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A NUMERICAL APPROACH FOR THE STUDY OF HEAT GENERATION IN THE PRESENCE OF THERMAL BOUNDARY LAYER FOR A FLAT PLATE

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ABSTRACT

In this study, we investigate the laminar boundary layer flow in two dimensions, steadiness, and incompressibility around a moving vertical flat plate in a uniform free stream of fluid with a convective surface boundary condition. The similarity transformation technique has been applied to convert the governing nonlinear partial differential equation into two nonlinear ordinary differential equations. By combining the finite difference method with the shooting technique, the problem is solved numerically. We present a tabular and graphical representation of the variation in dimensionless temperature and fluid-solid interface characteristics for various values of the Prandtl number. As a special case of the problem, a comparison between the current result and the previously published result demonstrates a good agreement.

Keywords: Heat Transfer, Similarity variable, Prandtl number, Dimensionless velocity, Stream function, Kinematics viscosity

INTRODUCTION

Following the investigation of the laminar flow through a flat plate with the effect of viscosity by German scientist Prandtl (1904), boundary layer flow attracted attention. Because of the effects of kinetic energy and the disregard for viscous energy dissipation, when considering boundary layer flow in the presence of heat transfer, most velocity and velocities gradients are assumed to be appropriately small (Desale and Pradhan, 2015). Prandtl, (1904) claimed that the frictional effects, known as boundary layers, are what caused the fluid to stick in a no-slip condition next to the surface of the solid body. The boundary layer solution is the outcome of research that was first conducted by Prandtl (1904) and Blasius's study from 1908, much as the boundary layer equation and boundary layer theory are related. In the year (2023), Omokhuale and Dange conducted research on the impact of heat absorption on the magnetohydrodynamic flow of jeffery fluid within an infinite vertical plate. Higher heat absorption and Jeffery parameter values were found to cause the momentum boundary layer to expand, whereas higher suction and chemical reaction parameter values caused the fluid's velocity to decrease. In a channel filled with porous medium, Makinde *et al.*, (2005) described how heat transfer affects MHD oscillatory flow. In the explanation of Makinde *et al.,* (2011) the authors discussed the effects of mass and heat transfer on a moving, isothermal vertical plate during chemical reactions. They assumed boundary layer flow with convection of heat transfer over a flat plate and the viscous dissipation effect. It is also important to consider that the velocity is extremely high. Viscosity dissipation fluid's mixed convection around a vertical plate was investigated by Makinde (2008). Taking into account the wall's thermal boundary condition and the direction of free stream flow, the author examined four distinct flow scenarios.

The classical problem of hydrodynamic and thermal boundary layers over a flat plate in a uniform stream of fluid with a viscous dissipation effect and a variable plate temperature was covered by Desale and Pradhan (2015). According to the authors, for an inconstant temperature, the temperature distribution grows and decreases as the Eckert number (Ec) and Prandtl number (Pr) increase, respectively. By taking the fluid's viscous dissipation into account, Pantokratos (2005) investigated the steady laminar boundary flow along a vertical, motionless heated plate. Nonetheless, Aziz (2009) proposed a similarity solution through a convective surface of a boundary condition for a laminar thermal boundary layer along a flat plate. As a result, a similarity variable (η) can be used to convert the boundary equation to an ordinary differential equation. The authors assume that as the Prandtl number increases, the thermal boundary layer thickness decreases, leaving less energy for the back heat. They computed it numerically and arrived at their proposed result. The authors study the internal heat generation effect on thermal boundary layer with convective surface boundary condition using 4th order Runge-kutta method. Hussein and Hani (2018) established a new method for numerical solution for boundary layer theory of fluid flow past a flat plate using MATLAB. Basant *et al*., (2016) used Laplace transformation method to investigate the combine effect suction/injection on MHD free convection flow in vertical channel through thermal radiation. The effects of internal heat generation/absorption and viscous dissipation on combined heat and mass transfer MHD viscous fluid flow over a moving wedge in the presence of mass suction/injection with the convective boundary condition were investigated by Ahmad and Ahmed (2014). According to the authors, the results indicate that the parameters related to mass suction/injection, heat generation/dissipation, pressure, stretching/shrinking, magnetic, and Prandtl number have a significant impact on the flow field. Olanrewaju (2012) investigates the likeness solution for natural convection from a moving vertical plate with internal heat generation and convective boundary condition. In this study, we examine the momentum laminar boundary layers of an incompressible fluid flow caused by a uniform free stream (Blasius flow) across a flat plate. The laminar boundary layer equation is a nonlinear third order ordinary deferential equation, so we use the Newton-Rapson method to solve it after first linearizing it using the Jacobian transformation.

MATERIALS AND METHOD

The Governing Equations

We study the 2D laminar boundary layer fluid flow along a semi-infinite (very thin) flat plate as shown in figure 1

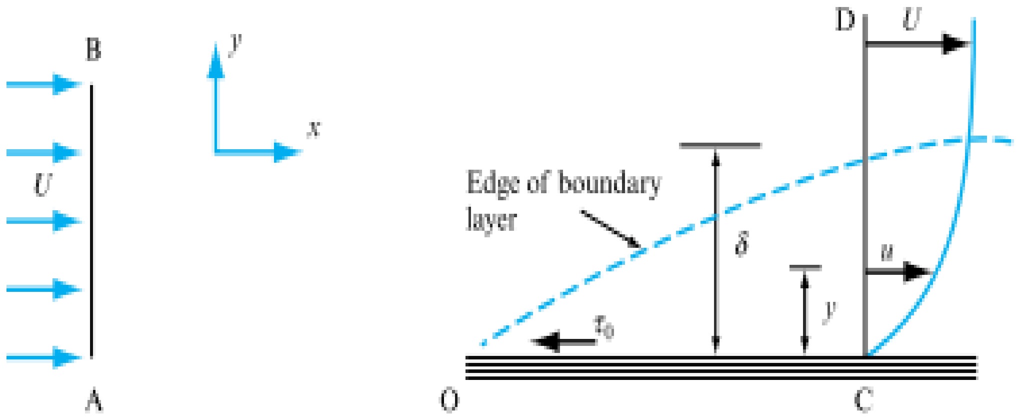

Figure 1: Heat transfer via a convective boundary layer

When a fluid flows over a motionless flat surface, such as the upper surface of the smooth flat plate as shown in figure 1 there will be a shear stress τ_0 between the surface of the plate and the fluid, acting to retard the fluid. At a section AB of the flow well upstream of the tip of the plate O, the velocity will be undisturbed and equal to U. The fluid in contact with the surface of the plate will be fixed, and at a cross-section such as CD, the velocity u of the adjacent fluid will increase gradually with the distance y away from the plate till it approaches the free stream velocity at the exterior of the boundary layer when $y = \delta$, which is the limit of this boundary layer, in which the drag of the motionless boundary affects the velocity of the fluid. The value of δ will increase from zero at the leading-edge O, subsequently the drag force D applied on the fluid due to the shear stress τ_0 will increase as x increases. Continuity equation focus on conservation of mass on a motion of fluid flow with the assumption made that flow is in steady condition which is not varying with the time. The incompressible form of continuity and momentum equations in the presence of viscous dissipation and heat generation or absorption can be written as (Ahmad and Ahmed, 2014);

Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Momentum in x- direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2)$$

Momentum in y- direction:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (3)$$

Where (x, y) are the Cartesian coordinates associated with the fluid velocities u, v and ν is the kinematics viscosity and ρ is the fluid pressure.

Under the boundary conditions

$$\text{At } y = 0 \qquad u = 0 \qquad v = 0 \qquad (4)$$

$$\text{At } y = \infty \qquad u = U_\infty \qquad \frac{\partial u}{\partial y} = 0 \qquad (5)$$

Equation (1) with the boundary conditions of equation (4) and (5) are in non-linear partial differential equation for unknown velocity field U and V.

Blasius reasoned that to solve them, the velocity profile $\frac{U}{U_\infty}$ should be similar for all values of x when plotted with non-dimensional distance from the wall. The boundary layer thickness δ.

Thus the solution,

$$\frac{U}{U_\infty} = f(\eta) \qquad \eta = \frac{y}{\delta} \qquad (6)$$

Base on solution of stokes (fox *et al*. (2009)) Blasius reasoned that

$\delta = \sqrt{\frac{\nu x}{U}}$ And set $\eta = \frac{y}{\frac{\sqrt{\nu x}}{\sqrt{U}}}$

this implies. $$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \qquad (7)$$

The stream function ψ were introduced, where

$$U = \frac{\partial \psi}{\partial y} \qquad V = -\frac{\partial \psi}{\partial x} \qquad (8)$$

Using continuity equation satisfies the continuity equation (1) now replacing for U and V into equation (2) reduce the equation to which ψ is the single dependent variable. The dimensionless stream function is defined as $f(\eta) = \frac{\psi}{\sqrt{\nu x U_\infty}}$ the dependent variable and as η the independent variable in equation (8) with ψ defined by equation (7), and η defined by equation (7). Which can be evaluated each of the terms;

From equation (1)

$$U = \frac{\partial \psi}{\partial x} \qquad V = \frac{\partial \psi}{\partial y} \qquad (9)$$

Integrate both side of equation (1) it yield,

$$\int \partial\psi = \int u\partial y \qquad \int v\partial x = \int \partial\psi \qquad (10)$$

$\psi = \int u\partial y$ But, $\frac{U}{U_\infty} = g(\eta) = f^I$

$$\int_0^y u\frac{dy}{d\eta}\,d\eta \qquad (11)$$

Where, $$\frac{dy}{d\eta} = \sqrt{\frac{\nu x}{U_\infty}} \qquad (12)$$

Substituting equation (12) in equation (11)

$$\psi = \int U_\infty f(\eta)\sqrt{\frac{\nu x}{U_\infty}}\,d\eta$$

$\psi = \sqrt{\nu x_\infty}\int f(\eta)d\eta$ by simplified U_∞

$$\psi = \sqrt{\nu x U_\infty}\, f(\eta) + C(x) \qquad (13)$$

Where $f(\eta) = \int F(\eta)d\eta$ and $c(x) = 0$, if the stream function at the solid surface is set to be zero (0).

Now from equation (8)

$$U = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} = U_\infty f^I(\eta)$$

and $$V = -\frac{\partial\psi}{\partial x} = -\left[\sqrt{\nu x U_\infty}\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}f\right] \qquad (14)$$

$$= -\left[\sqrt{\nu x U_\infty}\left(-\frac{1}{2}\eta\frac{1}{x}\right) + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}f\right] \qquad (15)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta f^I(\eta) - f(\eta)\right]$$

$$= -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}f(\eta) + \frac{1}{2}\frac{U_\infty}{x}yf^I(\eta) \qquad (16)$$

By differentiating the velocity u components with respect to x and y, the result can be shown as

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x}\eta f^{II}(\eta) \qquad (17)$$

$\frac{\partial u}{\partial y} = U_{\infty}\sqrt{\frac{U_\infty}{Vx}} f^{II}(\eta)$ (18)

$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{vx} f^{III}(\eta)$ (19)

Substituting this equation (19) in equation (2) above, this yield

$2f^{III}(\eta) + f(\eta)f^{II}(\eta) = 0$ (20)

$f^{III}(\eta) + \frac{1}{2}f(\eta)f^{II}(\eta) = 0$ (21)

$f^{III} + \frac{1}{2}ff^{II} = 0$ (22)

This equation (22) is known as ***Blasius Equation*** (or the laminar boundary layer equation). The Blasius is basically obtained from the governing equation of the fluid flow (Navier Stokes equation) through similarity transformation to third order nonlinear ordinary differential equation (ODE) as shown above.

And the boundary condition

At $y = 0$ $u = v = 0$

At $\eta = 0$ $f^I(\eta) = 0$ $f(\eta) = 0$.

Similarly at $y \to \infty$ $\eta = \infty$ $f^I(\eta) = 1$ (23)

The primes refer differentiation with to respect η.

For the temperature equation of the plate

The boundary equation for temperature

$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \propto \frac{\partial^2 T}{\partial y^2} + Q\theta$ (24)

The boundary conditions

At $y = 0$ $T = T_w$

$y \to 0$ $T = T_\infty$

$x \to 0$ $T = T_\infty$

To defined the temperature θ

i.e if $\theta = \frac{T - T_w}{T_\infty - T_w}$, at $y = 0$, $\theta = 0$, and $y \to \infty$ $\theta = 1$

Writing equation (24) in terms of θ

We have, $U\frac{\partial \theta}{\partial x} + V\frac{\partial \theta}{\partial y} = \propto \frac{\partial^2 \theta}{\partial y^2} + Q\theta$.

if $Pr = 1$ and $\propto = \vartheta$

$\theta = \frac{u}{U_\infty}(x, y)$, which is exactly identical to momentum equation.

So the solution for $\theta(x, y) = \frac{u}{U_\infty}(x, y)$ for a flat plate if $\theta = \theta(\eta)$ which is similarity variable $\frac{u}{U_\infty} = g(\eta)$, i.e

$\eta = \frac{y}{\delta} = y\sqrt{\frac{U_\infty}{\vartheta_x}}$ (25)

We know U and V already.

To find $\frac{\partial \theta}{\partial x}$ using similarity transformation techniques

$\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\eta}\frac{d\eta}{dx} = y\sqrt{\frac{U_\infty}{\vartheta_x}}\frac{-1}{2x}\frac{d\theta}{d\eta}$ (26)

But $y\sqrt{\frac{U_\infty}{\vartheta_x}} = \eta$, so we get $\frac{d\theta}{d\eta}\frac{d\eta}{dx} = \frac{-\eta}{2x}\frac{d\theta}{d\eta}$

$\frac{\partial \theta}{\partial y} = \frac{d\theta}{d\eta}\frac{d\eta}{dy} = \sqrt{\frac{U_\infty}{\vartheta_x}}\frac{d\theta}{d\eta}$ (27)

If we square equation (27) above, we get

$\frac{\partial^2 \theta}{\partial y^2} = \frac{U_\infty}{\vartheta_x}\frac{d^2\theta}{d\eta^2}$ (28

But U and V are similarity expression, substituting equation (26), (27) and (28) in equation (24).

We obtained the heat equation

$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2}Pr\frac{d\theta}{d\eta} + Q\theta$ (29)

With the boundary condition

$\eta = 0$ $\theta = 0$, $\eta \to \infty$ $\theta = 1$

$\eta \to \infty$ Means two things either $y \to \infty$ or $x = 0$ from equation (20) above, in both cases η must be 1.

Numerical Solution

Equation (22) with the boundary condition $f(0) = f^I(0) = 0$, $f^I(\infty) = 1$ are solved numerically by finite difference method. Since the boundary condition are at infinity and we are considering a thin boundary layer being induced over finite flat plate, we therefore let our infinity, $\infty = 3$. using step size $h = 1.0$, we divide the interval into equally four spaced mesh points. to determine the value of the function f's for the internal mesh point only, we therefore use the shooting-technique

RESULTS AND DISCUSSION

The proposed problem was solved by using finite difference method and the results obtained were computed numerical. These results are shown in table 1 and 2 which are also represented in figure 1, 2, 3 and discuss accordingly.

Table 1: Summary of the result obtained from finite difference method.

η	F_1	F_2
0	0.5	1.0
1	0.4388762882	1.75894663
2	2.07555438	2.821153758
3	0.544036709	3.326031669
4	0.534112418	0.371468641
5	0.0242025	0.501314024
6	0.828440589	1.21332267
7	0.702454183	0.112038316
8	0.848466483	1.048633418

Since the boundary condition of the blasius equation are at infinity and we are considering a thin boundary layer being induced over finite flat plate, we let our infinity, $\infty = 3$. using step size $h = 1.0$, we divide the interval into equally four spaced mesh points. Consequently, we proceed by replacing their derivative by central difference then we notice that f_{-1} and f_4 from the central difference are outside the domain. These mesh points are called the fictions node or ghost node, we replace the derivative for Blasius equation by central difference to obtained equation (30) as follows;

$\frac{f_{n+2} - 2f_{n+1} + 2f_{n-1} - f_{n-2}}{2h^3} = \frac{-1}{2}\left(\frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}\right)f_n$ (30)

Meanwhile we are supposed to determine the value of the function f's for the internal mesh point only, we therefore use the shooting-technique using the boundary condition we obtained equation the following equations

$F_1(f_1 f_2) = f_1 + 2f_2 - f_1 f_2 + 2f^2{}_1 - 1.396842 = 0$ (31)

$F_2(f_1 f_2) = 2f_1 + 2.396842 f_2 - 2f^2{}_2 + f_1 f_2 - 0.793684 = 0$ (32)

This system of equations (31) and (32) were solved by iterative method (Newton-Rapson), the result of the iteration were summarize and tabulated as shown table 1

Table 2: Comparison of the result obtained with the results reported by other authors.

η	Present method	Parand *etal.,* (2009)	Rafael, (2010)	Ghorbani, (2015)	Sakiadis, (1961)
0	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.84866	0.16557	0.16557	0.18498	0.78620
2	1.04863	0.65003	0.65003	0.69365	1.21855
3	1.39682	1.39683	1.39689	1.44689	1.43273

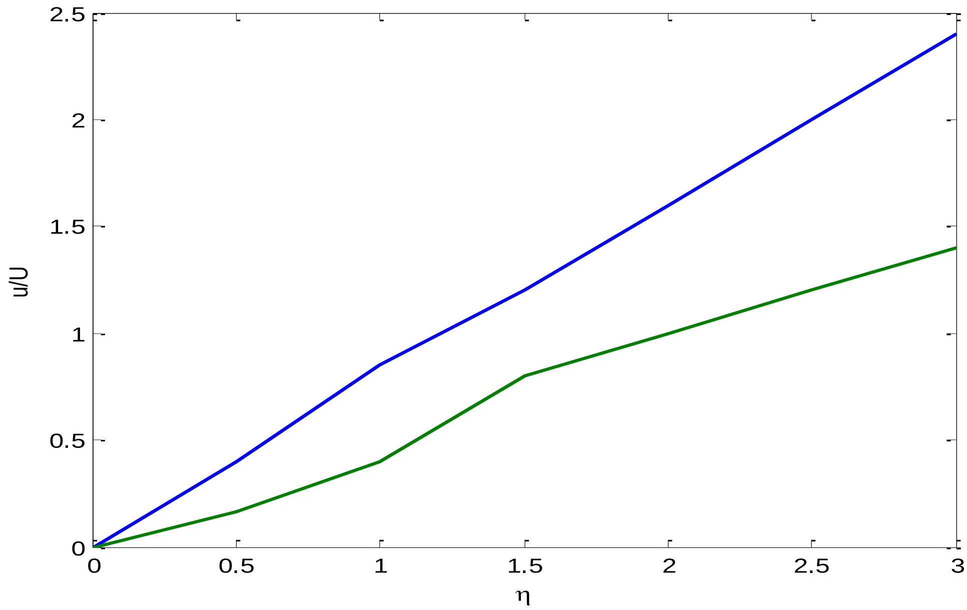

Figure 2: The effect of heat along a flat plate at $\eta = 1.0$ and $\frac{U}{U_\infty} = 0.84846$

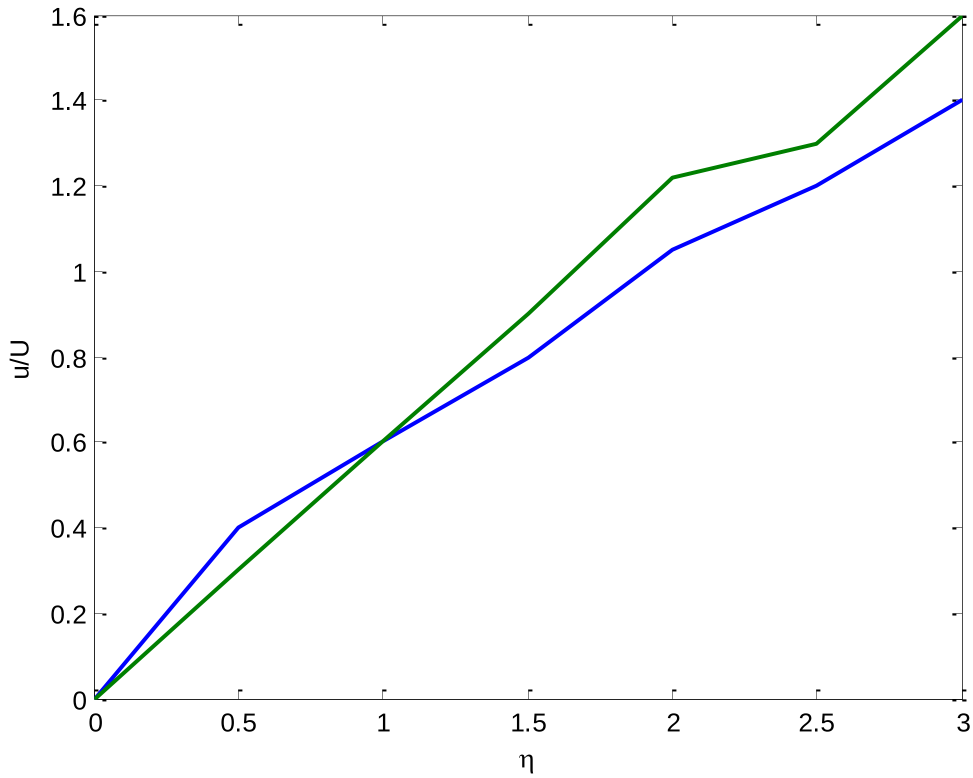

Figure 3: Effect of heat flow along a flat plate at $\eta = 2.0$ and $\frac{U}{U_\infty} = 1.0486$

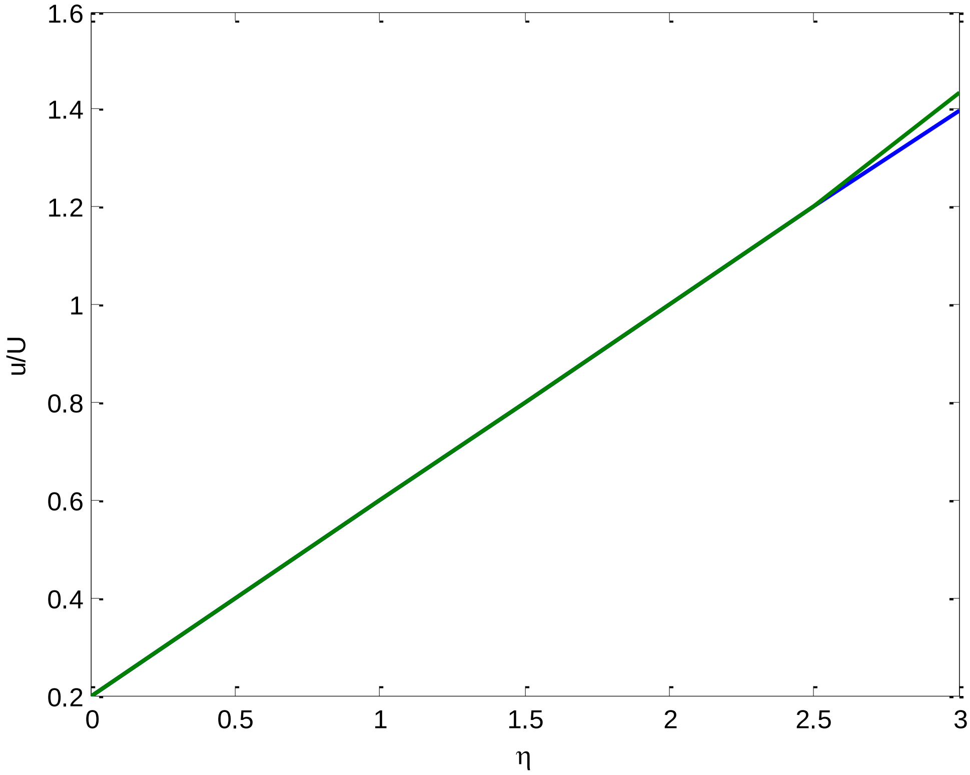

Figure 4: Effect of heat flow along a flat plate for a uniform temperature at $\eta = 3.0 \quad and \ \frac{U}{U_\infty} = 1.39684$

The results shown in figure (2) show some agreements, but when compared, they differ somewhat for a number of reasons. Each measurement began with the calibration of velocity first. This could be the reason for a small variation in the measurement. The finite difference method with shooting techniques yields the numerical solution of the ordinary differential equation (22) with the boundary conditions $f(0) = f^I(0) = 0$, $f^I(\infty) = 1$. Although it is not feasible to solve the boundary value problem for even very large finite intervals, it cannot be solved on infinite ones. $\eta_\infty = 3$ is the finite point at which we apply the infinite boundary condition in this work. There is now only one nonlinear equation, which is the system of two nonlinear ordinary differential equations with boundary conditions. The velocity profile and stream function have been presented in order to provide a clear understanding of the physical problem. To make the embedded parameter values realistic, they were assigned a numerical value. Rafael, (2010) stresses the importance of the stream function values ($(\eta)\ from\ 1\ to\ 3$), as well as the heat. Within Table 2, a comparison was conducted using a few fixed parameters, and Rafael, (2010) a unique instance of ours—shows perfect agreement. We continued in Figure 3, where we increased the parameter's value to look into the effect following the initial observation and then increased the value to look into as before. It is evident that whereas an increase in velocity causes the fluid flow behavior to change, the rate at which heat is induced into the plate causes the stream function at the wall plate to increase. It's interesting to see that the stream function at the wall plate increases as the velocity included in the flow model increases.

CONCLUSION

We have examined a well-known Blasius boundary layer equation in this work by considering the two-dimensional laminar viscous incompressible flow over a flat plate. The finite difference method was used to handle the boundary layer equation (Blasius) numerical solution. This equation is laborious because of the boundary condition that exists at infinity. We take into account the boundary condition at $\eta = 3$ instead of using infinity for an easy approximation of the solution because we are working with a very thin boundary layer. Since the solution at F(1) is exact and the result was marginally accurate when compared to Sakiadis's (1961) result, we let our $f(3) = 1.396842$ in accordance with Rafael, (2010) be our boundary condition and the nonlinear differential equation has been solved using the finite difference method. We have almost precisely obtained the result for $\eta \leq 3$ because we used a Taylor series method around $= 0$.

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