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NEW GENERALIZED ODD FRÉCHET-ODD EXPONENTIAL-G FAMILY OF DISTRIBUTION WITH STATISTICAL PROPERTIES AND APPLICATIONS

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ABSTRACT

A new lifetime continuous probability distribution called the new Generalized Odd Fréchet-Odd-Exponential-G Family of Distribution is developed using the principle of Alzaatreh. The developed distribution is flexible for studying positive real-life datasets. The statistical properties related to this family are obtained. The parameters of the family were estimated by using a technique of maximum likelihood. A NewGeneralized Odd Fréchet-Odd-Exponential-Weibull model is introduced. This distribution was fitted with a set of lifetime data. A Monte Carlo simulation is applied to test the consistency of the estimated parameters of this distribution in terms of their bias and mean squared error with a comparison of M.L.E and the maximum product spacing (MPS).The outcome of the Monte Carlo simulation shows that the M.L.E method is the best technique for estimating the parameter of the New Generalized Odd Frechet-Odd-Exponential-Weibull distribution and the New Generalized Odd Frechet-Odd-Exponential-Rayleigh distribution than the M.PS method. The outcomes of the application on the data set produce a higher flexibility than some of the competing distributions. The distributions serve as a viable alternative to other distributions available in the literature for modelling positive data.

Keywords: New Generalized Odd Fréchet-G Family, Moments, Hazard functions, Maximum Likelihood, Monte Carlo Simulations

INTRODUCTION

The novelty of developing a generalized form of probability distribution drew the attention of academicians and devoted statisticians to the flexibility possessions of the generalized distributions. The Fréchet distribution, also viewed as the EV distribution of type II, was introduced by the Western mathematician Maurice René (MR) Fréchet in the 1920s as a maximum value distribution. Sadiq *et al.* (2023), provide a detailed explanation of the GEV distribution and its extensive implementations in various disciplines such as sea currents, natural disasters, horse racing, heavy rainfall, supermarket queues, and wind speeds, among others.Alizadeh *et al.* (2017a),statistical models play a crucial role in describing and forecasting countless real-world events. To model data in different domains, several extended and comprehensive distributions have remained broadly employed over the last few decades. Recent advances in statistical literature have focused on describing innovative families of distributions that can outspread renowned distributions and, at the same time, deliver prodigious flexibility in demonstrating observational facts in practice. Therefore, different categories have been proposed for breeding novel distributions by accumulating one or more parameters. Some acknowledged families of distribution were the NGOF-G by Sadiq *et al.* (2023), a modified T-X family by Aslam *et al.* (2020), the Odd-Burr generalized family by Alizadeh *et al.* (2017b), on generating T-X family by Aljarrah *et al.* (2014), Logistic-X family by Tahir *et al.* (2016), TGOGEG by Reyad.*et al.* (2019), General Linear Model by Sadiq *et al.* (2020), the NOBPBX distribution by Suleiman *et al.* (2023) and Odd Gompertz-G family of distribution by Kajuru *et al.* (2023).

MATERIALS AND METHODS

Sadiq *et al.*(2023) defined a random variable Xas said to have a New Generalized Odd Fréchet-G (NGOF-G) family of distribution with scale parameter α and shapes parameter β and γ if its CDF (cumulative distribution function)ispresented as (for all $x,\alpha,\beta,\gamma,\xi>0$),

$$F_{NGOF-G}(x;\alpha,\beta,\gamma,\xi) = exp\left\{-\left(\alpha\left(F_{cdf}^{-\gamma}(x;\xi)-1\right)\right)^{\beta}\right\} \quad (1)$$

Similarly, according to Bourguignon *et al*. (2014); defined a CDF of the odd exponential-G family of distributions are given by:

$$F_{OEG}(x;\delta,\xi) = 1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}; \quad \forall x;\delta,\xi>0 \quad (2)$$

New Generalized Odd Fréchet-Odd Exponential-G (NGOF-OE-G) family

We "propose a new extension of the NGOF-G family developed by Sadiq *et al.*(2023) using the Odd Exponential-G family introduced by Bourguignon *et al*. (2014) as a baseline generator. The hybridization of these two families is expected to produce more flexible and robust distributions that can accommodate a wide range of datasets. However, using the direct substitution method, putting equation (2) into equation (1), our proposed family namedthe New Generalized Odd Fréchet-Odd Exponential-G (NGOF-OE-G) family is set as (for all "$x,\alpha,\beta,\gamma,\delta,\xi>0$")"

$$F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi) = exp\left\{-\left(\alpha\left(F_{OEG}^{-\gamma}(x;\delta)-1\right)\right)^{\beta}\right\}$$

$$= exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\} \quad (3)$$

The corresponding pdf of equation (3) is given by

$$f(x;\alpha,\beta,\gamma,\delta,\xi)=\beta\gamma\alpha^{\beta}\delta r(x;\xi)\big(1-R(x;\xi)\big)^{-2}\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-(\gamma+1)}$$
$$exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)^{\beta-1}$$
$$exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\} \tag{4}$$

where "$r(x;\xi)$ and $R(x;\xi)$ are the pdf and CDF of the baseline distribution and $\xi$is the parameter vector. However, a random variable X with density function and distribution function in equations (3) and (4) is denoted by $X\sim NGOF-OE-G\ (\alpha,\beta,\gamma,\delta,\xi)$".

Special NGOF-OE-G distributions

Lifetime distributions are crucial in various fields including "survival analysis, biomedical science, engineering, and social sciences". Generally, "lifetime refers to the length of human life, the lifespan of a device before its failure, or the survival time of a patient with a severe illness from diagnosis to death. In this article, we introduce two special NGOF-OE-G distributions that may come in handy" for applications

The NGOF-OE-Weibull distribution

The NGOF-OE-Weibull distribution "is defined from equations (3) and (4) by taking $R(x\ ;\ \xi)=1-exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}$ and $r(x\ ;\ \xi)=\omega\phi^{-\omega}x^{\omega-1}\,exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}$ to be the Weibull distribution with positive parametersϕ, ω and $\xi=(\omega,\phi)$. The CDF and pdf of the NGOF-OE-Weibull distribution are given by (for $x>0$)"

$$F_{NGOFOEW}(x;\alpha,\beta,\gamma,\delta,\phi,\omega)=exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\delta\left(exp\left\{\left(\frac{x}{\phi}\right)^{\omega}\right\}-1\right)\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\} \tag{5}$$

$$f_{NGOFOEW}(x;\alpha,\beta,\gamma,\delta,\phi,\omega)=\beta\gamma\alpha^{\beta}\delta\left(\omega\phi^{-\omega}x^{\omega-1}\,exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)\left(exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^{-2}$$
$$\left(1-exp\left\{-\delta\left(exp\left\{\left(\frac{x}{\phi}\right)^{\omega}\right\}-1\right)\right\}\right)^{-(\gamma+1)}exp\left\{-\delta\left(exp\left\{\left(\frac{x}{\phi}\right)^{\omega}\right\}-1\right)\right\}$$
$$\left(\left(1-exp\left\{-\delta\left(exp\left\{\left(\frac{x}{\phi}\right)^{\omega}\right\}-1\right)\right\}\right)^{-\gamma}-1\right)^{\beta-1} \tag{6}$$
$$exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\delta\left(exp\left\{\left(\frac{x}{\phi}\right)^{\omega}\right\}-1\right)\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\}$$

The NGOF-OE-Rayleigh distribution

The NGOF-OE-Rayleigh distribution "is defined from equations (3) and (4) by taking $R(x\ ;\ \xi)=1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}$and $r(x\ ;\ \xi\ )=\phi x\,exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}$ to be the Rayleigh distribution with positive parameters ϕ and $\xi=\phi$. The CDF and pdf of the NGOF-OE-Rayleigh distribution are given by (for $x>0$)"

$$F_{NGOFOER}(x;\alpha,\beta,\delta,\phi)=exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta\left(1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)}{exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\} \tag{7}$$

$$f_{NGOFOER}(x;\alpha,\beta,\gamma,\delta,\phi)=\beta\gamma\alpha^{\beta}\delta\left(\phi x\,exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)\left(exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)^{-2}$$
$$\left(1-exp\left\{-\frac{\delta\left(1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)}{exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}}\right\}\right)^{-(\gamma+1)}exp\left\{-\frac{\delta\left(1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)}{exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}}\right\}$$
$$\left(\left(1-exp\left\{-\frac{\delta\left(1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)}{exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}}\right\}\right)^{-\gamma}-1\right)^{\beta-1}$$
$$exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta\left(1-exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)}{exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\} \tag{8}$$

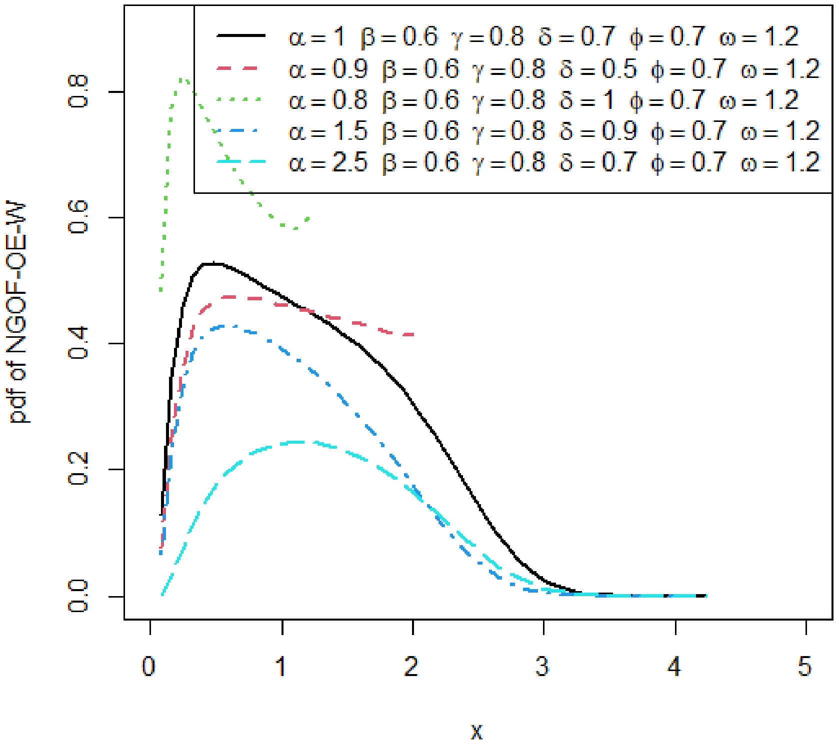
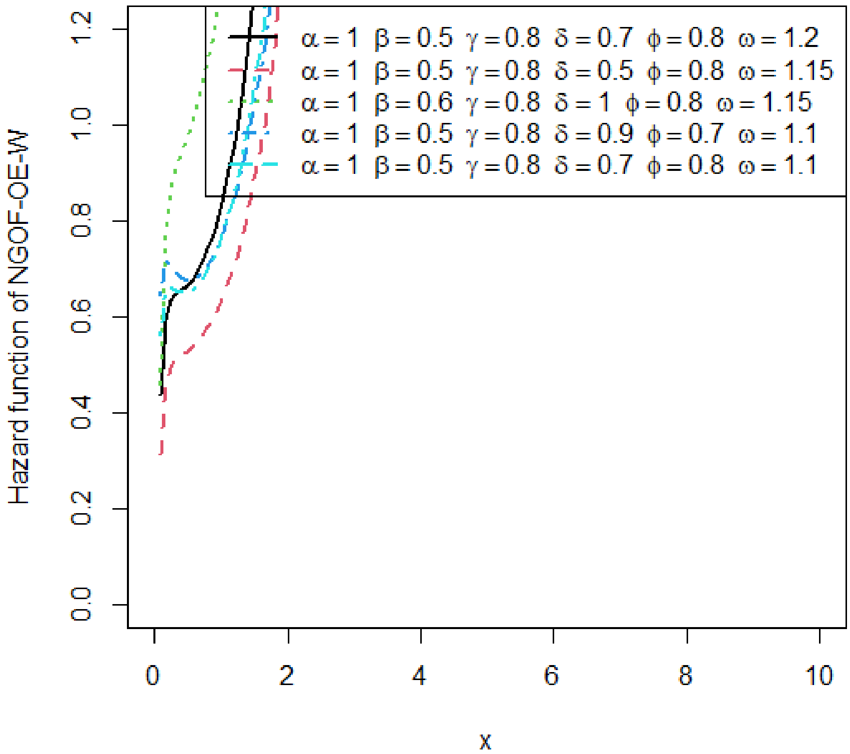


Figure 1: pdf Plot of the New Generalized Odd Fréchet-Odd Exponential-Weibull Distribution



Figures 2: HF Plot of the New Generalized Odd Fréchet-Odd Exponential-Weibull Distribution

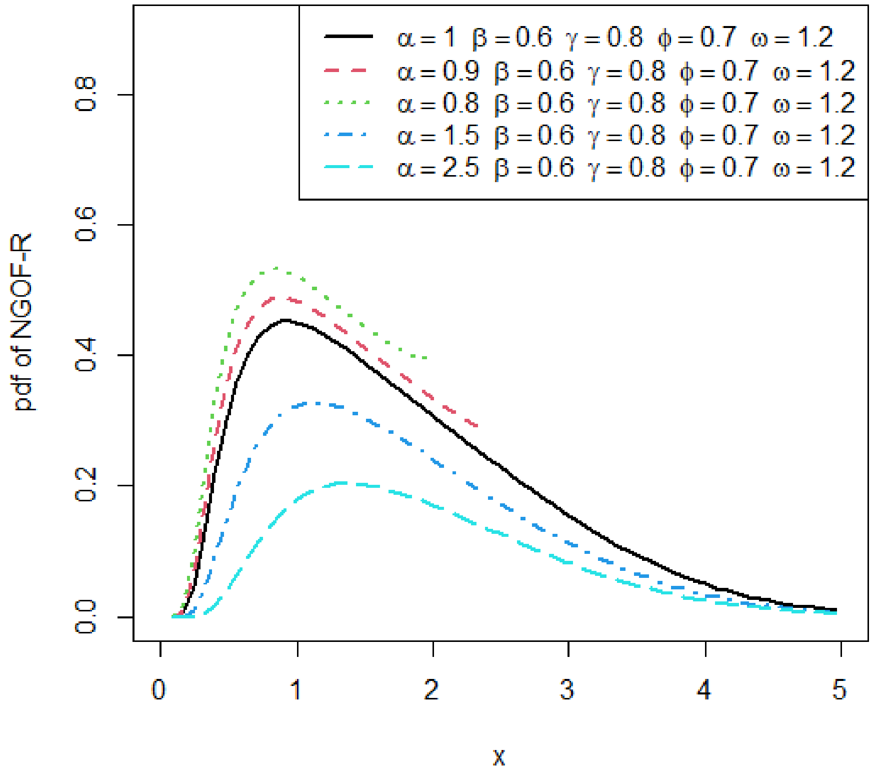


Figure 3: PDF Plot of New Generalized Odd Frechet-Odd-Exponential-Rayleigh Distribution

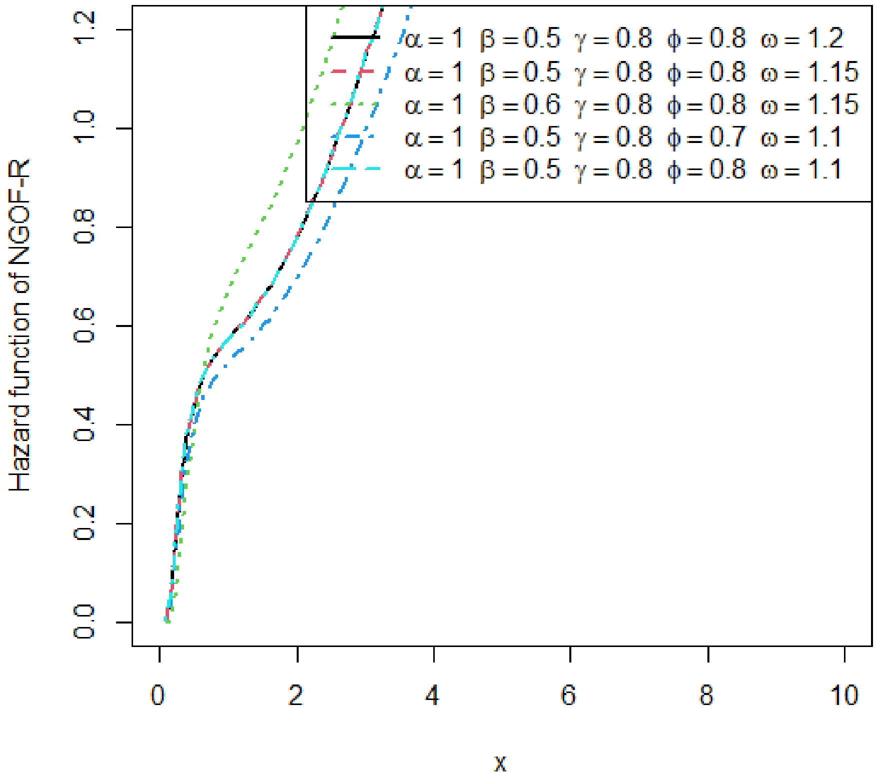


Figure 4: Hazard Plot of New Generalized Odd Frechet-Odd-Exponential-Rayleigh Distribution

So, Figures 1 and 3 show the density function of the NGOF-OE-Weibulland NGOF-OE-Rayleighmodels at different parameter values. The figures display the shapes and behaviour of the distribution, and how the parameters interact with one another. For example, if the parameters have equal values, the distribution is symmetrical. However, if the values differ, the distribution becomes more positively skewed. Additionally, the greater the difference between the parameter values, the less pronounced the bell shape of the distribution. Furthermore, Figures 2 and 4 show the hazard functions of the NGOF-OE-Weibulland NGOF-OE-Rayleigh at various parameter values. The graph displays the modified unimodal and modal shapes of hazard rates at different parameter values.

Expansions

Let's take a closer look at the terms in the NGOF-OE-G family's CDF presented in equations (3). We can use standard mathematical expansions such as the generalized binomial expansion for negative and positive powers, the power series expansion, and more to break down each term.

$$F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)=exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\}$$

$$=exp\left\{-\left(\alpha\left(R_{OEG}^{-\gamma}(x;\xi)-1\right)\right)^{\beta}\right\},$$

where $R_{OEG}(x;\xi)=\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)$

$$=e^{-\left(\alpha\left(\frac{1-R_{OEG}^{\gamma}(x;\xi)}{R_{OEG}^{\gamma}(x;\xi)}\right)\right)^{\beta}}$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\alpha\left(\frac{1-R_{OEG}^{\gamma}(x;\xi)}{R_{OEG}^{\gamma}(x;\xi)}\right)\right)^{i\beta}$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^{i\beta}\left(1-R_{OEG}^{\gamma}(x;\xi)\right)^{i\beta}R_{OEG}^{-\gamma i\beta}(x;\xi)$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^{i\beta}\sum_{j=0}^{\infty}(-1)^j\binom{i\beta}{j}\left(R_{OEG}^{\gamma i\beta}(x;\xi)\right)^{j}R_{OEG}^{-\gamma i\beta}(x;\xi)$$

$$=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!}\alpha^{i\beta}\binom{i\beta}{j}R_{OEG}^{\gamma i\beta(j-1)}(x;\xi) \quad (9)$$

Therefore, equation (9) reduces to,

$$F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)=\sum_{i,j=0}^{\infty}S_{i,j}R_{OEG}^{\gamma i\beta(j-1)}(x;\xi) \quad (10)$$

where $S_{i,j}=\frac{(-1)^{i+j}}{i!}\alpha^{i\beta}\binom{i\beta}{j}$

Differentiating equation (10) w.r.t. x we have the corresponding pdf as:

$$f_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)=\sum_{i,j=0}^{\infty}S_{i,j}\,\gamma i\beta(j-1)r_{OEG}(x;\xi)R_{OEG}^{\gamma i\beta(j-1)-1}(x;\xi) \quad (11)$$

Further simplification of equation (10) is as,

$$F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)=\sum_{k=0}^{\infty}v_kW_k(x) \quad (12)$$

where $v_k=\sum_{i,j=0}^{\infty}S_{i,j}$ and $W_k(x)=R_{OEG}^{\gamma i\beta(j-1)}(x;\xi)$

Differentiate equation (12) w.r.t. x we obtained the corresponding pdf as:

$$f_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)=\sum_{k=0}^{\infty}v_kw_k(x) \quad (13)$$

where $w_k(x)=kr_{OEG}(x;\xi)R_{OEG}^{k-1}(x;\xi)$

Moments

The role of moments in application to statistics is clear, and "the most essential characteristics of a probability model can be examined using moments. Evaluation in statistical inference is necessary, the most vital properties of the distributions were derived using the moments". The rth ordinary moment of a random variable X that follows the New Generalized Odd Frechet-Odd Exponential-G (NGOF-OE-G) family by using equation (13) we have

$$\mu_r^{'}=E(X^r)$$

$$=\int_0^{\infty}x^r f_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)dx$$

$$=\int_0^{\infty}x^r\sum_{k=0}^{\infty}v_kw_k(x)\,dx$$

$$=\sum_{k=0}^{\infty}v_k\int_0^{\infty}x^rw_k(x)\,dx$$

$$=\sum_{k=0}^{\infty}v_kE[Z_k^r] \quad (14)$$

where $E[Z_k^r]=\int_0^{\infty}x^rkr_{OEG}(x;\xi)R_{OEG}^{k-1}(x;\xi)\,dx$

Moment-Generating Function

Moment-generating functions "offer a clear and elegant framework for understanding and analyzing probability distributions and random variables, making them an essential tool in many branches of statistics and applied mathematics". The moment-generating function of a random variable X that follows the New Generalized Odd Frechet-Odd Exponentiated-G (NGOF-OE-G) family by using equation (13) we have,

$$M_X^{NFOFOEG}(t)=E(e^{tx})$$

$$=\int_0^{\infty}x^r f_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)dx$$

$$=\int_0^{\infty}x^r\sum_{k=0}^{\infty}v_kw_k(x)\,dx$$

$$=\sum_{k=0}^{\infty}v_k\int_0^{\infty}x^rw_k(x)\,dx$$

$$=\sum_{k=0}^{\infty}v_kE[Z_k^r] \quad (15)$$

where $E[e^{tZ_k}]=\int_0^{\infty}e^{tx}kr_{OEG}(x;\xi)R_{OEG}^{k-1}(x;\xi)\,dx$

Entropies

Entropy, "a term borrowed from thermodynamics, is a measure of how uncertainty or randomness of a physical system. Entropy is a measure of the randomness or uncertainty of a random variable or a probability distribution that is used in statistics and information theory. It provides a means of measuring how informational the outcomes of a random process are. It is used in a variety of disciplines, including cryptography, machine learning, data analysis, and information theory".

The entropy of any "random variable X is a measure of indecisiveness, variability, and details properties of the probable results of the variable". The entropy of the NGOF-OE-G family using equation (13) we have,

$$I_R(\varpi) = \frac{1}{1-\varpi} log\left(\int_0^{\infty} f^{\varpi}{}_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)\ dx\right)$$

$$= \frac{1}{1-\varpi} log\left(\int_0^{\infty} (\textstyle\sum_{k=0}^{\infty}\ v_k w_k(x))^{\varpi}\ \ dx\right) \quad (16)$$

where $\varpi > 0$and $\varpi \neq 1$

The nth entropy is defined by

$$I_{nth}(\varpi) = \frac{1}{\varpi - 1} log\left(1 - \int_0^{\infty} f^{\varpi}{}_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)\ dx\right)$$

$$= \frac{1}{1-\varpi} log\left(\ 1 - \int_0^{\infty} (\textstyle\sum_{k=0}^{\infty}\ v_k w_k(x))^{\varpi}\ \ dx\right) \quad (17)$$

where $\varpi > 0$and $\varpi \neq 1$

Order Statistics

Briefly stated order statistics are "a set of values obtained by placing the observations from a sample in either ascending or descending order. They are important in many statistical analyses, assisting in describing the distribution, drawing conclusions, and examining the extreme values of a dataset. It offers information about the distribution and properties of the data".

Suppose $X_1,\ X_2,\ X_3,\ \ldots\ \ ,\ X_n$ is a random sample from the NGOF-OE-G distribution and $X_{i\ :\ n}$ represent the ith order statistic, then, using equations (12) and (13) we have

$$f_{i\ :\ n}(x;\alpha,\beta,\gamma,\delta,\xi) = \frac{n!}{[(i-1)!\,(n-i)!]}[f_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)]$$

$$[F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)]^{i-1}[1 - F_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)]^{n-i}$$

$$= \frac{n!}{[(i-1)!(n-i)!]}[\textstyle\sum_{k=0}^{\infty}\ v_k w_k(x)]\ [\sum_{k=0}^{\infty}\ v_k W_k(x)]^{i-1}[1 - \sum_{k=0}^{\infty}\ v_k W_k(x)] \quad (18)$$

Estimation of Parameters

For estimating the parameters of a statistical model, "maximum likelihood estimation (MLE) is a frequently used technique in statistics. MLE aims to identify the model parameter values that maximize the likelihood function, which assesses how well the model accounts for the observed data. Numerous statistical and scientific disciplines, such as econometrics, biostatistics, machine learning, and others, use MLE extensively".

Suppose that $x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_n$ are the observed values from the proposed NGOF-OE-G family with parameters $\alpha,\ \beta,\gamma,\delta$ and ξ. Suppose that $\Phi\ =\ [\alpha,\ \beta,\ \gamma,\delta,\xi]^T$ is the $[m\ \times\ 1]$ vector of the parameter. The log-likelihood function Φ using equation (4) is expressed by

$$\ell_n = \ell_n(\Phi) = n\,log(\beta) + n\,log(\gamma) + n\beta\,log(\alpha) + n\,log(\delta) + \sum_{i=1}^{n} log[r(x;\ \xi)]$$

$$-2\sum_{i=1}^{n} log[1 - R(x;\ \xi)] - (\gamma+1)\sum_{i=1}^{n} log\left[1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right]$$

$$-\delta\sum_{i=1}^{n}\left[\frac{R(x;\xi)}{1-R(x;\xi)}\right] + (\beta-1)\sum_{i=1}^{n} log\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)$$

$$- \sum_{i=1}^{n}\left(\alpha\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)\right)^{\beta} \quad (19)$$

Taking "the partial derivative of equation (19) w.r.t. the parameters $(\alpha;\ \beta;\ \gamma;\delta;\xi)$ are respectively given" as:

$$\frac{\partial\ell_n(\Phi)}{\partial\alpha} = \frac{n\beta}{\alpha} - \sum_{i=1}^{n}\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)^{\beta} \quad (20)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\beta} = \frac{n}{\beta} + n\,log(\alpha) + \sum_{i=1}^{n} log\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)$$

$$-\sum_{i=1}^{n}\left(\alpha\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)\right)^{\beta} ln\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right) \quad (21)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\gamma} = \frac{n}{\gamma} - (\delta+1)\sum_{i=1}^{n} log[R(x;\ \xi)] - \sum_{i=1}^{n} log\left[1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right]$$

$$-(\beta-1)\sum_{i=1}^{n}\frac{\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} ln\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)}{\left(\left(1 - exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma} - 1\right)}$$

$$-\sum_{i=1}^{n}\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta} ln\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right) \quad (22)$$

$$\frac{\partial \ell_n(\Phi)}{\partial \delta}=\frac{n}{\delta}-\sum_{i=1}^{n}\left[\frac{R(x;\xi)}{1-R(x;\xi)}\right]-(\beta-1)\sum_{i=1}^{n}\frac{\exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\frac{R(x;\xi)}{1-R(x;\xi)}}{\left(\left(1-\exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)}$$

$$-\sum_{i=1}^{n}\left(\alpha\left(\left(1-\exp\left\{-\frac{R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta} \quad (23)$$

$$\frac{\partial \ell_n(\Phi)}{\partial \xi}=\frac{r'(x;\ \xi)}{r(x;\ \xi)}+2\sum_{i=1}^{n}\frac{R'(x;\xi)}{[1-R(x;\ \xi)]}-(\gamma+1)\sum_{i=1}^{n}\frac{\frac{R'(x;\xi)}{(1-R'(x;\xi))}}{\left[1-exp\left\{-\frac{\delta G(x;\xi)}{1-G(x;\xi)}\right\}\right]}$$

$$-\delta\sum_{i=1}^{n}\left[\frac{R(x;\xi)}{1-R(x;\xi)}\right]\frac{R'(x;\xi)}{(1-R'(x;\xi))}+\ (\beta-1)\sum_{i=1}^{n} log\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\frac{R'(x;\xi)}{(1-R'(x;\xi))}$$

$$-\ \sum_{i=1}^{n}\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\frac{R'(x;\xi)}{(1-R'(x;\xi))} \quad (24)$$

The MLEs of the parameters $(\alpha;\ \beta;\ \gamma;\delta;\xi)$, says $(\hat{\alpha};\ \hat{\beta};\ \hat{\gamma};\hat{\delta};\hat{\xi})$are the simultaneous solution of equations (20), (21), (22), (23) and (24) when equated to zero, i.e.$\frac{\partial\ell_n(\Phi)}{\partial\alpha}=0;\ \frac{\partial\ell_n(\Phi)}{\partial\beta}=0;\ \frac{\partial\ell_n(\Phi)}{\partial\gamma}=0;\ \frac{\partial\ell_n(\Phi)}{\partial\delta}=0;\frac{\partial\ell_n(\Phi)}{\partial\xi}$. These equations are intractable and non-linear and can only be solved using a numerical iterative method.

Hazard Function

The hazard function, and cumulative hazard function random variable is X which follows the NGOF-OE-G family are respectively given as,
are respectively given as,

$$h_{NGOFOEG}(x;\alpha,\beta,\gamma,\delta,\xi)\ =\beta\gamma\alpha^{\beta}\delta r(x;\xi)(1-R(x;\xi))^{-2}\ \left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-(\gamma+1)}$$
$$exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)^{\beta-1}$$
$$exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\}$$
$$\left(1-exp\left\{-\left(\alpha\left(\left(1-exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right)^{-\gamma}-1\right)\right)^{\beta}\right\}\right)^{-1} \quad (25)$$

Quantile Function

The "quantile function of the NGOF-OE-G family is obtained by inverting the CDF in equation (3). Supposed the variable U is uniformly distributed on (0,1)", then

$$x=\Phi(u)=R^{-1}\left(\frac{log\left(1-\left(\frac{\alpha}{\alpha+(-log(u))^{\frac{1}{\beta}}}\right)^{\frac{1}{\gamma}}\right)}{log\left(1-\left(\frac{\alpha}{\alpha+(-log(u))^{\frac{1}{\beta}}}\right)^{\frac{1}{\gamma}}\right)-\delta}\right) \quad (26)$$

where R^{-1} is the quantile function of the baseline distribution $R(x;\xi)$. And $0<u<1$.

RESULTS AND DISCUSSION

Simulation Study

The vast class of computational algorithms known as "Monte Carlo simulations" uses replicated random sampling to produce numerical results. The basic idea is to employ randomness to address problems that could be theoretically deterministic.

M.L.E and M.P.S Techniques for the NGOF-OE-Weibull Distribution

To evaluate the consistency of the new family's parameters, "the simulation study was pilotedusing the Monte Carlo Simulation technique by computing the bias, variance and mean square error of the estimated parameters from the maximum likelihood estimates and the maximum product spacing estimate". The Simulated data is generated using the quantile function in equation (26) and the likelihood function in equation (19) for different sample sizes n = 20, and 50with replicate 200 times each. For the NOGF-OE-Weibull distribution parameter values are $(\alpha,\beta,\gamma,\delta,\phi,\omega) = (11, 1.0, 2.5, 1.0, 2.0, 3.0)$.

Table 1: Results of the simulated data from the NGOF-OE-Weibull Distribution.

Sample Sizes	Parameters (Actual Values)	M.L.E. Techniques			M.P.S. Techniques		
		Estimates	Bias	RMSE	Estimates	Bias	RMSE
20	α (11)	10.5149	-0.4851	0.7028	11.1391	0.1391	0.3981
	β (1.0)	0.1809	-0.8191	0.8199	0.1681	-0.8319	0.8327
	γ (2.5)	2.7102	0.2102	0.5764	3.6658	1.1658	1.3041
	δ (1.0)	1.4722	0.4722	0.6784	1.0813	0.0813	0.5060
	ϕ (2.0)	2.7378	0.7378	0.8290	1.4186	-0.581	0.7164
	ω (3.0)	3.7965	0.7965	1.0644	3.4093	0.4093	0.6585
50	α (11)	10.6819	-0.3181	0.5800	11.1677	0.1677	0.3894
	β (1.0)	0.1771	-0.8229	0.8238	0.2010	-0.7990	0.8004
	γ (2.5)	2.5992	0.0992	0.4439	3.9635	1.4635	1.5934
	δ (1.0)	1.6763	0.6763	0.8367	1.2994	0.2994	0.6296
	ϕ (2.0)	2.6243	0.6243	0.7132	1.3182	-0.6818	0.7990
	ω (3.0)	3.5471	0.5471	0.8318	3.0721	0.0721	0.4148

Table 1 presents the results obtained from the Monte Carlo Simulation study. The results indicated that the bias and root mean square error decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost equal at different sample sizes and iterative levels for the M.L.E technique. This proves the consistency of the MLE parameter estimates. For the M.P.S technique, the actual value of the parameters and the estimated values are almost not equal at different sample sizes and iterative levels. This proves the least consistency of the M.P.S parameter estimates. The result also means that the M.L.E technique is the best technique for estimating the parameter of New Generalized Odd Frechet-Odd Exponential-Weibull distribution than the M.PS technique.

M.L.E and M.P.S Techniques for the NGOF-OE-Rayleigh Distribution

To evaluate the consistency of the new family's parameters, "the simulation study was piloted using the Monte Carlo Simulation technique by computing the bias, variance and mean square error of the estimated parameters from the maximum likelihood estimates and the maximum product spacing estimate". The Simulated data is generated using the quantile function in equation (26) and the likelihood function in equation (19) for different sample sizes n = 50, 100, 250, 500 and 1000 with replicate 200 times each. For the NOGF-OE-Rayleigh distribution parameter values are $(\alpha,\beta,\gamma,\delta,\phi) = (1.0, 1.0, 2.5, 1.0, 2.0)$.

Table 2: Results of the simulated data from the NGOF-OE- Rayleigh Distribution.

Sample Sizes	Parameters (Actual Values)	M.L.E. Techniques			M.P.S. Techniques		
		Estimates	Bias	RMSE	Estimates	Bias	RMSE
50	α (1.0)	0.9792	-0.0208	0.0308	0.9998	-0.0002	0.0232
	β (1.0)	1.0424	0.0424	0.1448	0.9769	-0.0231	0.1189
	γ (2.5)	2.5605	0.0605	0.1390	2.5468	0.0468	0.1383
	δ (1.0)	1.0051	0.0051	0.0531	1.0264	0.0264	0.0863
	ϕ (2.0)	1.9927	-0.0073	0.0658	1.9984	-0.0016	0.0869
100	α (1.0)	0.9906	-0.0094	0.0144	1.0004	0.0004	0.0113
	β (1.0)	1.0169	0.0169	0.0747	0.9839	-0.0161	0.0689
	γ (2.5)	2.5527	0.0527	0.1120	2.5475	0.0475	0.1128
	δ (1.0)	1.0039	0.0039	0.0296	1.0130	0.0130	0.0409
	ϕ (2.0)	1.9987	-0.0013	0.0420	2.0068	0.0068	0.0452
	Q						
250	α (1.0)	0.9964	-0.0036	0.0054	1.0000	0.0000	0.0040
	β (1.0)	1.0027	0.0027	0.0396	0.9854	-0.0146	0.0416
	γ (2.5)	2.5412	0.0412	0.0860	2.5414	0.0414	0.0815
	δ (1.0)	1.0041	0.0041	0.0171	1.0069	0.0069	0.0193
	ϕ (2.0)	2.0008	0.0008	0.0236	2.0076	0.0076	0.0253
500	α (1.0)	0.9984	-0.0016	0.0027	1.0001	0.0001	0.0021
	β (1.0)	1.0004	0.0004	0.0229	0.9897	-0.0103	0.0294
	γ (2.5)	2.5283	0.0283	0.0640	2.5293	0.0293	0.0586
	δ (1.0)	1.0039	0.0039	0.0128	1.0049	0.0049	0.0118
	ϕ (2.0)	2.0009	0.0009	0.0175	2.0050	0.0050	0.0165

	A						
	α (1.0)	0.9994	-0.0006	0.0012	1.0001	0.0001	0.0010
	β (1.0)	0.9983	-0.0017	0.0158	0.9924	-0.0076	0.0191
1000	γ (2.5)	2.5247	0.0247	0.0523	2.5221	0.0221	0.0480
	δ (1.0)	1.0035	0.0035	0.0104	1.0037	0.0037	0.0084
	ϕ (2.0)	2.0019	0.0019	0.0131	2.0042	0.0042	0.0116

Table 2 presents the results obtained from the Monte Carlo Simulation study. The results indicated that the bias and root mean square error decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost equal at different sample sizes and iterative levels for the M.L.E technique. This proves the consistency of the MLE parameter estimates. For the M.PS technique, the actual value of the parameters and the estimated values are almost not equal at different sample sizes and iterative levels. This proves the least consistency of the M.P.S parameter estimates. The result also means that the M.L.E technique is the best technique for estimating the parameter of New Generalized Odd Frechet-Odd-Exponential-Rayleigh distribution than the M.PS technique.

Applications

Here we used some existing real-life data sets to assess the flexibility of our developed family using Weibull distribution as a baseline.

Dataset 1

The data set was originally reported by Sadiq et al. (2023) which represents "the Maximum Annual Flood Discharges of North Saskatchewan in units of 1000 cubic feet per second, of the North Saskatchewan River at Edmonton, for 47 years". The data are:"19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560".

Table 3: Parameters Estimates and Goodness of Fit Measures for Dataset 1

Model	Parameter Estimates and Goodness of Fit							
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\omega}$	LL	AIC
NGOFOEW	10.980	0.1577	0.4999	0.0244	1.5605	0.2944	215.175	**442.129**
GOFW	1.3e15	1.4e-01	-	-	5.3e-02	1.3e-01	414.8936	835.78
OFW	2.6391	-	-	-	0.06794	0.64733	285.7757	583.551
WD	-	-	-	-	0.00074	1.7724	225.7065	459.413

Table 3 provides the parameter estimates and goodness of fit measures for the New Generalized Odd Frechet-Weibull distribution with other competing models using the maximum annual flood discharges dataset. Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) are the performance metrics. A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. The new generalized odd Frechet-Odd Exponential Weibull distribution is the best model that outperforms other competitors based on the data set.

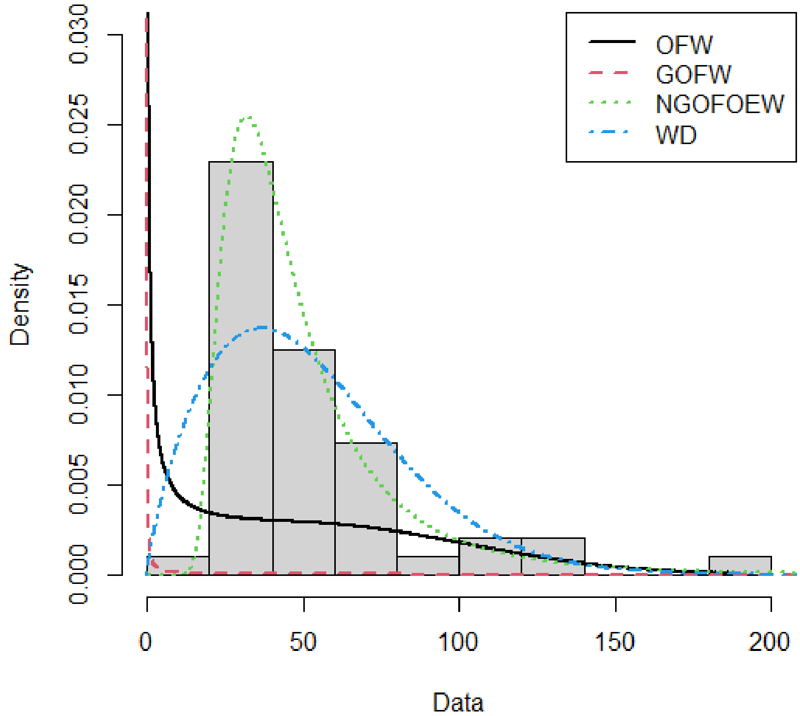


Figure 5: Histogram Plots of the Distribution of Maximum Annual Flood Discharges Data

Dataset 2
We used the data set that was analyzed by Fulment *et al.* (2023) which represents "the survival times (in years) of a group of patients given chemotherapy treatment". The data are: "0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033"

Table 4: Parameters Estimates and Goodness of Fit Measures for the Dataset 2

Model	Parameter Estimates and Goodness of Fit						
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\phi}$	AIC	Rank
NGOFOER	0.9583	0.8819	0.1619	0.0971	0.3415	123.252	1
GOFR	-	8.3235	0.0287	-	0.0033	28149.8	4
OFR	-	1.345	-	-	0.5213	197.412	3
RD	-	-	-	-	0.6025	163.83	2

Table 4 provides the parameter estimates and goodness of fit measures for the New Generalized Odd Frechet-Odd Exponential-Rayleigh distribution with other competing models using the survival times (in years) of a group of patients given a chemotherapy treatment dataset. Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) are the performance metrics. A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. The New Generalized Odd Frechet-Odd Exponentiatial-Rayleigh distribution is the best model that outperforms other competitors based on the data set.

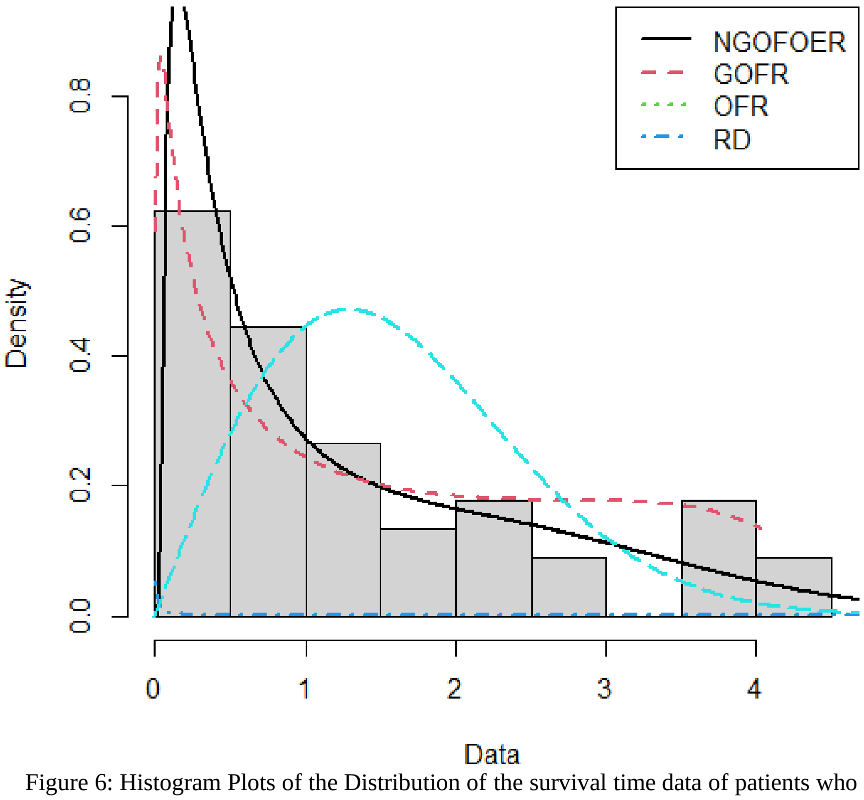


Figure 6: Histogram Plots of the Distribution of the survival time data of patients who received chemotherapy treatment

CONCLUSION

In this research paper, we introduce the NGOF-OE-G family of distributions and explore its statistical properties, including the survival function, hazard function, cumulative hazard function, moments, moment-generating function, entropies, order statistics, and MLE. We also plot the pdf and the hazard rate function to observe the shapes and behaviour of the models at different parameter values. To test the consistency of the MLE and MPS of the parameters, we conduct simulation studies. We then apply the NGOF-OE-W distribution to the survival time data of patients who received chemotherapy treatment and the NGOF-OE-R distribution to the data representing Maximum Annual Flood Discharges employing Rayleigh and Weibull as the baseline distribution, respectively. Our analysis shows that the NGOF-OE-W is the "best fit" model for the group of patients given chemotherapy treatment and the NGOF-OE-R is the "best fit" model for the maximum annual flood discharge data.

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