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BEHIND WEERAKOON AND FERNANDO'S SCHEME: IS WEERAKOON AND FERNANDO SCHEME VERSION COMPUTATIONALLY BETTER THAN ITS POWER-MEANS VARIANTS?

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ABSTRACT

The Weerakoon and Fernando scheme for estimating the solution of nonlinear equations is a modification of the Newton iteration scheme (NIS) with better convergence order and efficiency. It was developed based on the composition of the NIS with a corrector iterative function that is based on the use of arithmetic mean. In this article, we put forward family of power-means variants of the Weerakoon and Fernando iterative scheme. The family is shown to have convergence order three. Numerical studies on the family enabled us to decide whether the classical Weerakoon and Fernando scheme version is computationally better than its power-means variants versions. From the numerical results, it is discovered that there are some highly efficient and competitive elements in the developed family of Weerakoon and Fernando scheme version.

Keywords: Iterative scheme, Weerakoon and Fernando scheme, Power means, Nonlinear equations

INTRODUCTION

The nonlinear equations of the form

$$f(\upsilon) = 0, \ \upsilon \in \mathbb{R} \quad (1)$$

are often experienced in the field of science and engineering. There is no general analytic formulation for solving the nonlinear equation in (1) in literature. Consequently, iterative schemes were utilized to obtain the solution of (1) up to some level of precision. An iterative scheme recursively computes the solution of nonlinear equation via iteration cycles using an initial approximation to the solution. It is expected that at the end of each complete iteration cycle, a better estimation of the true solution is obtained until convergence. A classical one point iterative scheme for solving (1) is the Newton Iterative scheme (Traub,1964) given as:

$$\upsilon_{j+1} = \upsilon_j - \varphi_j, \ j = 0,1,2,\ldots \quad (2)$$

where $\varphi_j = \frac{f(\upsilon_j)}{f'(\upsilon_j)}$. The Newton scheme iteratively decides the solution of (1) with convergence order (CO) two. To overcome the Newton scheme CO, two-points and multi-points iterative schemes such as in (Weerakoon & Fernando,2000; Jarrats,1966) and many more, have been developed in literature, see also the books (Petkovic *et al*.,2013; Amat & Busquier,2016) and the articles (Ogbereyivwe & Izevbizua,2023; Ogbereyivwe & Ojo-Orobosa,2021) for more detailed overview.

The Weerakoon and Fernando scheme developed in Weerakoon and Fernando (2000) is an excellent modification of the Newton scheme developed using two-step composition technique. Its corrector iterative function require the assessment of the arithmetic mean of the derivative of function of values at two iteration points. Although the Weerakoon and Fernando scheme has been existing for over two decades and highly cited in literature, much work has not been done in developing its variants family and investigation on performance measure among its variants. Consequently, this paper is aimed at putting forward a new family of Weerakoon and Fernando scheme variants that is based on the power-means generating function. In this case, we replace the arithmetic mean used in the Weerakoon and Fernando scheme corrector iteration function, with a generalized power-means type generating function to produce infinitely many variants of the classical Weerakoon and Fernando scheme versions. Furthermore, numerical studies on whether the classical Weerakoon and Fernando scheme version is computationally best among elements of its power-means family versions was also considered.

MATERIALS AND METHODS

The family of iterative schemes

Consider an iterative scheme put forward as:

$$\upsilon_{j+1} = \upsilon_j - \frac{f(\upsilon_j)}{M_G[f'(\upsilon_j), f'(y_j)]} \quad (3)$$

where $M_G[f'(\upsilon_j), f'(y_j)]$, is power means-type generating function involving the data obtained using $f'(\upsilon_j)$ and $f'(y_j)$ at iteration points υ_j and y_j respectively. The power-means generating function is defined as:

$$M_G[f'(\upsilon_j), f'(y_j)] = \left(\frac{\left(f'(\upsilon_j)\right)^m + \left(f'(y_j)\right)^m}{2}\right)^{\frac{1}{m}} \quad (4)$$

We note that $m \in \mathbb{R}$ and $m \neq 0$ so as not to annihilate the contributions of $f'(\upsilon_j)$ and $f'(y_j)$. Consequently, $\forall$ value m, a power-mean type is eminently obtained. For instance, Table 1 shows some power-means types obtained when m is assigned values.

Table 1: $m -$ values and corresponding power-mean

m	Power-means type
1	Arithmetic mean
-1	Harmonic mean
2	Root mean square
-2	Inverse-root, inverse-square mean
$1/2$	Square mean-root
3	Cube-root mean cube

Clearly, for $m = 1$ in the iterative scheme (3), the famous Weerakoon and Fernando scheme version is obtained and for other values of m other than 0, power means variants of Weerakoon and Fernando schemes are obtained. The question is, will these power-means variants of the Weerakoon and Fernando scheme, computationally performed better than the classical Weerakoon and Fernando scheme version or is the classical Weerakoon and Fernando scheme version the best of its power means variants? To answer these questions, we first determine the convergence of the family of schemes in (3).

The scheme convergence analysis

This section establishes the convergence of sequence of iteration results generated by any element of the scheme (3) that is generated by any value of m. This is achieved via the proof of the next theorem.

Theorem 2.1: *Let the function $f: D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be defined and sufficiently differentiable on the domain D such that $f'(\cdot) \neq 0$. Suppose that v_0 is sufficiently close to v_* the solution of (1) and if v_0 is used as an iteration starting value in (3), then a sequence of approximations $\{v_j\}$ will be generated $\ni \lim_{j \longrightarrow n}\{v_j\} = v_*$.*

Proof: The third order Taylor's expansion of $f(v)$ about v_i is:

$$f(v) = f(v_j) + \sum_{k=1}^{3} \frac{1}{k!} f^{(k)}(v_j)(v - v_j)^k + O\left(|v - v_j|^4\right) \tag{5}$$

where $f^{(k)}(\cdot)$ is kth derivative of $f(\cdot)$.

Suppose $e_j = |v - v_j|$ is the jth iteration error of the scheme and set $v = v_*$ in (5), then

$$f(v_j) = \sum_{k=1}^{3}\left[(-1)^{k+1} \frac{1}{k!} f^{(k)}(v_j) e_j^k\right] + O\left(|e_j|^4\right). \tag{6}$$

When the expression in (6) is multiplied by $\frac{1}{f'(v_j)}$, the next equation is obtained.

$$\frac{f(v_j)}{f'(v_j)} = e_j + \sum_{k=1}^{3}\left[(-1)^{k+1} \frac{1}{k!} \frac{f^{(k)}(v_j)}{f(v_j)} e_j^k\right] + O\left(|e_j|^4\right). \tag{7}$$

From the first step of (3), we have

$$y_j - v_j = -e_j + \sum_{k=1}^{3}\left[(-1)^{k+1} \frac{1}{k!} \frac{f^{(k)}(v_j)}{f(v_j)} e_j^k\right] + O\left(|e_j|^4\right). \tag{8}$$

Using (8), the next equations are obtained.

$$(y_j - v_j)^2 = e_j^2 + \frac{f''(v_j)}{f'(v_j)} e_j^3 + O\left(|e_j|^4\right) \tag{9}$$

and

$$(y_j - v_j)^3 = -e_j^3 + O\left(|e_j|^4\right). \tag{10}$$

Now, the Taylor's expansion of $f'(y_j)$ about v_j is:

$$f'(y_j) = \sum_{k=1}^{3}\left[\frac{1}{k!} f^{(k)}(v_j)(y_j - v_j)^{k-1}\right] + O\left(|y_j - v_j|^4\right). \tag{11}$$

Now, (3) can be re-written as:

$$M_G[f'(v_j), f'(y_j)]\, e_{j+1} = M_G[f'(v_j), f'(y_j)] e_j - f(v_j). \tag{12}$$

Consequent upon (12), we need to show that the right hand side of (12) is of error order three. Using equations (4), (6) and (11) in (12) and after simplifications, the next equation is obtained.

$$M_G[f'(v_j), f'(y_j)]\, e_{j+1} = \left(\frac{f^{(3)}(v_j)}{6} + \frac{f'(v_j)\left(2f'(v_j)f^{(3)}(v_j) + \left(f^{('')}(v_j)\right)^2\right)}{8\left(f^{(')}(v_j)\right)^2}(1+m)\right) e_j^3 + O\left(|e_j|^4\right). \tag{13}$$

The expression in (13) is the error equation of the family of iterative scheme (3) and has convergence order three. This concludes the proof.

Remark 2.1

Note that, for all m value used in (3) will produce a variant version of the Weerakoon and Fernando scheme version that is of order three and the factor $(1+m)$ in (13) is responsible for error difference in their respective nonlinear solution approximations.

RESULTS AND DISCUSSION

This subsection offers numerical experiments conducted on the family of schemes in (3) with the aim of establishing whether the Weerakoon and Fernando scheme version which is a typical member of the scheme (3) is computationally best of its variants. In all conducted experiments, 200 digits precision and $|f(v_{j+1})| \leq 10^{-100}$ were used as stopping criterion in the designed computational program written in Maple 2017 version environment. The numerical outputs of some typical elements of the family of iterative schemes in (3), obtained by varying the value of m were compared on the basis of number of iterations (*N*), residual error function ($|f(v_{j+1})|$), computational order of convergence (ρ_{coc}) given in Jay (2001) as $\rho_{coc} = \frac{\log|f(v_{i+1})|}{\log|f(v_i)|}$, efficiency index (*E.I*) and approximate computational cost (computational complexity) *ACC*. In this case, the ACC of an iterative scheme per an iteration cycle is approximated as:

$ACC = nFE + nFO$

where, nFE and nFO represents number of functions evaluation and functions arithmetic operations respectively. According to Traub in Traub (1964), the metrics for $E.I$ of an iterative scheme is obtained by $E.I = \rho^{\frac{1}{T}}$, where T is total number distinct function assessment in the scheme. For sufficiency, we modify the Traub's $E.I$ as $EI = \rho^{\frac{1}{nFE+nFO}}$. Here, we assume that the computational complexity for FE and FO are same. Consequently, for one complete iteration cycle, the computational complexity of the various schemes obtained from (3) are given in Table 1.

Table 1: Schemes computational complexity per cycle

m	ρ	ACC	EI
1	3	9	$3^{\frac{1}{9}}$
$m \neq 1$	3	12	$3^{\frac{1}{12}}$

The functions taken from Ogbereyivwe and Izevbizua (2023) and used for computational test are presented next.
$f_1(\upsilon) = (\upsilon - 1)^3 - 2 = 0, \upsilon_* = 2.2599 \ldots$
$f_2(\upsilon) = \sin(\upsilon) - \upsilon^2 + 1 = 0,\ \ \upsilon_* = 1.40449 \ldots$
$f_3(\upsilon) = \upsilon^2 - \exp(x) - 3\upsilon + 2 = 0, \quad \upsilon_* = 0.2575 \ldots$
$f_4(\upsilon) = tan^{-1}\upsilon = 0,\ \ \upsilon_* = 0,$

Table 2 provides the computational outcomes when some concretes members of the family of power-means type Weerakoon and Fernando schemes put forward in (3) where computationally experimented on some nonlinear equations. The members were designed by assigning $m = 1, -1, 2, -2, 3$ and -3 at different instances. Note that the case $m = 1$, produces the Weerakoon and Fernando scheme version and for other values, its variants were obtained. Observe that the scheme obtained with $m = 1$ (i.e Weerakoon and Fernando scheme version), solved all the tested problems with the lowest iteration number N to converge to solution of nonlinear equations. It also produced the lowest corresponding total computation complexity in the $TACC$ column. In addition, it had the highest computational order of convergence ρ_{coc}. Furthermore, the corresponding ρ_{coc} for all the schemes are in the neigbourhood of 3. This confirmed the theoretical CO obtained in Subsection 2.1. Also, the Weerakoon and Fernando scheme version gives the highest $E.I$ value. Finally, although the schemes obtained with negative values of m (in particular $m = -1$) converge faster than when $m = 1$ (i.e Weerakoon and Fernando scheme version) and with lower error , its $E.I$ is always lower than the later scheme

Table 2: Schemes performance comparison

$f_i(\upsilon)$	υ_0	m	N	$TACC$	$\lvert f(\upsilon_{i+1})\rvert$	ρ_{coc}	$E.I$
$f_1(\upsilon)$	1.5	1	7	63	$2.7e-096$	3.0478	1.1318
		-1	6	72	$1.9e-122$	3.0409	1.0971
		2	8	96	$4.4e-135$	3.0303	1.0968
		-2	6	72	$3.7e-133$	3.0466	1.0973
		3	8	96	$6.6e-090$	3.0418	1.0971
		-3	6	72	$1.9e-122$	3.0409	1.0971
$f_2(\upsilon)$	1	1	5	45	$1.9e-198$	3.0320	1.1308
		-1	4	48	$1.9e-198$	3.0226	1.0966
		2	9	108	$3.0e-070$	3.0657	1.0979
		-2	7	84	$1.6e-133$	3.0385	1.0970
		3	9	108	$5.8e-221$	3.0191	1.0965
		-3	8	96	$1.9e-198$	3.0226	1.0966
$f_3(\upsilon)$	0	1	5	45	$7.8e-106$	3.0756	1.1330
		-1	5	60	$4.3e-112$	3.0745	1.0981
		2		Failed to converge			
		-2		Failed to converge			
		3	11	132	$1.2e-186$	3.0333	1.0084
		-3	10	120	$2.7e-100$	3.0772	1.0094
$f_4(\upsilon)$	0	1	4	36	$2.5e-073$	3.0303	1.0313
		-1	5	60	$5.8e-196$	3.0056	1.0185
		2	4	48	$3.3e-070$	3.0110	1.0232
		-2	5	60	$4.1e-182$	3.0015	1.0185
		3	4	48	$3.9e-070$	3.0022	1.0232
		-3	5	60	$3.3e-172$	2.9990	1.0185

Note: $A.Be - C$ represents $A.B \times 10^{-C},\ A, B, C \in \mathbb{R}$.

CONCLUSION

This paper offers a family of power-means type based Weerakoon and Fernando iterative scheme for solving nonlinear equation. The Taylor's series technique was utilized to prove that the convergence order of the family of schemes is three. After numerical studies on the family, we conclude that classical Weerakoon and Fernando scheme version performed better among all the members of its power-means based family versions. In future research, further investigation can be done on the chaotic behavior and stability of members of the family. Also, extension of the family to tackle the solution of multidimensional nonlinear equation can be considered.

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