NEW GENERALIZED ODD FRÉCHET-ODD EXPONENTIAL-G FAMILY OF DISTRIBUTION WITH STATISTICAL PROPERTIES AND APPLICATIONS

Ibrahim Abubakar Sadiq, Doguwa S. I. S., Abubakar Yahaya and Jamilu Garba

Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria

ABSTRACT

A new lifetime continuous probability distribution called the new Generalized Odd Fréchet-Odd-Exponential-G Family of Distribution is developed using the principle of Alzaatreh. The developed distribution is flexible for studying positive real-life datasets. The statistical properties related to this family are obtained. The parameters of the family were estimated by using a technique of maximum likelihood. A NewGeneralized Odd Fréchet-Odd-Exponential-Weibull model is introduced. This distribution was fitted with a set of lifetime data. A Monte Carlo simulation is applied to test the consistency of the estimated parameters of this distribution in terms of their bias and mean squared error with a comparison of M.L.E and the maximum product spacing (MPS). The outcome of the Monte Carlo simulation shows that the M.L.E method is the best technique for estimating the parameter of the New Generalized Odd Fréchet-Odd-Exponential-Weibull distribution and the New Generalized Odd Fréchet-Odd-Exponential-Rayleigh distribution than the M.P.S method. The outcomes of the application on the dataset produce a higher flexibility than some of the competing distributions. The distributions serve as a viable alternative to other distributions available in the literature for modelling positive data.

Keywords: New Generalized Odd Fréchet-G Family, Moments, Hazard functions, Maximum Likelihood, Monte Carlo Simulations

INTRODUCTION

The novelty of developing a generalized form of probability distribution drew the attention of academicians and devoted statisticians to the flexibility possessions of the generalized distributions. The Fréchet distribution, also viewed as the EV distribution of type II, was introduced by the Western mathematician Maurice René (MR) Fréchet in the 1920s as a maximum value distribution. Sadiq et al. (2023), provide a detailed explanation of the GEV distribution and its extensive implementations in various disciplines such as sea currents, natural disasters, horse racing, heavy rainfall, supermarket queues, and wind speeds, among others. Alizadeh et al. (2017a), statistical models play a crucial role in describing and forecasting countless real-world events. To model data in different domains, several extended and comprehensive distributions have remained broadly employed over the last few decades. Recent advances in statistical literature have focused on describing innovative families of distributions that can outspread renowned distributions and, at the same time, deliver prodigious flexibility in demonstrating observational facts in practice. Therefore, different categories have been proposed for breeding novel distributions by accumulating one or more parameters. Some acknowledged families of distribution were the NGOF-G by Sadiq et al. (2023), a modified T-X family by Aslam et al. (2020), the Odd-Burr generalized family by Alizadeh et al. (2017b), on generating T-X family by Aljarrah et al. (2014), Logistic-X family by Tahir et al. (2016), TGOGEG by Reyad et al. (2019), General Linear Model by Sadiq et al. (2020), the NOBPBX distribution by Suleiman et al. (2023) and Odd Gompertz-G family of distribution by Kajuru et al. (2023).

MATERIALS AND METHODS

Sadiq et al. (2023) defined a random variable \( X \) as said to have a New Generalized Odd Fréchet-G (NGOF-G) family of distribution with scale parameter \( \alpha \) and shapes parameter \( \beta \) and \( \gamma \) if its CDF (cumulative distribution function) is presented as (for all \( x, \alpha, \beta, \gamma, \xi > 0 \)),

\[
F_{\text{NGOF-OG}}(x;\alpha,\beta,\gamma,\xi) = \exp\left\{-\left(\alpha F_{\text{cdf}}(x;\xi) - 1\right)^{\beta}\right\}
\]

(1)

Similarly, according to Bourguignon et al. (2014); defined a CDF of the odd exponential-G family of distributions are given by:

\[
F_{\text{OEG}}(x;\delta,\xi) = 1 - \exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}; \quad \forall x; \delta, \xi > 0
\]

(2)

New Generalized Odd Fréchet-odd Exponential-G (NGOF-OG) family

We propose a new extension of the NGOF-G family developed by Sadiq et al. (2023) using the Odd Exponential-G family introduced by Bourguignon et al. (2014) as a baseline generator. The hybridization of these two families is expected to produce more flexible and robust distributions that can accommodate a wide range of datasets. However, using the direct substitution method, putting equation (2) into equation (1), our proposed family named the New Generalized Odd Fréchet-odd Exponential-G (NGOF-OG) family is set as (for all \( x, \alpha, \beta, \gamma, \delta, \xi > 0 \))

\[
F_{\text{NGOF-OG}}(x;\alpha,\beta,\gamma,\delta,\xi) = \exp\left\{-\left(\alpha F_{\text{OEG}}(x;\delta) - 1\right)^{\beta}\right\}
\]

\[
= \exp\left\{-\left(\alpha \left[1 - \exp\left\{-\frac{\delta R(x;\xi)}{1-R(x;\xi)}\right\}\right]^{\gamma} - 1\right)^{\beta}\right\}
\]

(3)
The corresponding pdf of equation (3) is given by
\[
f(x; \alpha, \beta, \gamma, \delta, \xi) = \beta \gamma a^\beta \delta \Psi \left(1 - R(x; \xi)\right)^{-2} \left(1 - \exp \left[-\frac{\delta R(x; \xi)}{1 - R(x; \xi)}\right] \right)^{-(\gamma + 1)} \]
\[
\exp \left\{-\left(1 - \exp \left[-\frac{\delta R(x; \xi)}{1 - R(x; \xi)}\right] \right)^{\gamma} \right\}^{-1} \right) \right)^{\beta} \right(1 - \exp \left[-\left(\frac{x}{\phi}\right)^{\omega_1}\right] \right) \right) \right)^{\beta-1} \}
\]
where \(\gamma(x; \xi)\) and \(R(x; \xi)\) are the pdf and CDF of the baseline distribution and \(\xi\) is the parameter vector. However, a random variable \(X\) with density function and distribution function in equations (3) and (4) is denoted by \(X \sim N\text{GOF} - O\text{E} - G \left(\alpha, \beta, \gamma, \delta, \xi\right)\).

Special N\text{GOF}-O\text{E}-G distributions

Lifetime distributions are crucial in various fields including “survival analysis, biomedical science, engineering, and social sciences”. Generally, “lifetime refers to the length of human life, the lifespan of a device before its failure, or the survival time of a patient with a severe illness from diagnosis to death. In this article, we introduce two special N\text{GOF}-O\text{E}-G distributions that may come in handy” for applications

The N\text{GOF}-O\text{E}-Weibull distribution

The N\text{GOF}-O\text{E}-Weibull distribution “is defined from equations (3) and (4) by taking \(R(x; \xi) = 1 - \exp \left\{-\frac{x}{\phi}\right\}\) and \(r(x; \xi) = \omega_2 a^{\omega_2} x^{\omega_2-1} \exp \left\{-\frac{x}{\phi}\right\}\) to be the Weibull distribution with positive parameters \(\phi, \omega_2\) and \(\xi = (\omega, \phi)\). The CDF and pdf of the N\text{GOF}-O\text{E}-Weibull distribution are given by (for \(x > 0\))
\[
F_{N\text{GOF}\text{WE}(x; \alpha, \beta, \gamma, \delta, \phi, \omega)} = \exp \left\{-\left(1 - \exp \left[-\left(\frac{x}{\phi}\right)^{\omega_1}\right] \right) \right)^{\beta} \right(1 - \exp \left[-\left(\frac{x}{\phi}\right)^{\omega_1}\right] \right) \right) \right)^{\beta-1} \}
\]
\[
f_{N\text{GOF}\text{WE}(x; \alpha, \beta, \gamma, \delta, \phi, \omega)} = \beta \gamma a^\beta \delta \left(\omega_2 a^{\omega_2} x^{\omega_2-1} \exp \left\{-\frac{x}{\phi}\right\}\right) \left(1 - \exp \left[-\left(\frac{x}{\phi}\right)^{\omega_1}\right] \right) \right) \right)^{\beta-1} \}
\]
The N\text{GOF}-O\text{E}-Rayleigh distribution

The N\text{GOF}-O\text{E}-Rayleigh distribution “is defined from equations (3) and (4) by taking \(R(x; \xi) = 1 - \exp \left\{-\frac{x}{\phi}\right\}\) and \(r(x; \xi) = \phi x \exp \left\{-\frac{x}{\phi}\right\}\) to be the Rayleigh distribution with positive parameters \(\phi\) and \(\xi = \phi\). The CDF and pdf of the N\text{GOF}-O\text{E}-Rayleigh distribution are given by (for \(x > 0\))
\[
F_{N\text{GOF}\text{RE}(x; \alpha, \beta, \gamma, \delta, \phi)} = \exp \left\{-\left(1 - \exp \left[-\frac{x}{\phi}\right]\right) \right)^{\beta} \right(1 - \exp \left[-\frac{x}{\phi}\right]\right) \right)^{\beta-1} \}
\]
\[
f_{N\text{GOF}\text{RE}(x; \alpha, \beta, \gamma, \delta, \phi)} = \beta \gamma a^\beta \delta \left(\phi x \exp \left\{-\frac{x}{\phi}\right\}\right) \left(1 - \exp \left[-\frac{x}{\phi}\right]\right) \right)^{\beta-1} \}
\]
Figure 1: pdf Plot of the New Generalized Odd Fréchet-Odd Exponential-Weibull Distribution

Figure 2: HF Plot of the New Generalized Odd Fréchet-Odd Exponential-Weibull Distribution

Figure 3: PDF Plot of New Generalized Odd Frechet-Odd-Exponential-Rayleigh Distribution
NEW GENERALIZED ODD FRÉCHET...

Sadiq et al.,

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So, Figures 1 and 3 show the density function of the NGOF-OE-Weibull and NGOF-OE-Rayleigh models at different parameter values. The figures display the shapes and behaviour of the distribution, and how the parameters interact with one another. For example, if the parameters have equal values, the distribution is symmetrical. However, if the values differ, the distribution becomes more positively skewed. Additionally, the greater the difference between the parameter values, the less pronounced the bell shape of the distribution.

Furthermore, Figures 2 and 4 show the hazard functions of the NGOF-OE-Weibull and NGOF-OE-Rayleigh at various parameter values. The graph displays the modified unimodal and modal shapes of hazard rates at different parameter values.

Expansions

Let's take a closer look at the terms in the NGOF-OE-G family’s CDF presented in equations (3). We can use standard mathematical expansions such as the generalized binomial expansion for negative and positive powers, the power series expansion, and more to break down each term.

\[ F_{NGOFEG}(x; \alpha, \beta, \gamma, \delta, \xi) = \exp \left\{ - \left( \alpha \left( \left( 1 - \exp \left\{ - \frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right \} \right)^{-\gamma} - 1 \right) \right \} \]

where \( R_{DEG}(x; \xi) = \left( 1 - \exp \left\{ - \frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right \} \right) \)

\[ = e^{-\alpha \left( \frac{1}{R_{DEG}(x; \xi)} \right)^{\beta}} \]

\[ = \sum_{i=0}^{\infty} (-1)^i \frac{\alpha}{i!} \left( \frac{1}{R_{DEG}(x; \xi)} \right)^{i\beta} \]

\[ = \sum_{i=0}^{\infty} (-1)^i \frac{\alpha^{i\beta}}{i!} \sum_{j=0}^{\infty} (-1)^j \left( R_{DEG}(x; \xi) \right)^{j\beta} R_{DEG}^{-j\beta}(x; \xi) \]

\[ = \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{\alpha^{i\beta}}{i!} \frac{1}{j!} R_{DEG}^{j\beta-i\beta}(x; \xi) \]

Therefore, equation (9) reduces to:

\[ F_{NGOFEG}(x; \alpha, \beta, \gamma, \delta, \xi) = \sum_{i=0}^{\infty} S_{i,j} R_{DEG}^{j\beta-i\beta}(x; \xi) \]

where \( S_{i,j} = (-1)^{i+j} \frac{\alpha^{i\beta}}{i!} \frac{1}{j!} \)

Differentiating equation (10) w.r.t. \( x \) we have the corresponding pdf as:

\[ f_{NGOFEG}(x; \alpha, \beta, \gamma, \delta, \xi) = \sum_{i=0}^{\infty} v_{k} W_k(x) \]

where \( v_k = \sum_{j=0}^{\infty} S_{i,j} \) and \( W_k(x) = R_{DEG}^{j\beta-i\beta}(x; \xi) \)

Differentiate equation (12) w.r.t. \( x \) we obtained the corresponding pdf as:

\[ f_{NGOFEG}(x; \alpha, \beta, \gamma, \delta, \xi) = \sum_{k=0}^{\infty} v_{k} W_k(x) \]
where \( w_k(x) = k \gamma \omega \Omega(x; \xi) \Omega^{k-1}(x; \xi) \)

**Moments**

The role of moments in application to statistics is clear, and “the most essential characteristics of a probability model can be examined using moments. Evaluation in statistical inference is necessary, the most vital properties of the distributions were derived using the moments”. The rth ordinary moment of a random variable \( X \) that follows the New Generalized Odd Frechet-Odd Exponential-G (NGOF-OE-G) family by using equation (13) we have

\[
\mu_r = \int_0^\infty x^r f_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi) dx
\]

\[
= \sum_{k=0}^\infty v_k \int_0^\infty x^r w_k(x) dx
\]

\[
= \sum_{k=0}^\infty v_k E[Z_k^r]
\]

where \( E[Z_k^r] = \int_0^\infty x^r k \gamma \omega \Omega(x; \xi) \Omega^{k-1}(x; \xi) dx \)

**Moment-Generating Function**

Moment-generating functions “offer a clear and elegant framework for understanding and analyzing probability distributions and random variables, making them an essential tool in many branches of statistics and applied mathematics”. The moment-generating function of a random variable \( X \) that follows the New Generalized Odd Frechet-Odd Exponentialiated-G (NGOF-OE-G) family by using equation (13) we have,

\[
M_{NGOF\text{-}OE\text{-}G}(t) = \int_0^\infty e^{tx} f_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi) dx
\]

\[
= \sum_{k=0}^\infty v_k \int_0^\infty x^r w_k(x) dx
\]

\[
= \sum_{k=0}^\infty v_k E[Z_k^r]
\]

**Entropies**

Entropy, “a term borrowed from thermodynamics, is a measure of how uncertainty or randomness of a physical system. Entropy is a measure of the randomness or uncertainty of a random variable or a probability distribution that is used in statistics and information theory. It provides a means of measuring how informational the outcomes of a random process are. It is used in a variety of disciplines, including cryptography, machine learning, data analysis, and information theory”. The entropy of any “random variable \( X \) is a measure of indecisiveness, variability, and details properties of the probable results of the variable”.

The entropy of the NGOF-OE-G family using equation (13) we have,

\[
l_{NGOF\text{-}OE\text{-}G}(x) = \frac{1}{\omega} \log \left( \int_0^\infty f_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi) dx \right)
\]

\[
= \frac{1}{\omega} \log \left( \sum_{k=0}^\infty v_k w_k(x) \right) dx
\]

where \( \omega > 0 \) and \( \omega \neq 1 \)

The nth entropy is defined by

\[
l_{n\text{th}}(x) = \frac{1}{(n-1)!} \log \left( 1 - \int_0^\infty f_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi) dx \right)
\]

\[
= \frac{1}{\omega} \log \left( 1 - \sum_{k=0}^\infty v_k w_k(x) \right) dx
\]

where \( \omega > 0 \) and \( \omega \neq 1 \)

**Order Statistics**

Briefly stated order statistics are “a set of values obtained by placing the observations in a sample from either ascending or descending order. They are important in many statistical analyses, assisting in describing the distribution, drawing conclusions, and examining the extreme values of a dataset. It offers information about the distribution and properties of the data”.

Suppose \( X_1, X_2, X_3, \ldots, X_n \) is a random sample from the NGOF-OE-G distribution and \( X_{(i)} \) represent the ith order statistic, then, using equations (12) and (13) we have

\[
f_{(i)}(x; \alpha, \beta, \gamma, \delta, \xi) = \frac{n!}{[(i-1)!][(n-i)!]} \int_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi) \]

\[
= \frac{[F_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi)]^{i-1}[1 - F_{NGOF\text{-}OE\text{-}G}(x; \alpha, \beta, \gamma, \delta, \xi)]^{n-i}}{[(i-1)!][n-i]!} \sum_{k=0}^\infty v_k w_k(x) \]

where \( \alpha, \beta, \gamma, \delta, \xi > 0 \)

**Estimation of Parameters**

For estimating the parameters of a statistical model, “maximum likelihood estimation (MLE) is a frequently used technique in statistics. MLE aims to identify the model parameter values that maximize the likelihood function, which assesses how well
the model accounts for the observed data. Numerous statistical and scientific disciplines, such as econometrics, biostatistics, machine learning, and others, use MLE extensively.”

Suppose that \( x_1, x_2, \ldots, x_n \) are the observed values from the proposed NGOF-OE-G family with parameters \( \alpha, \beta, \gamma, \delta \) and \( \xi \). Suppose that \( \Phi = [\alpha, \beta, \gamma, \delta, \xi]^T \) is the \([n \times 1]\) vector of the parameter. The log-likelihood function \( \ell \) using equation (4) is expressed by

\[
\ell_n = \ell_n(\Phi) = n \log(\beta) + n \log(\gamma) + n \beta \log(\alpha) + n \log(\delta) + \sum_{i=1}^{n} \log[R(x; \xi)] \\
-2 \sum_{i=1}^{n} \log[1 - R(x; \xi)] - (\gamma + 1) \sum_{i=1}^{n} \log \left[ 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right] \\
-\delta \sum_{i=1}^{n} \left[ \frac{R(x; \xi)}{1 - R(x; \xi)} \right] + (\beta - 1) \sum_{i=1}^{n} \log \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) \\
- \sum_{i=1}^{n} \left( a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right)^{\beta} 
\]

(19)

Taking the partial derivative of equation (19) w.r.t. the parameters \( (\alpha; \beta; \gamma; \delta; \xi) \) respectively given as:

\[
\frac{\partial \ell_n(\Phi)}{\partial \alpha} = \frac{n \beta}{\alpha} - \sum_{i=1}^{n} \left( \frac{1}{1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right)} \right)^{\gamma} - 1 \\
- \sum_{i=1}^{n} \left[ a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right]^{\beta} \ln \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) 
\]

(20)

\[
\frac{\partial \ell_n(\Phi)}{\partial \beta} = \frac{n}{\beta} + n \log(\alpha) + \sum_{i=1}^{n} \log \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) \\
- \sum_{i=1}^{n} \left( a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right)^{\beta} \ln \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) 
\]

(21)

\[
\frac{\partial \ell_n(\Phi)}{\partial \gamma} = -\delta + \sum_{i=1}^{n} \log[R(x; \xi)] - \sum_{i=1}^{n} \log \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) \\
- (\beta - 1) \sum_{i=1}^{n} \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{\gamma} \ln \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) \\
- \sum_{i=1}^{n} \left( a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right)^{\beta} \ln \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) 
\]

(22)

\[
\frac{\partial \ell_n(\Phi)}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \left[ \frac{R(x; \xi)}{1 - R(x; \xi)} \right] - (\beta - 1) \sum_{i=1}^{n} \left[ \frac{\exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right)}{1 - R(x; \xi)} \right] \\
- \sum_{i=1}^{n} \left( a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right)^{\beta} \ln \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) 
\]

(23)

\[
\frac{\partial \ell_n(\Phi)}{\partial \xi} = \frac{r(x; \xi)}{R(x; \xi)} + 2 \sum_{i=1}^{n} \left( \frac{R'(x; \xi)}{1 - R(x; \xi)} \right) - (\gamma + 1) \sum_{i=1}^{n} \left( \frac{R'(x; \xi)}{1 - R(x; \xi)} \right) \\
- \delta \sum_{i=1}^{n} \left[ \frac{R(x; \xi)}{1 - R(x; \xi)} \right] + (\beta - 1) \sum_{i=1}^{n} \log \left( \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right) \\
- \sum_{i=1}^{n} \left( a \left( 1 - \exp \left( -\frac{\delta R(x; \xi)}{1 - R(x; \xi)} \right) \right)^{-1} \right)^{\beta} \frac{R'(x; \xi)}{(1 - R'(x; \xi))} 
\]

(24)

The MLEs of the parameters \( (\alpha; \beta; \gamma; \delta; \xi) \) are the simultaneous solution of equations (20), (21), (22), (23) and (24) when equated to zero, i.e. \( \frac{\partial \ell_n(\Phi)}{\partial \alpha} = 0; \frac{\partial \ell_n(\Phi)}{\partial \beta} = 0; \frac{\partial \ell_n(\Phi)}{\partial \gamma} = 0; \frac{\partial \ell_n(\Phi)}{\partial \delta} = 0; \frac{\partial \ell_n(\Phi)}{\partial \xi} = 0 \). These equations are intractable and non-linear and can only be solved using a numerical iterative method.

**Hazard Function**

The hazard function, and cumulative hazard function random variable is \( X \) which follows the NGOF-OE-G family are respectively given as,

- \( \lambda(x; \alpha, \beta, \gamma, \delta, \xi) \)
- \( \Lambda(x; \alpha, \beta, \gamma, \delta, \xi) \)
\[ h_{NGOFEG}(x; \alpha, \beta, \gamma, \delta, \xi) = \beta \gamma \alpha^\beta \delta R(x; \xi) \left(1 - R(x; \xi)\right)^{-\gamma} \left(1 - e^{\frac{-\delta R(x; \xi)}{1 - R(x; \xi)}}\right)^{y+1} \]

where \( R(x; \xi) = \frac{1}{\Gamma(\alpha, \beta x \gamma)} \), and \( \xi = \frac{\log a}{\log (1 - \delta)} \).

Quantile Function

The “quantile function of the NGOF-OE-G family is obtained by inverting the CDF in equation (3). Supposed the variable \( U \) is uniformly distributed on \((0,1)\), then

\[ x = \Phi(u) = R^{-1} \left( \frac{\log \left(1 - \frac{u}{\log \left(\frac{1}{1 - \delta} \right)}\right)}{\log \left(1 - \frac{1}{\log \left(\frac{1}{1 - \delta} \right)}\right)} \right) \]

where \( R^{-1} \) is the quantile function of the baseline distribution \( R(x; \xi) \). And \( 0 < u < 1 \).

RESULTS AND DISCUSSION

Simulation Study

The vast class of computational algorithms known as “Monte Carlo simulations” uses replicated random sampling to produce numerical results. The basic idea is to employ randomness to address problems that could be theoretically deterministic.

MLE and M.P.S Techniques for the NGOF-OE-Weibull Distribution

To evaluate the consistency of the new family’s parameters, the simulation study was piloted using the Monte Carlo simulation technique by computing the bias, variance and mean square error of the estimated parameters from the maximum likelihood estimates and the maximum product spacing estimate. The simulated data is generated using the quantile function in equation (26) and the likelihood function in equation (19) for different sample sizes \( n = 20 \), and 50 with replicate 200 times each. For the NOG-OE-Weibull distribution parameter values are \((\alpha, \beta, \gamma, \delta, \phi, \omega) = (1, 1.0, 2.5, 1.0, 2.0, 3.0)\).

Table 1: Results of the simulated data from the NOG-OE-Weibull Distribution.

<table>
<thead>
<tr>
<th>Sample Sizes</th>
<th>Parameters (Actual Values)</th>
<th>M.L.E. Techniques</th>
<th>M.P.S. Techniques</th>
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<td></td>
<td>Estimates</td>
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<td>( \gamma ) (2.5)</td>
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<td>0.0992</td>
</tr>
<tr>
<td></td>
<td>( \delta ) (1.0)</td>
<td>1.6763</td>
<td>0.6763</td>
</tr>
<tr>
<td></td>
<td>( \phi ) (2.0)</td>
<td>2.6243</td>
<td>0.6243</td>
</tr>
<tr>
<td></td>
<td>( \omega ) (3.0)</td>
<td>3.5471</td>
<td>0.5471</td>
</tr>
</tbody>
</table>

Table 1 presents the results obtained from the Monte Carlo Simulation study. The results indicated that the bias and root mean square error decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost equal at different sample sizes and iterative levels for the M.L.E technique. This proves the consistency of the MLE parameter estimates. For the M.P.S technique, the actual value of the parameters and the estimated values are almost not equal at different sample sizes and iterative levels. This proves the least consistency of the M.P.S parameter estimates. The result also means that the M.L.E technique is the best technique for estimating the parameter of New Generalized Odd Frechet-Odd Exponential-Weibull distribution than the M.P.S technique.

M.L.E and M.P.S Techniques for the NOG-OE-Rayleigh Distribution

To evaluate the consistency of the new family’s parameters, the simulation study was piloted using the Monte Carlo Simulation technique by computing the bias, variance and mean square error of the estimated parameters from the maximum likelihood estimates and the maximum product spacing estimate. The simulated data is generated using the quantile function in equation (26) and the likelihood function in equation (19) for different sample sizes \( n = 20 \), and 50 with replicate 200 times each. For the NOG-OE-Rayleigh distribution parameter values are \((\alpha, \beta, \gamma, \delta, \phi, \omega) = (1, 1.0, 2.5, 1.0, 2.0, 3.0)\).
Table 2: Results of the simulated data from the NGOF-OE-Rayleigh Distribution.

<table>
<thead>
<tr>
<th>Sample Sizes</th>
<th>Parameters (Actual Values)</th>
<th>M.L.E Techniques</th>
<th>M.P.S. Techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimates</td>
<td>Bias</td>
<td>RMSE</td>
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<tr>
<td>50</td>
<td>α (1.0)</td>
<td>0.9792</td>
<td>-0.0208</td>
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<tr>
<td></td>
<td>β (1.0)</td>
<td>1.0424</td>
<td>0.0424</td>
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<tr>
<td></td>
<td>γ (2.5)</td>
<td>2.5605</td>
<td>0.0605</td>
</tr>
<tr>
<td></td>
<td>δ (1.0)</td>
<td>1.0051</td>
<td>0.0051</td>
</tr>
<tr>
<td></td>
<td>φ (2.0)</td>
<td>1.9927</td>
<td>-0.0073</td>
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<tr>
<td>100</td>
<td>α (1.0)</td>
<td>0.9906</td>
<td>-0.0094</td>
</tr>
<tr>
<td></td>
<td>β (1.0)</td>
<td>1.0169</td>
<td>0.0169</td>
</tr>
<tr>
<td></td>
<td>γ (2.5)</td>
<td>2.5527</td>
<td>0.0527</td>
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<tr>
<td></td>
<td>δ (1.0)</td>
<td>1.0039</td>
<td>0.0039</td>
</tr>
<tr>
<td></td>
<td>φ (2.0)</td>
<td>1.9987</td>
<td>-0.0013</td>
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<tr>
<td>250</td>
<td>α (1.0)</td>
<td>0.9964</td>
<td>-0.0036</td>
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<tr>
<td></td>
<td>β (1.0)</td>
<td>1.0027</td>
<td>0.0027</td>
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<tr>
<td></td>
<td>γ (2.5)</td>
<td>2.5412</td>
<td>0.0412</td>
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<tr>
<td></td>
<td>δ (1.0)</td>
<td>1.0041</td>
<td>0.0041</td>
</tr>
<tr>
<td></td>
<td>φ (2.0)</td>
<td>2.0008</td>
<td>0.0008</td>
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<tr>
<td>500</td>
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<td>-0.0016</td>
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<tr>
<td></td>
<td>β (1.0)</td>
<td>1.0004</td>
<td>0.0004</td>
</tr>
<tr>
<td></td>
<td>γ (2.5)</td>
<td>2.5283</td>
<td>0.0283</td>
</tr>
<tr>
<td></td>
<td>δ (1.0)</td>
<td>1.0039</td>
<td>0.0039</td>
</tr>
<tr>
<td></td>
<td>φ (2.0)</td>
<td>2.0009</td>
<td>0.0009</td>
</tr>
<tr>
<td>1000</td>
<td>α (1.0)</td>
<td>0.9994</td>
<td>-0.0006</td>
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<tr>
<td></td>
<td>β (1.0)</td>
<td>0.9983</td>
<td>-0.0017</td>
</tr>
<tr>
<td></td>
<td>γ (2.5)</td>
<td>2.5247</td>
<td>0.0247</td>
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<td></td>
<td>δ (1.0)</td>
<td>1.0035</td>
<td>0.0035</td>
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<tr>
<td></td>
<td>φ (2.0)</td>
<td>2.0019</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

Table 2 presents the results obtained from the Monte Carlo Simulation study. The results indicated that the bias and root mean square error decrease toward zero with an increase in sample size. However, the actual value of the parameters and the estimated values are almost equal at different sample sizes and iterative levels for the M.L.E technique. This proves the consistency of the MLE parameter estimates. For the M.P.S technique, the actual value of the parameters and the estimated values are almost not equal at different sample sizes and iterative levels. This proves the least consistency of the M.P.S parameter estimates. The result also means that the M.L.E technique is the best technique for estimating the parameter of New Generalized Odd Frechet-Odd-Exponential-Rayleigh distribution than the M.P.S technique.

Applications
Here we used some existing real-life data sets to assess the flexibility of our developed family using Weibull distribution as a baseline.

Dataset 1
The data set was originally reported by Sadiq et al. (2023) which represents “the Maximum Annual Flood Discharges of North Saskatchewan in units of 1000 cubic feet per second, of the North Saskatchewan River at Edmonton, for 47 years”.

The data are: “19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 64.400, 70.400, 70.450, 72.500, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560”.

Table 3: Parameters Estimates and Goodness of Fit Measures for Dataset 1

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter Estimates and Goodness of Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α</td>
</tr>
<tr>
<td>NGFOEW</td>
<td>10.980</td>
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<tr>
<td>GOFW</td>
<td>1.3e15</td>
</tr>
<tr>
<td>OFW</td>
<td>2.6391</td>
</tr>
<tr>
<td>WD</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 3 provides the parameter estimates and goodness of fit measures for the New Generalized Odd Frechet-Weibull distribution with other competing models using the maximum annual flood discharges dataset. Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) are the performance metrics. A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. The new generalized odd Frechet-Odd Exponential Weibull distribution is the best model that outperforms other competitors based on the data set.

![Histogram Plots of the Distribution of Maximum Annual Flood Discharges Data](image)

Table 4 provides the parameter estimates and goodness of fit measures for the New Generalized Odd Frechet-Odd Exponential-Rayleigh distribution with other competing models using the survival times (in years) of a group of patients given a chemotherapy treatment dataset. Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) are the performance metrics. A distribution with the lowest information or performance metrics is regarded as the best in terms of goodness of fit. The New Generalized Odd Frechet-Odd Exponential-Rayleigh distribution is the best model that outperforms other competitors based on the data set.

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter Estimates and Goodness of Fit</th>
<th>AIC</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGOFOER</td>
<td>0.9583, 0.8819, 0.1619, 0.0971, 0.3415</td>
<td>123.252</td>
<td>1</td>
</tr>
<tr>
<td>GOFR</td>
<td>- , 8.3235, 0.0287, - , 0.0033</td>
<td>28149.8</td>
<td>4</td>
</tr>
<tr>
<td>OFR</td>
<td>- , 1.345, - , - , 0.5213</td>
<td>197.412</td>
<td>3</td>
</tr>
<tr>
<td>RD</td>
<td>- , - , - , - , 0.6025</td>
<td>163.83</td>
<td>2</td>
</tr>
</tbody>
</table>
CONCLUSION
In this research paper, we introduce the NGOF-OE-G family of distributions and explore its statistical properties, including the survival function, hazard function, cumulative hazard function, moments, moment-generating function, entropies, order statistics, and MLE. We also plot the pdf and the hazard rate function to observe the shapes and behaviour of the models at different parameter values. To test the consistency of the MLE and MPS of the parameters, we conduct simulation studies. We then apply the NGOF-OE-W distribution to the survival time data of patients who received chemotherapy treatment and the NGOF-OE-R distribution to the data representing Maximum Annual Flood Discharges employing Rayleigh and Weibull as the baseline distribution, respectively. Our analysis shows that the NGOF-OE-W is the “best fit” model for the group of patients given chemotherapy treatment and the NGOF-OE-R is the “best fit” model for the maximum annual flood discharge data.

REFERENCES


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