+d) & 0 \\ -(1-\sigma)\omega & -\eta & (\gamma+v+\mu+d) \end{pmatrix} \quad (23)$$

$$V_x^{-1} = \begin{pmatrix} \frac{1}{(\omega+\mu)} & 0 & 0 \\ \frac{\sigma\omega}{a_1(\omega+\mu)} & \frac{1}{a_1} & 0 \\ \frac{\eta\sigma\omega + a_1(1-\sigma)\omega}{a_1 a_2(\omega+\mu)} & \frac{\eta}{a_1 a_2} & \frac{1}{a_2} \end{pmatrix} \quad (24)$$

where,

$a_1 = (\gamma + \eta + \mu + d)$
$a_2 = (\gamma + v + \mu + d)$
so that
$F_x V_x^{-1} =$

$$\begin{pmatrix} \beta(1-\theta q)\frac{\Pi}{\mu N}\left[\frac{\sigma\omega}{a_1(\omega+\mu)}\right] + \beta(1-\theta q)\frac{\Pi}{\mu N}\left[\frac{\eta\sigma\omega + a_1(1-\sigma)\omega}{a_1 a_2(\omega+\mu)}\right] & \beta(1-\theta q)\frac{\Pi}{\mu N}\left[\frac{1}{a_2}\right] + \beta(1-\theta q)\frac{\Pi}{\mu N}\left[\frac{\eta}{a_1 a_2}\right] & \beta(1-\theta q)\frac{\Pi}{\mu N}\left[\frac{1}{a_1}\right] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (25)$$

The maximum eigenvalue (FV^{-1}) is the basic reproduction number R_0, is

$$R_0 = \frac{\beta(1-\theta q)\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)} + \frac{(1-\sigma)\omega}{(\gamma+v+\mu+d)}\right] \quad (26)$$

Analysis of the Local Asymptotic Stability (LAS) of the Disease-Free Equilibrium (DFE) State
To establish the local and asymptotic stability of the Disease-Free Equilibrium (DFE) state, the associated matrix of partial derivatives of the model (2) – (7) , that is, the Jacobian matrix is obtained at the DFE state.
The matrix of the partial derivatives (Jacobian matrix) at the DFE state, $J(\mathcal{E}_0)$ is given by

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu & 0 & -\beta(1-\theta q)\frac{\Pi}{\mu N} & -\beta(1-\theta q)\frac{\Pi}{\mu N} & 0 & 0 \\ 0 & -(\omega+\mu) & \beta(1-\theta q)\frac{\Pi}{\mu N} & \beta(1-\theta q)\frac{\Pi}{\mu N} & 0 & 0 \\ 0 & \sigma\omega & -(\gamma+\eta+\mu+d) & 0 & 0 & 0 \\ 0 & (1-\sigma)\omega & \eta & -(\gamma+v+\mu+d) & 0 & 0 \\ 0 & 0 & 0 & v & -(\gamma+\mu+d) & 0 \\ 0 & 0 & \gamma & \gamma & \gamma & -\mu \end{pmatrix} \quad (27)$$

Theorem 3: The DFE state of the equations (2) – (7), is locally asymptotically stable whenever $R_0 < 1$ and is unstable if $R_0 > 1$.

Proof:
It is enough to prove that all the eigenvalues of the characteristic equation of $J(\mathcal{E}_0)$, have negative real parts. The eigenvalues are determined by solving the characteristics equation $\det(J(\mathcal{E}_0) - \lambda I) = 0$. Thus,

$$(-\mu-\lambda)(-\mu-\lambda)(-a_3-\lambda)\begin{vmatrix} -(\omega+\mu)-\lambda & \beta(1-\theta q)\frac{\Pi}{\mu N} & \beta(1-\theta q)\frac{\Pi}{\mu N} \\ \sigma\omega & -a_1-\lambda & 0 \\ (1-\sigma)\omega & \eta & -a_2-\lambda \end{vmatrix} = 0 \quad (28)$$

$a_1 = (\gamma + \eta + \mu + d)$
$a_2 = (\gamma + v + \mu + d)$
$a_3 = (\gamma + \mu + d)$
Simplifying to obtain

$$(-\mu-\lambda)\,(-\mu-\lambda)(-a_3-\lambda)\left[\lambda^3 + \lambda^2(a_1 + a_2 + \omega + \mu) + \lambda\left[-\beta(1-\theta q)\frac{\Pi\sigma\omega}{\mu N} - \beta(1-\theta q)\frac{\Pi(1-\sigma)\omega}{\mu N} + a_1 a_2 + (a_1 + a_2)(\omega+\mu)\right] - \beta(1-\theta q)\frac{\Pi}{\mu N}[\eta\sigma\omega + a_1\omega - a_1\sigma\omega + a_2\sigma\omega] + a_1 a_2(\omega+\mu)\right] = 0 \quad (29)$$

Thus, the eigenvalues are
$\lambda = -\mu$ (twice), $\lambda = -a_3$ and the root of

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (30)$$

where,

$$A = a_1 + a_2 + \omega + \mu, \quad (31)$$

$$B = a_1 a_2 + a_1(\omega+\mu) + a_2(\omega+\mu) - \beta(1-\theta q)\frac{\Pi\sigma\omega}{\mu N} - \beta(1-\theta q)\frac{\Pi(1-\sigma)\omega}{\mu N}, \quad (32)$$

$$C = -\frac{\beta(1-\theta q)\frac{\Pi}{\mu N}}{(\omega+\mu)}\left[\frac{\eta\sigma\omega + a_1(1-\sigma)\omega}{a_1 a_2} + \frac{\sigma\omega}{a_1}\right] + a_1 a_2(\omega+\mu), \quad (33)$$

All the roots of the characteristic equation (30) have negative real parts if, $A, B, C > 0$, $AB - C > 0$ according to Routh-Hurwitz stability criteria (Routh-Hurwitz, 1964).
Obviously,

$$A = a_1 + a_2 + \omega + \mu > 0 \quad (34)$$

$$B = a_1 a_2 + a_1(\omega+\mu) + a_2(\omega+\mu) - \beta(1-\theta q)\frac{\Pi\sigma\omega}{\mu N} - \beta(1-\theta q)\frac{\Pi(1-\sigma)\omega}{\mu N}$$

$$B = a_1 a_2 + a_1(\omega+\mu)\left[1 - \frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}\right] + a_2(\omega+\mu)\left[1 - \frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right]$$

$$B = a_1a_2 + a_1(\omega+\mu)\left[1-(R_0-(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N})\right] + a_2(\omega+\mu)\left[1-(R_0-(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N})\right]$$

$$B = a_1a_2 + a_1(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]+a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)\right]>0, \text{if } R_0<1 \quad (35)$$

$$C = -\frac{\beta(1-\theta q)\frac{\Pi}{\mu N}}{(\omega+\mu)}\left[\frac{\eta\sigma\omega+a_1(1-\sigma)\omega}{a_1a_2}+\frac{\sigma\omega}{a_1}\right]+a_1a_2(\omega+\mu)$$

$$C = a_1a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]>0, if\ R_0<1 \quad (36)$$

Next, consider $AB-C$,

$$AB-C =$$
$$(a_1+a_2\,\omega+\mu\left\{a_1a_2+a_1(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]+a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)\right]\right\} - a_1a_2(\omega+\mu)[1-R_0]$$
$$= a_1^2a_2+a_1^2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right] + +a_1a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)\right]+a_1a_2^2+a_1a_2(\omega+\mu)[1-R_0 + \left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)]+a_2^2(\omega+\mu)[1-R_0 + \left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)]+a_1a_2(\omega+\mu)+a_1(\omega+\mu)^2[1-R_0 + \left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)]+a_2(\omega+\mu)^2\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right) - a_1a_2(\omega+\mu)[1-R_0]$$

$$= a_1^2a_2+a_1^2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]++a_1a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)\right]+a_1a_2^2+a_1a_2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]+a_2^2(\omega+\mu)\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)\right]+a_1a_2(\omega+\mu)+a_1(\omega+\mu)^2\left[1-R_0+\left(\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi(1-\sigma)\omega}{a_2(\omega+\mu)\mu N}\right)\right]+a_2(\omega+\mu)^2\left(\frac{\beta(1-\theta q)\Pi\sigma\omega}{a_1(\omega+\mu)\mu N}+\frac{\beta(1-\theta q)\Pi\sigma\eta\omega}{a_1a_2(\omega+\mu)\mu N}\right)-a_1a_2(\omega+\mu)+a_1a_2(\omega+\mu)R_0>0,\ \text{if } R_0<1 \quad (37)$$

Global Asymptotic Stability of the Disease-Free Equilibrium (DFE) State

To certify that COVID-19 disease eradication is not dependent on the initial size of the model population, it is imperative to prove that the DFE of the equations (2) - (7) *is* globally asymptotically stable (GAS). To achieve this, we will use the following results by (Castillo-Chavez et al., 2002).

Lemma 4: (Castillo-Chavez et al., 2002). Let systems of equation (2) – (7) be put as follows

$$\frac{dX_1}{dt} = W(X_1, X_2) \quad (38)$$

$$\frac{dX_2}{dt} = G(X_1, X_2), G(X_1, 0) = 0 \quad (39)$$

where $X_1 \in \mathbb{R}^m$ denotes (its components), the population of uninfected classes and $X_2 \in \mathbb{R}^n$ denotes (its components), the population of infected compartments including latent, infectious, etc. Also, the conditions H_1 and H_2 are assumed as follows

(H_1), $\frac{dX_1}{dt} = W(X_1, 0), X_1^*$ is global asymptotic stable (GAS),

(H_2), $G(X_1, X_2) = QY - \hat{G}(X_1, X_2), \hat{G}(X_1, X_2) \geq 0$ for $(X_1, X_2) \in$ Q,

where the Jacobian $Q = \left(\frac{\partial G}{\partial Y}\right)_{X_0}$ is a Metzler matrix.

Lemma 5: The fixed point $X_0 = (X_1^*, 0)$ is a global asymptotically stable (GAS) of the system of equations (2) - (7) provided that $R_0 < 1$ and that the assumptions H_1 and H_2 are fulfilled.

We present the following theorem.

Theorem 6: The fixed point of the model (2) – (7) is globally asymptotically stable provided that $R_0 < 1$.

Proof: To show this we implement the notations in Lemma 4 and verify the conditions $(H_1)\ and\ (H_2)$.
From our model, $X_1 = (S, R)^T$, $X_2 = (E, I_A, I, I_S)$, and $X_1^* = (\frac{\pi}{\mu}, 0)$.
The uninfected subsystem is

$$\frac{d}{dt}\begin{bmatrix} S \\ R \end{bmatrix} = W = \begin{bmatrix} \Pi - \beta(1-\theta q)S\frac{(I+I_A)}{N} - \mu S \\ Y(I_A + I + I_S) - \mu R \end{bmatrix} \tag{40}$$

And the infected subsystem is

$$\frac{d}{dt}\begin{bmatrix} E \\ I_A \\ I \\ I_S \end{bmatrix} = G = \begin{bmatrix} \beta(1-\theta q)S\frac{(I+I_A)}{N} - (\omega+\mu)E \\ \sigma\omega E - (\gamma+\eta+\mu+d)I_A \\ (I-\sigma)\omega E + \eta I_A - (\gamma+v+\mu+d) \\ vI - (\gamma+\mu+d)I_S \end{bmatrix} \tag{41}$$

In addition, this follows that $Q = \left(\frac{\partial G}{\partial Y}\right)_{X_0}$
When $E = I_A = I = I_S = 0$, the uninfected subsystem (41) becomes,

$$\frac{d}{dt}\begin{bmatrix} S \\ R \end{bmatrix} = \begin{bmatrix} \Pi - \mu S \\ -\mu R \end{bmatrix} \tag{42}$$

and its solution is,

$$R(t) = R(0)e^{-\mu t}, S(t) = S(0)e^{-\mu t} + \frac{\Pi}{\mu}(1 - e^{-\mu t}) \tag{43}$$

clearly, $R(t) \longrightarrow 0\ and\ S(t) \longrightarrow \frac{\Pi}{\mu}\ as\ t \longrightarrow \infty$,
irrespective of the values of $R(0)\ and\ S(0)$. Thus, $X^* = (\frac{\Pi}{\mu}, 0)$ is a global asymptotical stable equilibrium for the subsystem,

$$\frac{dX_1}{dt} = W(X_1, 0). \tag{44}$$

Next, we have

$$Q = \begin{pmatrix} -(\omega+\mu) & \beta(1-\theta q)\frac{\Pi}{\mu N} & \beta(1-\theta q)\frac{\Pi}{\mu N} & 0 \\ \sigma\omega & -(\gamma+\eta+\mu+d) & 0 & 0 \\ (1-\sigma)\omega & \eta & -(\gamma+v+\mu+d) & 0 \\ 0 & 0 & v & -(\gamma+\mu+d) \end{pmatrix} \tag{45}$$

From $G(X_1, X_2) = QX_2 - \hat{G}(X_1, X_2)$
$\hat{G}(X_1, X_2) = QX_2 - G(X_1, X_2)$

$$\begin{pmatrix} -(\omega+\mu) & \beta(1-\theta q)\frac{\Pi}{\mu N} & \beta(1-\theta q)\frac{\Pi}{\mu N} & 0 \\ \sigma\omega & -(\gamma+\eta+\mu+d) & 0 & 0 \\ (1-\sigma)\omega & \eta & -(\gamma+v+\mu+d) & 0 \\ 0 & 0 & v & -(\gamma+\mu+d) \end{pmatrix}\begin{pmatrix} E \\ I_A \\ I \\ I_S \end{pmatrix} - \begin{pmatrix} \beta(1-\theta q)\frac{\Pi}{\mu N}(I_A + I) - (\omega+\mu)E \\ \sigma\omega E - (\gamma+\eta+\mu+d)I_A \\ (1-\sigma)\omega E + \eta I_A - (\gamma+v+\mu+d)I \\ vI - (\gamma+\mu+d)I_S \end{pmatrix} \tag{46}$$

$$\hat{G}(X_1, X_2) = \begin{pmatrix} \beta(1-\theta q)\frac{(I+I_A)}{N}\left[\frac{\Pi}{\mu} - S\right] \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{47}$$

Hence, $\hat{G}(X_1, X_2) \geq 0$ for $(X_1, X_2) \in \Omega$. We also note that G is an M-matrix. Thus, the DFE $\mathcal{E}_0 = (\frac{\Pi}{\mu}, 0,0,0,0)$ is a global asymptotically stable of the system (2) – (7) provided $R_0 < 1$.

Local Stability Analysis of the Endemic Equilibrium State
The model equations (2) – (7) have an endemic equilibrium given by

$$\mathcal{E}_1 = (S^*, E^*, I_A^*, I^*, I_S^*, R^*) = \left(\frac{a_1 a_2^2 N(\omega+\mu)}{\beta(1-\theta q)[(1-\sigma)\omega a_2^2 + \eta\sigma\omega a_2 + \sigma\omega a_1 a_2]}\right), \frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1], \frac{\sigma\omega}{a_2}\frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1], \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1], \frac{v}{a_3}\left[\frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1], \frac{\gamma}{\mu}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2} + \frac{v}{a_3}\left[\frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\right]\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1] \tag{48}$$

To establish the existence and local asymptotic stability of the Endemic Equilibrium (EE) state, the associated matrix of partial derivatives of the model (2) – (7) , that is, the Jacobian matrix is obtained at the EE state, $\mathcal{E}_1$.
The matrix of the partial derivatives (Jacobian matrix) at the EE state, $J(\mathcal{E}_1)$ is given by

$$J(\mathcal{E}_1) == \begin{pmatrix} -\beta(1-\theta q)\frac{(I^*+I_A^*)}{N} - \mu & 0 & -\beta(1-\theta q)\frac{S^*}{N} & -\beta(1-\theta q)\frac{S^*}{N} & 0 & 0 \\ \beta(1-\theta q)\frac{(I^*+I_A^*)}{N} & -(\omega+\mu) & \beta(1-\theta q)\frac{S^*}{N} & \beta(1-\theta q)\frac{S^*}{N} & 0 & 0 \\ 0 & \sigma\omega & -(\gamma+\eta+\mu+d) & 0 & 0 & 0 \\ 0 & (1-\sigma)\omega & \eta & -(\gamma+v+\mu+d) & 0 & 0 \\ 0 & 0 & 0 & v & -(\gamma+\mu+d) & 0 \\ 0 & 0 & \gamma & \gamma & \gamma & -\mu \end{pmatrix} \quad (49)$$

Theorem 7: The EE state $\mathcal{E}_1$, of the equations (2) – (7) is locally asymptotically stable provided $R_0 > 1$.

Proof:

It is enough to show that all the eigenvalues of the characteristic equation of $J(\mathcal{E}_1)$, have negative real parts. The characteristics equation $\det(J(\mathcal{E}_1) - \lambda I) = 0$ is given by

$$\begin{vmatrix} -\beta(1-\theta q)\frac{(I^*+I_A^*)}{N} - \mu - \lambda & 0 & -\beta(1-\theta q)\frac{S^*}{N} & -\beta(1-\theta q)\frac{S^*}{N} & 0 & 0 \\ \beta(1-\theta q)\frac{(I^*+I_A^*)}{N} & -(\omega+\mu)-\lambda & \beta(1-\theta q)\frac{S^*}{N} & \beta(1-\theta q)\frac{S^*}{N} & 0 & 0 \\ 0 & \sigma\omega & -a_1-\lambda & 0 & 0 & 0 \\ 0 & (1-\sigma)\omega & \eta & -a_2-\lambda & 0 & 0 \\ 0 & 0 & 0 & v & -a_3-\lambda & 0 \\ 0 & 0 & \gamma & \gamma & \gamma & -\mu-\lambda \end{vmatrix} = 0 \quad (50)$$

where

$a_1 = (\gamma+\eta+\mu+d)$,

$a_2 = (\gamma+v+\mu+d)$,

$a_3 = (\gamma+\mu+d)$,

Evaluating equation (50) to obtain,

$$(-\mu-\lambda)(-(\gamma+\mu+d)-\lambda) \begin{vmatrix} \frac{-\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_1} + \frac{(1-\sigma)\omega a_1 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1] - \mu - \lambda & 0 & -\frac{a_1 a_2(\omega+\mu)}{[(1-\sigma)\omega a_2 + \eta\sigma\omega + \sigma\omega a_1]} & -\frac{a_1 a_2(\omega+\mu)}{[(1-\sigma)\omega a_2 + \eta\sigma\omega + \sigma\omega a_1]} \\ \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_1} + \frac{(1-\sigma)\omega a_1 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1] & -(\omega+\mu)-\lambda & \frac{a_1 a_2(\omega+\mu)}{[(1-\sigma)\omega a_2 + \eta\sigma\omega + \sigma\omega a_1]} & \frac{a_1 a_2(\omega+\mu)}{[(1-\sigma)\omega a_2 + \eta\sigma\omega + \sigma\omega a_1]} \\ 0 & \sigma\omega & -(\gamma+\eta+\mu+d)-\lambda & 0 \\ 0 & (1-\sigma)\omega & \eta & -(\gamma+v+\mu+d)-\lambda \end{vmatrix} = 0 \quad (51)$$

That is,

$$(-\mu-\lambda)(-a_3-\lambda) \begin{vmatrix} a_4-\mu-\lambda & 0 & b & b \\ -a_4 & -(\omega+\mu)-\lambda & -b & -b \\ 0 & \sigma\omega & -a_1-\lambda & 0 \\ 0 & (1-\sigma)\omega & \eta & -a_2-\lambda \end{vmatrix} = 0$$

where,

$$b = \frac{-a_1 a_2(\omega+\mu)}{[(1-\sigma)\omega a_2 + \eta\sigma\omega + \sigma\omega a_1]},$$

$$a_4 = \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[1-R_0]$$

Simplifying to have,

$$(-\mu-\lambda)(-a_3-\lambda)[\lambda^4 + [(\omega+\mu)+(a_1+a_2)+\mu-a_4)]\lambda^3 + [(\omega+\mu)(a_1+a_2)+a_1a_2+b\sigma\omega+\mu(\omega+\mu)+\mu(a_1+a_2)-a_4(\omega+\mu)+(a_1+a_2)]\lambda^2 + [(\omega+\mu)a_1a_2+b\sigma\omega a_2+b\sigma\omega\eta+b(1-\sigma)\omega a_2+\mu((\omega+\mu)(a_1+a_2))+a_1a_2+b\sigma\omega-a_4((\omega+\mu)(a_1+a_2)+a_1a_2+b\sigma\omega)+b(\sigma\omega+(1-\sigma)\omega+a_4\sigma\omega+a_4(1-\sigma)\omega]\lambda + [\mu(\omega+\mu)a_1a_2+b\sigma\omega a_2+b\sigma\omega\eta+b(1-\sigma)\omega a_2-a_4[b\sigma\omega a_2+b\sigma\omega\eta+b(1-\sigma)\omega a_2+(\omega+\mu)a_1a_2]+b(\sigma\omega a_2+\sigma\omega\eta+(1-\sigma)\omega a_1+a_4\sigma\omega a_2+a_4\sigma\omega\eta+a_4(1-\sigma)\omega a_1 = 0 \quad (52)$$

Thus, the eigenvalues are

$\lambda_1 = -\mu, \lambda_2 = -a_3$ and $\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$ (53)

where,

$$A = (\omega+\mu)+(a_1+a_2)+\mu-a_4)$$

$$A = (\omega+\mu)+(a_1+a_2)+\mu+\frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1] > 0$$

$if\ R_0 > 1$ (54)

$$B = (\omega+\mu)(a_1+a_2)+a_1a_2+b\sigma\omega+\mu(\omega+\mu)+\mu(a_1+a_2)-a_4(\omega+\mu)+(a_1+a_2)]$$

$$B = (\omega+\mu)(a_1+a_2)+a_1a_2-\frac{a_1a_2(\omega+\mu)}{[(1-\sigma)\omega a_2+\eta\sigma+\sigma\omega a_1]}\sigma\omega+\mu(\omega+\mu)+\mu(a_1+a_2) + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1 a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1](\omega+\mu)+(a_1+a_2)$$

$$B = (\omega+\mu)(a_1+a_2) + a_1a_2 + \mu(\omega+\mu) + \mu(a_1+a_2) + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2 + \eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1](\omega+\mu) + (a_1+a_2)] > \frac{a_1a_2(\omega+\mu)\sigma\omega}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]} \; if\; R_0 > 1 \quad (55)$$

$$C = (\omega+\mu)a_1a_2 + b\sigma\omega a_2 + b\sigma\omega\eta + b(1-\sigma)\omega a_2 + \mu((\omega+\mu)(a_1+a_2) + a_1a_2 + b\sigma\omega) - a_4((\omega+\mu)(a_1+a_2) + a_1a_2 + b\sigma\omega)) + b(\sigma\omega + (1-\sigma)\omega + a_4\sigma\omega + a_4(1-\sigma)\omega)$$

$$C = (\omega+\mu)a_1a_2 - a_4\big((\omega+\mu)(a_1+a_2) + a_1a_2\big)\big) + \mu\big((\omega+\mu)(a_1+a_2) + a_1a_2\big) + b(\sigma\omega\eta + \omega a_2 + \mu\sigma\omega + \omega) + a_4b(\omega+\sigma\omega)$$

$$C = (\omega+\mu)a_1a_2 + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1]\big((\omega+\mu)(a_1+a_2)+a_1a_2\big)\big) + \mu\big((\omega+\mu)(a_1+a_2)+a_1a_2\big) - \frac{a_1a_2(\omega+\mu)}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]}(\sigma\omega\eta+\omega a_2+\mu\sigma\omega+\omega) + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1]\left[\frac{a_1a_2(\omega+\mu)}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]}\right](\omega+\sigma\omega)$$

$$C = (\omega+\mu)a_1a_2 + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1]\big((\omega+\mu)(a_1+a_2)+a_1a_2\big)\big) + \mu\big((\omega+\mu)(a_1+a_2)+a_1a_2\big) + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1]\left[\frac{a_1a_2(\omega+\mu)}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]}\right](\omega+\sigma\omega) > \frac{a_1a_2(\omega+\mu)[(\sigma\omega\eta+\omega a_2+\mu\sigma\omega+\omega)]}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]} \; if\; R_0 > 1 \quad (56)$$

$$D = \mu(\omega+\mu)a_1a_2 + b\sigma\omega a_2 + b\sigma\omega\eta + b(1-\sigma)\omega a_2 - a_4[b\sigma\omega a_2 + b\sigma\omega\eta + b(1-\sigma)\omega a_2 + (\omega+\mu)a_1a_2] + b(\sigma\omega a_2 + \sigma\omega\eta + (1-\sigma)\omega a_1 + a_4\sigma\omega a_2 + a_4\sigma\omega\eta + a_4(1-\sigma)\omega a_1$$

$$D = \mu(\omega+\mu)a_1a_2 - a_4(\omega+\mu)a_1a_2 + a_4b[(1-\sigma)\omega a_1] - a_4b[(1-\sigma)\omega a_2] + b[2\sigma\omega\eta + \omega a_2 + \sigma\omega a_2 + (1-\sigma)\omega a_1]$$

$$D = \mu(\omega+\mu)a_1a_2 + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0-1](\omega+\mu)a_1a_2 + \frac{\beta(1-\theta q)}{N}\left[\frac{\sigma\omega}{a_2} + \frac{(1-\sigma)\omega a_2+\eta\sigma\omega}{a_1a_2}\right]\frac{\Pi}{(\omega+\mu)R_0}[R_0 - 1]\left[\frac{a_1a_2(\omega+\mu)}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]}\right][(1-\sigma)\omega a_1] > \frac{\beta(1-\theta q)\Pi[R_0-1][(1-\sigma)\omega a_2]}{NR_0} + \left[\frac{a_1a_2(\omega+\mu)[2\sigma\omega\eta+\omega a_2+\sigma\omega a_2+(1-\sigma)\omega a_1]}{[(1-\sigma)\omega a_2+\eta\sigma\omega+\sigma\omega a_1]}\right] \; if\; R_0 > 1 \quad (57)$$

All the roots of the characteristic equation (53) have negative real parts if: $A, B, C, D > 0, ABC - C^2 - A^2D > 0, and\; AB - C > 0$, by the Routh-Hurwitz stability criteria (Routh-Hurwitz, 1964).

Global Asymptotic Stability of Endemic Equilibrium (EE) State

Theorem 8: The system of equations (2) – (7) contains no periodic orbits.

Proof: We employ Dulac's stability criterion was implemented.

Let $X = (S, E, I_A, I, I_S, R)$. Taking the Dulac's function:

$$G = \frac{1}{SE} \quad (58)$$

we have,

$$G\frac{dS}{dt} = \frac{\Pi}{SE} - \beta(1-\theta q)\frac{(I+I_A)}{NE} - \frac{\mu}{E} \quad (59)$$

$$G\frac{dE}{dt} = \beta(1-\theta q)\frac{(I+I_A)}{NE} - \frac{(\omega+\mu)}{S} \quad (60)$$

$$G\frac{dI_A}{dt} = \frac{\sigma\omega}{S} - \frac{(\gamma+\eta+\mu+d)I_A}{SE} \quad (61)$$

$$G\frac{dI}{dt} = \frac{(1-\sigma)\omega}{S} + \frac{\eta I_A}{SE} - \frac{(\gamma+v+\mu+d)I}{SE} \quad (62)$$

$$G\frac{dI_S}{dt} = \frac{vI}{SE} - \frac{(\gamma+\mu+d)I_S}{SE} \quad (63)$$

$$G\frac{dR}{dt} = \frac{\gamma I_A}{SE} + \frac{\gamma I}{SE} + \frac{\gamma I_S}{SE} - \frac{\mu R}{SE} \quad (64)$$

Thus,

$$\frac{dGX}{dt} = \frac{\partial}{\partial S}\left(G\frac{dS}{dt}\right) + \frac{\partial}{\partial E}\left(G\frac{dE}{dt}\right) + \frac{\partial}{\partial I_A}\left(G\frac{dI_A}{dt}\right) + \frac{\partial}{\partial I}\left(G\frac{dI}{dt}\right) + \frac{\partial}{\partial I_S}\left(G\frac{dI_S}{dt}\right) + \frac{\partial}{\partial R}(G\frac{dR}{dt})$$

$$= \frac{\partial}{\partial S}\left(\frac{\Pi}{SE} - \beta(1-\theta q)\frac{(I+I_A)}{NE} - \frac{\mu}{E}\right) + \frac{\partial}{\partial E}\left(\beta(1-\theta q)\frac{(I+I_A)}{NE} - \frac{(\omega+\mu)}{S}\right) + \frac{\partial}{\partial I_A}\left(\frac{\sigma\omega}{S} - \frac{(\gamma+\eta+\mu+d)I_A}{SE}\right)$$
$$+ \frac{\partial}{\partial I}\left(\frac{(1-\sigma)\omega}{S} + \frac{\eta I_A}{SE} - \frac{(\gamma+v+\mu+d)I}{SE}\right) + \frac{\partial}{\partial I_S}\left(\frac{vI}{SE} - \frac{(\gamma+\mu+d)I_s}{SE}\right)$$
$$+ \frac{\partial}{\partial R}\left(\frac{\gamma I_A}{SE} + \frac{\gamma I}{SE} + \frac{\gamma I_s}{SE} - \frac{\mu R}{SE}\right)$$

$$= -\frac{\Pi}{S^2E} - \frac{\beta(1-\theta q)(I+I_A)}{N^2E} - \frac{(\gamma+\eta+\mu+d)}{SE} - \frac{(\gamma+v+\mu+d)}{SE} - \frac{(\gamma+\mu+d)}{SE} - \frac{\mu}{SE}$$
$$= -\left(\frac{\Pi}{S^2E} + \frac{(\gamma+\eta+\mu+d)+(\gamma+v+\mu+d)+(\gamma+\mu+d)+\mu}{SE}\right)$$

$< 0.$ (65)

Hence, the systems of equation (2) - (7) have no periodic orbit. Thus proven. Since Ω is positive and attracting, then, from Poincare Bendixson theorem, all solutions of the systems of equation (2) – (7) start and stay in Ω for all t. Hence, the following theorem.

Theorem 9: The systems of equations (2) – (7) have an endemic equilibrium that is globally asymptotically stable provided $R_0 > 1$.

Sensitivity Analysis of R_0

We carried out sensitivity index on R_0 for the population influx rate (Π), infection transmission rate (β), proportion of individuals who wear face-masks in public (θ), Efficacy of face-masks use (q), clinical diagnosis of asymptomatic individuals (η) and isolation rate (v). We adopt the normalized forward sensitivity index using the following formula (Chitnis et al., 2008).

$$\Lambda_{\mathrm{Q}}^{R_0} = \frac{\partial R_0}{\partial Q}\left(\frac{Q}{R_0}\right) \quad (66)$$

where Q denotes the model parameter.

The results are given as follows,

$$\Delta_{\boldsymbol{\Pi}}^{\boldsymbol{R_0}} = \frac{\beta(1-\theta q)}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega + (\gamma+\eta+\mu+d)(1-\sigma)\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)}\right].\frac{\boldsymbol{\Pi}}{\boldsymbol{R_0}} = 1, \quad (67)$$

$$\Delta_{\boldsymbol{\beta}}^{\boldsymbol{R_0}} = \frac{(1-\theta q)\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega + (\gamma+\eta+\mu+d)(1-\sigma)\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)}\right].\frac{\boldsymbol{\beta}}{\boldsymbol{R_0}} = 1, \quad (68)$$

$$\Delta_{\boldsymbol{\theta}}^{\boldsymbol{R_0}} = \frac{-\beta q\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega + (\gamma+\eta+\mu+d)(1-\sigma)\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)}\right].\frac{\boldsymbol{\theta}}{\boldsymbol{R_0}} = \frac{-q\theta}{(1-\theta q)}, \quad (69)$$

$$\Delta_{\mathbf{q}}^{\boldsymbol{R_0}} = \frac{-\beta\theta\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega + (\gamma+\eta+\mu+d)(1-\sigma)\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)}\right].\frac{\mathbf{q}}{\boldsymbol{R_0}} = \frac{-q\theta}{(1-\theta q)}, \quad (70)$$

$$\Delta_{\boldsymbol{\eta}}^{\boldsymbol{R_0}} = \frac{\beta(1-\theta q)\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{-\sigma\omega}{(\gamma+\eta+\mu+d)^2} + \frac{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)[\sigma\omega+(1-\sigma)\omega] - [\eta\sigma\omega+(\gamma+\eta+\mu+d)(1-\sigma)\omega](\gamma+v+\mu+d)}{[(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)]^2}\right].\frac{\boldsymbol{\eta}}{\boldsymbol{R_0}}, \quad (71)$$

$$\Delta_{\boldsymbol{v}}^{\boldsymbol{R_0}} = \frac{\beta(1-\theta q)\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{-(\eta\sigma\omega+(\gamma+\eta+\mu+d)(1-\sigma)\omega).(\gamma+\eta+\mu+d)}{[(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)]^2}\right].\frac{\mathbf{v}}{\boldsymbol{R_0}}, \quad (72)$$

Discussion

We developed a deterministic mathematical model of COVID-19 infection dynamics incorporating vital/demographic dynamics, face-mask use, asymptomatically and symptomatically infectious individuals, clinical diagnosis of asymptomatic individuals and isolation of infected individuals.

The analytical result shows that the model is bounded, positive, and attracting. Fundamental to our result is the basic reproduction number R_0, given by

$$R_0 = \frac{\beta(1-\theta q)\Pi}{\mu \mathrm{N}(\omega+\mu)}\left[\frac{\sigma\omega}{(\gamma+\eta+\mu+d)} + \frac{\eta\sigma\omega}{(\gamma+\eta+\mu+d)(\gamma+v+\mu+d)} + \frac{(1-\sigma)\omega}{(\gamma+v+\mu+d)}\right]$$

as a tool for effective disease management. The basic reproduction number, R_0 has three different constituents, namely, the multiplication of the infection transmission parameter of the susceptible population who are not face-masks compliant at the critical point $\frac{\beta(1-\theta q)\Pi}{\mu \mathrm{N}}$ and

i. the rate of latent population with no COVID symptoms moves into asymptomatic class $\sigma\omega$, average duration in the latent and asymptomatic compartment $\frac{1}{(\omega+\mu)(\gamma+\eta+\mu+d)}$.
ii. the proportion of the latent population with no COVID symptoms but who have been diagnosed transfers into the asymptomatic category. $\eta\sigma\omega$, average duration in latent compartment, asymptomatic compartment and symptomatic compartment $\frac{1}{(\omega+\mu)(\gamma+\eta+\mu+d)(\gamma+\nu+\mu+d)}$.

the proportion of the latent population with OVID symptoms moves to the symptomatic category. $(1-\sigma)\omega$, the average duration in the latent compartment and the symptomatic compartment $\frac{1}{(\omega+\mu)(\gamma+\nu+\mu+d)}$.

The result from the stability analysis of the critical points is shown to be local asymptotic stable and also, globally asymptotically stable provided $R_0 < 1$, as shown in Theorem 3 and Theorem 6. The inference of Theorem 3 and Theorem 6 is that, the slight influx of COVID-19 cases will not create a COVID-19 epidemic provided $R_0 < 1$. Theorem 7 and Theorem 8 revealed that the endemic equilibrium (EE) state is local asymptotic stable and also, globally asymptotically stable provided $R_0 > 1$. We can deduce that, COVID-19 will persist in the population and can result in a pandemic.

The sensitivity index on R_0 for the population influx rate (Π), infection transmission rate (β), the proportion of face-masks compliance in public (θ), Efficiency of face-masks (q), clinical diagnosis of asymptomatic individuals (η) and isolation rate (ν) was computed. The results of the analysis show that the recruitment rate Π, and the infection transmission rate β are the most sensitive parameters. Thus, by decreasing recruitment, infection parameters will decrease R_0 and vice-versa. The sensitivity index for the vital/demographic indicates that the COVID-19 infection will remain in the population provided $R_0 > 1$. This may as a result of not restricting the influx of those prone to the disease into the population. This enables the infection to get likely targets to infect. This is in contrast with the models with no demography, where the epidemic diminishes over time, the value of the basic reproduction number notwithstanding.

CONCLUSION

We extended and analyzed a deterministic mathematical model of COVID-19 infection by assessing the impact of vital/demographic dynamics. We further assessed the impact of the infection transmission, face-mask use and compliance, clinical diagnosis of infectious asymptomatic individuals, and isolation of infected individuals on the disease burden. It was evident that the recruitment rate and infection transmission (contact) rate play a significant role in disease management and eradication. Thus, efforts geared at reducing the recruitment of susceptible individuals and infection transmission rate will significantly eliminate the disease burden. It was further established that the basic reproduction number, R_O can be employed by public health or control agencies in the effort to eradicate the disease.

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