FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 7 No. 1, February, 2023, pp 323 - 329
DOI: https://doi.org/10.33003/fjs-2023-0701-2070

NEW LIFE-TIME CONTINUOUS PROBABILITY DISTRIBUTION WITH FLEXIBLE FAILURE RATE FUNCTION

***F. Manu, U. Usman and A. Audu**

Department of Statistics, Usmanu Danfodiyo University, Sokoto.

*Corresponding authors' email: farukmanu9@gmail.com

ABSTRACT

Recently, the area of distribution theory has been receiving increased interest in generating or defining new classes of continuous probability distributions by way of extending the existing distributions. The new generated distribution is expected to be more flexible and have wider acceptability in modeling and predicting real world data sets. In this research, we proposed and study an extension of Ikum distribution using Zubai G-Family (2018) of distribution called Z-Ikum distribution. Expression of some basic structural properties of the new distribution such as cdf, pdf, quantile functions, moments, moment generating function, characteristics function and order statistics was derived. Survival function, hazard rate function, commutative hazard rate function and reversed hazard rate function was also discussed. Plots of the hazard rate function showincrease, decrease and bathtub shapes.Estimation of the proposed distribution parameters was carried out using MLE method. Performance of the parameter estimation was also evaluated via simulation studies. Result of the simulation studies indicates that our estimator is consistent. Three life data sets were used to evaluate the performance of our proposed distribution over some existing distributions. Result of the empherical study revealed that our proposed distribution performwell in modeling real life data than the competing distributions.

Keywords: Ikum, Bathtub, Hazard rate, Failure rate, Simulation

INTRODUCTION

Continuous statistical distributions are important for parametric inferences and commonly appliedto describe real world phenomena. Some of these distributions include Beta Distribution, Burr Distributions, Logistic Distribution, Half logistic Distribution, Exponential distribution, Normal distribution, Weibulldistribution, Rayleigh distribution, Lognormal distribution, Gamma distribution,Frechet distribution, Gumbel distribution, Kumaraswamy distribution, Pareto distribution,Topp-Leone distribution, and many others. Numerous of these classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, medical sciences and in reliability studies.

The quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distribution that the random variable follows. Many lifetime data used for statistical analysis follow a particular probability distribution and therefore knowledge of the appropriate distribution that any phenomenon follows greatly improves the sensitivity, reliability and efficiency of the statistical analysis associated with it. Furthermore, it is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. There is a strong desire among statisticians to create statistical distributions that are more flexible and there have been many different types of generalized distributions developed and applied to various phenomena (Olalekan et al., 2021).

Some well-known methods in the early days for generating univariate continuous distributions include methods based on differential equations developed by Pearson (1895), methods of translation developed by Johnson (1949), and the methods based on quantile functions developed by Tukey (1960). The interest in developing new methods for generating new or more flexible distributions continues to be active in the modern decades. Lee *et al*. (2013) indicated that the majority of methods developed after 1980s are the methods of 'combination' for the reason that these new methods are based on the idea of combining two existing distributions or by adding additional parameters to an existing distribution to generate a new family of distributions. As a result,many new families of distributions have been developed and studied by researchers. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data.

The Ikum distribution which is the inverse form of the Kumaraswamy distribution is obtained using transformation $X = X^{-1}$ by Abd AL Fattah *et al*, (2017) has been used in modeling lifetime data. However, in many applied instances, the IKum distribution fails to give adequate fits to lifetime data such as the life cycle of machines, human mortality and biomedical data which show non-monotone failure rates.

Some Existing Distributions

Ikum Distribution

The random variable X is said to have the IKum distribution if its CDF and corresponding PDF are as given in (1) and (2) respectively.

$$G(x;\lambda,\beta) = \left(1-(1+x)^{-\lambda}\right)^{\beta}, \qquad x>0,\ \lambda>0,\ \beta>0 \tag{1}$$

$$g(x,\lambda,\beta) = \lambda\beta(1+x)^{-(\lambda+1)}\left(1-(1+x)^{-\lambda}\right)^{\beta-1}, \qquad x>0,\ \lambda,\beta>0 \tag{2}$$

where$\lambda,\beta$are the two shape parameters.

Zubair G Family of distribution

A family of life distributions, called the Zubair-G family was introduced by Zubair (2018). The benefit of using this family is that its cdf has a closed form solution and capable of data modeling with monotonic and non-monotonic failure rates. TheCDF and PDF of the new family defined by Zubair (2018) for random Variable X is given in (3) and (4) respectively.

$$F(x,\alpha\varphi) = \frac{\exp\{\alpha G(x;\varphi)^2\}-1}{exp(\alpha)-1}, \qquad \alpha > 0, x \quad \varepsilon\Re \tag{3}$$

$$f(x,\alpha,\varphi) = \frac{2\alpha g(x;\phi)G(x;\varphi)exp\{\alpha G(x;\varphi)^2\}}{exp(\alpha)-1}, \qquad \alpha > 0, x \quad \varepsilon\Re \tag{4}$$

Where φ is vector of the baseline distribution parameter, α is the parameter of Zubair G-family $g(x;\phi)$ and $G(x;\varphi)$are pdf and cdf of the baseline distribution respectively.

Proposed Distribution

Zubair-Ikum (Z-IKUM) Distribution.

To obtain the CDF of the Z-Ikum distribution, we let$G(x)$ be cdf of the Ikum random variable given by $G(x;\lambda,\beta) = \left(1-(1+x)^{-\lambda}\right)^{\beta}$and substitute in (3.1). Then the cdf of Zubair–Ikum (Z-Ikum) distribution is obtained as in (5).

$$F(x;\alpha\lambda,\beta)_{Z-Ikum} = \frac{exp\left(\alpha\left((1-(1+x)^{-\lambda})^{\beta}\right)^2\right)-1}{exp(\alpha)-1} \tag{5}$$

The corresponding probability density function (pdf) of Zubair–IKum (Z-Ikum) distribution denoted by $f(x;\alpha,\lambda,\beta)_{Z-Ikum}$ is obtained by differentiating (45) with respect to$x$.

From the definition

$$f(x;\alpha,\lambda,\beta)_{Z-Ikum} = \frac{dF(x;\alpha,\lambda,\beta)_{Z-Ikum}}{dx} \tag{6}$$

$$f(x;\alpha,\lambda,\beta)_{Z-Ikum} = \frac{1}{exp(\alpha)-1}\left(\frac{d}{dx}\left(exp\left(\alpha\left((1-(1+x)^{-\lambda})^{\beta}\right)^2\right) - \frac{d}{dx}(1)\right)\right) \tag{7}$$

$$f(x;\alpha,\lambda,\beta)_{Z-Ikum} = \frac{2\lambda\beta\, exp\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right)\alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{exp(\alpha)-1} \tag{8}$$

Theorem 1:$f(x;\alpha,\lambda,\beta)_{Z-Ikum}$ is a pdf. That is $\int_0^\infty f(x;\alpha,\lambda,\beta)_{Z-Ikum}\,dx = 1$ (9)

Proof: Let $T_3 = \int_0^\infty f(x;\alpha,\lambda,\beta)_{Z-Ikum}\,dx$ then,

$$T_3 = \frac{2\lambda\beta\alpha}{-1+exp(\alpha)}.\int_0^\infty exp(1-(1+x)^{-\lambda}+1)^{2\beta}\alpha(1+x)^{-1-\lambda}(1-(x+1)^{-\lambda}+1)^{-1+2\beta}\,dx \tag{10}$$

Apply u substitution let

$$u_3 = \alpha\left(1-(1+x)^{-\lambda}\right)^{2\beta}, \frac{du_3}{dx} = 2\alpha\beta\left(1-(1+x)^{-\lambda}\right)^{2\beta-1}\left(\lambda(1+x)^{-\lambda-1}\right),$$

$$dx = \frac{du_3}{2\alpha\beta\left(1-(1+x)^{-\lambda}\right)^{2\beta-1}\left(\lambda(1+x)^{-\lambda-1}\right)}, 0 < u_3 < \infty \qquad \text{Then}$$

$$T_3 = \frac{exp(\alpha)}{-1+exp(\alpha)} - \frac{exp(\alpha)-1}{-1+exp(\alpha)} = 1 \tag{11}$$

Hence, the proof.

Figures **1** and **2** below displayed the plots of the *pdf* and *cdf* of the Z-Ikum distributionfor some selected parameter values respectively.

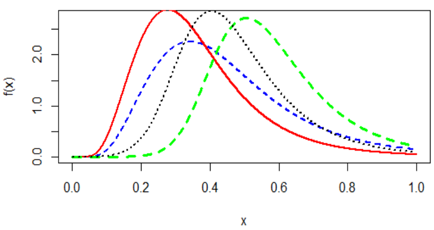

Figure 1: Plot of of Z-Ikum PDF

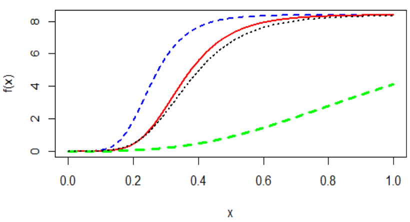

Figure 2: Plot of of Z-Ikum CDF

Statistical Properties of Z-Ikum distribution

This section studies the statistical properties of the proposed distributions such as the quintile functions,order statistics and moments. Also reliability analysis of the proposed distribution is discussed in details.

Quantile Function

The quantile function Z-Ikum distribution is obtain by inverting (5) as given in (12)

$$Q(U) = \left(1 - \left(\frac{1}{\alpha} ln(u(exp(\alpha) - 1) + 1)\right)^{\frac{1}{2\beta}}\right)^{-\frac{1}{\lambda}} - 1 \quad (12)$$

$where u \quad \sim U(0,1)$

To obtained, the first quartile, the median, and the third quartile, we replace u with 0.25, 0.5 and 0.75 in (12) respectively. .

Order Statistics

Suppose $X_1, X_2, \ldots\ldots\ldots\ldots, X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1n}, X_{2n}, \ldots\ldots\ldots\ldots, X_{in}$ denote the corresponding order statistic obtained from this sample. The i^{th} order statistic of the proposed distributions can be obtained as in (13)

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}(1 - F(x))^{n-i} \quad (13)$$

using binomial expansion,

$$(1 - F(x))^{n-i} = \sum_{k=0}^{n-i}(-1)^k \binom{n-1}{k} F(x)^k \quad (14)$$

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1} \sum_{k=0}^{n-i}(-1)^k \binom{n-1}{k} F(x)^k \quad (15)$$

Inserting (5) and (5) in (13), the pdf of the ith order statistics can be given as in (16)

$$f_{in}(x) = \frac{n!}{(i-1)!\,(n-i)!}\left(\frac{2\lambda\beta\, \boldsymbol{exp}\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right) \alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{\boldsymbol{exp}(\alpha) - 1}\right)$$
$$\times \left(\frac{\boldsymbol{exp}\left(\alpha\left((1-(1+x)^{-\lambda})^{\beta}\right)^2\right)-1}{\boldsymbol{exp}(\alpha)-1}\right)^{i-1} \sum_{k=0}^{n-i}(-1)^k \binom{n-1}{k} \left(\frac{\boldsymbol{exp}\left(\alpha\left((1-(1+x)^{-\lambda})^{\beta}\right)^2\right)-1}{\boldsymbol{exp}(\alpha)-1}\right)^{k} \quad (17)$$

Moments

The rth non-central moment of the IKum random variable is derived.

By definition

$$\mu_r^1 = E(X^r) = \int_{\infty}^{-\infty} X^r f(x,\theta)dx \quad (18)$$

Using (8), we have

$$\mu_r^1 = E(X^r) = \int_0^1 X^r . \frac{2\lambda\beta\, exp\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right)\alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{exp(\alpha)-1} dx \quad (19)$$

Let $y = (1 + x)^{-\lambda}$, $\frac{dy}{dx} = \lambda(1 + x)^{-\lambda-1}$, $\quad dx = \frac{dy}{\lambda(1+x)^{-\lambda-1}} \quad x = y^{\frac{-1}{\lambda}} - 1$

$$\mu_r^1 = E(X^r) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1} \int_0^1 \left(y^{-1/\lambda} - 1\right)^r exp(1 - y)^{2\beta}\, (1 - y)^{2\beta-1} dy \quad (20)$$

Using power series

$$\left(y^{-1/\lambda} - 1\right)^r = (-1)^r \sum_{j=o}^{r} \binom{r}{j} (-r)^j (y)^{j/\lambda}$$

$$exp(1 - y)^{2\beta} = \sum_{j=0}^{\infty} \frac{\left((1-y)^{2\beta}\right)^j}{j!}$$

$$\mu_r^1 = E(X^r) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1} (-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \binom{r}{j} \frac{1}{j!} \int_0^1 \left(y^{(-j/\lambda+1)-1}(1 - y)^{2\beta j+2\beta-1}\right) dy \quad (21)$$

Finally,

$$\mu_r^1 = E(X^r) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1} (-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \binom{r}{j} \frac{1}{j!} B\left(\frac{-j}{\lambda} + 1{,}2\beta - 1\right) \quad (22)$$

where

$$B\left(\frac{-j}{\lambda} + 1{,}2\beta - 1\right) = \int_0^1 \left(y^{(-j/\lambda+1)-1}(1 - y)^{2\beta j+2\beta-1}\right)$$

Mean=$(x) = \bar{\mu_1}$, $V(x) = \bar{\mu_2} - \bar{\mu_1}^2$, Skewness = $\frac{\bar{\mu_3} - 3\bar{\mu_2}\,\bar{\mu_1} + 4\bar{\mu_1}^3}{(\bar{\mu_2} - \bar{\mu_1}^2)^{\frac{3}{2}}}$,

Kurtosis = $\frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\,\mu_2' - 3\mu_1'^4}{(\mu_3' - \mu_1')^2}$

Moment Generating Function of Z-Ikum distribution.

Theorem 4.1 Let X have an Z-Ikum distribution the moment generating function is given by

$$M_X(t) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1} (-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \frac{t^r}{r!} \binom{r}{j} \frac{1}{j!} B\left(\frac{-j}{\lambda} + 1{,}2\beta - 1\right) \quad (23)$$

Proof: If the moment generating function premised on the support $(0, \infty)$, exist, then

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\, dx \quad (24)$$

Using Taylor series expansion, the moment generating function can be given as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} u_r \quad (25)$$

where u_r is the r^{th} non-central moment. Substituting the r^{th}non-central moment gives the moment generating function of Z-Ikum distribution as

$$M_X(t) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1}(-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \frac{t^r}{r!}\binom{r}{j}\frac{1}{j!} B\left(\frac{-j}{\lambda}+1, 2\beta-1\right) \quad (26)$$

Hence the proof.

Characteristic Function of Z-Ikum distribution

Theorem 4.2 Let X have an Z-Ikum distribution. Then the characteristic function is given by

$$\phi_X(t) = E(e^{itx}) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1}(-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \frac{(it)^r}{r!}\binom{r}{j}\frac{1}{j!} B\left(\frac{-j}{\lambda}+1, 2\beta-1\right) \quad (27)$$

Proof: If the characteristic function premised on the support$(0, \infty)$, exist, then

$$\phi_X(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)\, dx \quad (28)$$

Using Taylor series expansion, the moment generating function can be given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} u_r \quad (29)$$

where u_r is the r^{th} non-central moment. Substituting the r^{th}non-central moment gives the moment generating function of Z-Ikum distribution as

$$\varphi_X(t) = \frac{2\lambda\alpha\beta}{exp(\alpha)-1}(-1)^{r+j} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \frac{(it)^r}{r!}\binom{r}{j}\frac{1}{j!} B\left(\frac{-j}{\lambda}+1, 2\beta-1\right) \quad (30)$$

Survival Function of Zubair-Ikum (Z-Ikum) distribution

Using (31) the survival function of Z-Ikum distribution is given as in (32)

$$S(x) = 1 - F(x) \quad (31)$$

$$S(x) = \frac{exp(\alpha) - exp\left(\alpha\left(-(x+1)^{\lambda}+1\right)^{2b}\right)}{exp(\alpha)-1} \quad (32)$$

Hazard Rate Function of Zubair-Ikum (Z-Ikum) distribution

Using (33) the hazard rate functions of Z-Ikum distribution is given as in (34)

$$h(x) = \frac{f(x)}{S(x)} \quad (33)$$

$$h(x) = \frac{2\lambda\beta\, exp\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right)\alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{exp(\alpha) - exp\left(\alpha\left(-(x+1)^{\lambda}+1\right)^{2b}\right)} \quad (34)$$

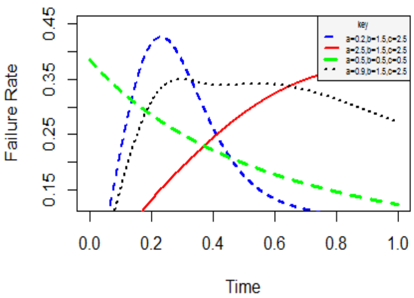

Figure 3: Plot of of Z-Ikum hr

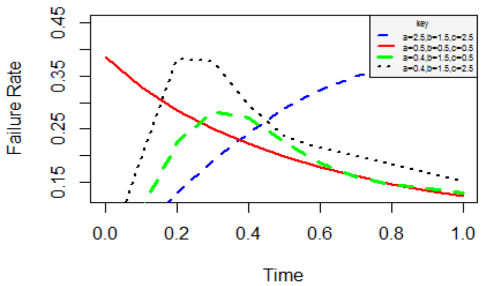

Figure 4: Plot of of Z-Ikum hr

The plot of the hazard rate function of Z-Ikum distribution for specific sets of parameter values exhibits Bath-Tub shape, right skewed, increasing, and decreasingshape.

Reverse Hazard Rate Function ofZ-Ikum distribution

Using (35) the Reverses hazard rate functions of Z-Ikum distribution is obtained as in (36)

$$r(x) = \frac{f(x)}{F(x)} \quad (35)$$

$$r(x) = \frac{2\lambda\beta\, exp\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right)\alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{exp\left(\alpha\left((1-(1+x)^{-\lambda})^{\beta}\right)^2\right)-1} \quad (36)$$

Cumulative Hazard Rate of Zubair-Ikum (Z-Ikum) distribution

Using (37) the cumulative hazard rate function of Zubair-Ikum (Z-Ikum) distribution is obtained as in (38)

$$H(x) = -\boldsymbol{log}\left(1 - F(x)\right) \quad (37)$$

$$H(x) = -\boldsymbol{ln}\left(\boldsymbol{exp}(\alpha) - \boldsymbol{exp}\left(\alpha\left(-(x+1)^{\lambda}+1\right)^{2b}\right)\right) - \boldsymbol{ln}(\boldsymbol{exp}(\alpha) - 1) \quad (38)$$

Parameters Estimation and Simulation Studies of Z-Ikum Distribution

In this section, estimators are developed for estimating the parameters of Z-Ikum distribution using the well known method of maximum likelihood estimate (MLE).

Parameter Estimation

Let $X_1, X_2 \ldots\ldots X_n$ be a random sample from Z-Ikum distribution with unknown parameter vector $\phi = (\alpha, \beta, \lambda,)^T$, the likelihood function of the distribution is obtained using (39)

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \quad (40)$$

$$L(\phi) = \prod_{i=1}^{n} \left(\frac{2\lambda\beta\, \boldsymbol{exp}\left(\alpha(1-(1+x)^{-\lambda})^{2\beta}\right)\alpha(x+1)^{-\lambda-1}(1-(1+x)^{-\lambda})^{2\beta-1}}{\boldsymbol{exp}(\alpha)-1}\right) \quad (41)$$

$$L(\phi) = \left(\frac{2^n\lambda^n\beta^n\, \boldsymbol{exp}\left(\alpha^n \sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta}\right)\alpha^n \prod_{i=1}^{n} (x+1)^{-\lambda-1} \prod_{i=1}^{n} (1-(1+x)^{-\lambda})^{2\beta-1}}{(\boldsymbol{exp}(\alpha)-1)^n}\right) \quad (42)$$

$$log(L(\phi)) = \left(n\,ln\,2 + n\,ln\,\lambda + n\,ln\,\beta + \left(\alpha \sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta}\right) + n\,ln\,\alpha + \sum_{i=1}^{n} ln(x+1)^{-\lambda-1} + (2\beta-1)\sum_{i=1}^{n} ln(1-(1+x)^{-\lambda})\right) - (exp(\alpha)-1)^n \quad (43)$$

$$\frac{\partial(log(\phi))}{\partial\alpha} = \sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta} + \frac{n}{2} - \frac{n\,exp(\alpha)}{exp(\alpha)-1} \quad (44)$$

$$\left(\sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta} + \frac{n}{2} - \frac{n\,exp(\alpha)}{exp(\alpha)-1}\right) = 0 \quad (45)$$

$$\frac{\partial(log(\phi))}{\partial\beta} = \frac{n}{\beta} + \alpha\sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta}\,ln\,\beta + 2\sum_{i=1}^{n} ln(1-(1+x)^{-\lambda}) \quad (46)$$

$$\left(\frac{n}{\beta} + \alpha\sum_{i=1}^{n} (1-(1+x_i)^{-\lambda})^{2\beta}\,ln\,\beta + 2\sum_{i=1}^{n} ln(1-(1+x)^{-\lambda})\right) = 0 \quad (47)$$

$$\frac{\partial(log(\phi))}{\partial\lambda} = \frac{n}{\lambda} - \alpha\sum_{i=1}^{n} 2\beta(1-(1+x_i)^{-\lambda})^{2\beta-1}\left((1+x_i)^{-\lambda}\,ln\,\lambda\right) + n\,ln\,\lambda\,(2\beta-1)\sum_{i=1}^{n}\frac{1}{(1-(1+x_i)^{-\lambda})}\left((1+x_i)^{-\lambda}\,ln\,\lambda\right) \quad (48)$$

$$\left(\frac{n}{\lambda} - \alpha\sum_{i=1}^{n} 2\beta(1-(1+x_i)^{-\lambda})^{2\beta-1}\left((1+x_i)^{-\lambda}\,ln\,\lambda\right) + n\,ln\,\lambda\,(2\beta-1)\sum_{i=1}^{n}\frac{1}{(1-(1+x_i)^{-\lambda})}\left((1+x_i)^{-\lambda}\,ln\,\lambda\right)\right) = 0 \quad (49)$$

Equations (45), (47) and (49) cannot be solved analytically, statistical software like R can be used to simultaneously solve them numerically using iterative methods. Solutions of these equations yields the maximum likelihood estimate $\hat{\phi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ of $\phi = (\alpha, \beta, \lambda,)^T$

Simulation Studies

The performance of the maximum likelihood estimates for theZ-Ikum distribution parameters was evaluated using Monte Carlo simulation for a three parameter combinations. Different sample sizes (n = 25, 50, and 75)and some selected parameter values (α = 0.10, β = 1.50, λ = 0.6)were used to perform the simulation. Result of the simulation is presented in the table below.

Table 1. Average MLEs, Variance and MSE of the MLEs of parameters of Z-Ikum distribution with actual parameter values(α = 0.10, β = 1.50, λ = 0.6)

N	Estimates			Variance			MSE		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
25	0.0946	1.5153	0.0252	1.12e-07	9.25e-07	6.37e-07	2.75e-05	0.0002	0.0012
50	0.0948	1.5152	0.0252	1.11e-07	9.05e-07	5.45e-07	2.74e-05	0.0002	0.0012
75	0.0949	1.5150	0.0253	1.04e-07	8.50e-07	5.43e-07	2.70e-05	0.0002	0.0011

Model Comparison and Selection Criteria

In this case, we will consider the generally well known criteria such as Akaike Information Critareion (AIC),the Bayesian Information Criterion (BIC), the Consistant Akaike Information Cretarion (CAIC) and Hannan-Quinn Information Criterion (HQIC) to compare Z-Ikum distributionwith some existing distributions using three sets of real life data.

Data set 1

The first data set was taken from Amal et al. (2016). The data is referring to the time between failures for a repairable item. The data are as follows.

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

Data set 2
The second data is on failure time of 84 particular model aircraft windshield. Noor et al. (2017) recently studied the data. The data are as follows.
0.040, 1.866,2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.395, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223,3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663

Data set 3
This data was used and analyze byMusa et al. (2021)
0.42909, 0.83559, 0.85818, 0.79042, 0.67751, 0.99368, 0.88076, 0.88076, 0.93722, 0.74526, 0.76784, 0.82429, 0.77913, 0.68879, 0.98238, 0.71138, 0.76784, 0.51942, 0.77913, 0.70009, 0.54200, 0.75655, 0.86947, 0.99368, 0.76784, 0.92593, 0.80172, 0.46296, 0.76784, 0.76784, 0.48555, 0.89205, 0.36134, 0.65492, 0.79042, 0.84688, 0.80172, 0.64363, 0.42909, 0.74526, 0.80172, 0.48555, 0.67751, 0.75655, 0.47425, 0.94851, 0.92593, 0.63234, 0.93722, 0.73397, 0.71138, 0.90334, 0.72267, 0.99368, 0.63234, 0.45167, 0.65492, 0.92593, 0.41779, 0.72267, 0.75655, 0.47425, 0.94851, 0.48555, 0.63234, 0.54201, 0.89205, 0.80172, 0.65492, 0.46296, 0.75655, 0.84688, 0.47425, 0.65492, 0.51942, 0.39521, 0.91463, 0.37263, 0.66621, 0.49684, 0.86947, 0.82429, 0.63234, 0.41779, 0.74526, 0.80172, 0.12421, 0.16938, 0.15808, 0.09033, 0.88076, 0.37263, 0.66621, 0.18067, 0.85818, 0.83559, 0.64363, 0.49684, 0.76784, 0.77913, 0.89205, 0.35005, 0.99368, 0.60976, 0.75655, 0.77913, 0.65492, 0.39521, 0.74526, 0.82429, 0.92593, 0.97109, 0.68879, 0.94851, 0.7904, 0.99368, 0.71138, 0.49684, 0.06775, 0.91463, 0.97109, 0.91463, 0.86947, 0.76784, 0.86947, 0.79042, 0.79042, 0.41779, 0.77913, 0.99368, 0.51942, 0.67751, 0.84688, 0.80172, 0.90334, 0.80172, 0.90334, 0.71138, 0.63234, 0.74526, 0.54201, 0.39295, 0.76784, 0.71138, 0.67751, 0.63234, 0.77913, 0.85818, 0.63234, 0.99368, 0.55329, 0.75655, 0.82429, 0.37263, 0.56459, 0.15808, 0.45167, 0.64363, 0.67751, 0.99368, 0.92593, 0.67751, 0.84689, 0.68879, 0.76784, 0.50813, 0.68879, 0.82429, 0.67751, 0.28229, 0.49684, 0.62105, 0.66621, 0.62105, 0.86947, 0.89205, 0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392

Table 2: Performance of Z-Ikum distribution's goodness of fit using data set 1

	MODEL		
	Z-IKUM	**MOK-IKUM**	**IKUM**
AIC	10.3176	141.8897	110.4512
CAIC	11.2063	142.7786	111.3401
BIC	14.6194	146.1917	114.7531
HQIC	11.7198	143.2921	111.8535
Rank	**1**	**2**	**3**

Table 3: Performance of Z-Ikum distribution's goodness of fit using data set 2

	MODEL		
	Z-IKUM	**MOK-IKUM**	**IKUM**
AIC	467.8207	561.0219	804.5428
CAIC	468.1207	561.3219	804.8428
BIC	475.1131	568.3144	811.8353
HQIC	470.7522	563.9534	807.4743
Rank	**1**	**2**	**3**

Table 4: Performance of Z-Ikum distribution's goodness of fit using data set 3

	MODEL		
	Z-IKUM	**MOK-IKUM**	**IKUM**
AIC	311.0659	398.3712	665.0520
CAIC	311.1765	398.4817	665.1626
BIC	321.2604	408.5656	675.2464
HQIC	315.1823	402.4875	669.1683
Rank	**1**	**2**	**3**

From Table **2, 3** and **4** we can observe that Z-Ikum distribution has the least value of statistics AIC, CAIC, BIC and HQIC than MOK-Ikum and Ikum distributions. This implies that Z-Ikum distribution is more flexible in modeling real life data.

RESULTS AND DISCUSSION

A three parameter distribution called Zubair- Ikum (Z-Ikum) distribution is proposed in this research. The proposed distribution is an extension of Ikum distribution using the Zubair G-Family (2018) of continuous probability distribution. Some Structural properties such as Quantile functions, moments, moment generating functions, characteristic functions, order statistics of the new distributions was derived. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. From the hazard rate plot its evident that Z-Ikkum distribution has increase, decrease, Bathtub and inverted Bathtub shape property. Maximum likelihood estimate was used to estimate the Z-Ikum distribution parameters, Monte Carlo simulation also was carried out to evaluate the performance of MLE in estimating our distribution parameters.

CONCLUSION

From the simulation studies it was observed that, the estimator is consistent which indicate appropriately the choice of MLE. To illustrate the applications of Z-Ikum distribution with regards to modeling real data sets, analytical measure of goodness of fit of some information criterion such as Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion(BIC), and Hannan-Quinn information criterion (HQIC) was considered using three different real life data sets. From the results obtained, it shows that Z-Ikum distribution has the least values of these information criterions, hence it is regarded as best distribution among the competing distributions in modeling and predicting real word phenomenon.

REFERENCES

Alzaghal, A., Famoye. F., & Lee, C. (2013) Exponentiated T-X family of distributions with some applications. *International Journal of Probability Statistics*. 2(3),31–49.

Jalmar, M., F. Carrasco & Silvia, L. P. Ferrari (2010) A New Generalized Kumaraswamy Distribution.

Johnson, N. L., (1949). “Systems of frequency curves generated by methods of translation”, Biometrika, 36, 149–176.

Lishamol, T.,&Jiju, G. (2018) The marshall-olkin ikum distribution *Biometrics & Biostatistics International Journa*,.7, (1) ,10-16

Lee, C., Famoye, F., & Alzaatreh, A., (2013). “Methods for generating families of univariate continuous distributions in the recent decades”, *WIREs Computational Statistics*, vol. 5(3), pp.219-238.

Musa, Y., Muhammad, A.B., Usman, U.,and Zakari, Y.(2021). On the properties of Burr X-Topp Leone Distribution and its application. *Lapai Journal of Applied and Natural Sciences,* 6(1),114-120

Olalekan, A. B., Sani I. D., Abukakar Y. and Haruna M. J. (2021) A Type II Half Logistic Exponentiated-G Family of Distributions with Applications to Survival Analysis. *FUDMA Journal of Sciences (FJS) Vol. 5 No. 3, September, 2021, pp 177 - 190*

Zubair, A. (2018). The Zubair-G family of distributions: Properties and applications.*Annals of Data Science.*