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SOME PROPERTIES OF AN EXTENDED Γ_1-NON DERANGED PERMUTATION GROUP

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ABSTRACT

A Γ_1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}(p \geq 5$ and $p =$ prime) is a permutation group such that $\mathcal{G}_p = \{\omega_i \mid 1 \leq i \leq p-1\}$ where $\omega_i = ((1)(1+i)_{mP}(1+2i)_{mP} \dots (1+(p-1)i)_{mP})$. In this paper, an extension of $\mathcal{G}_p^{\Gamma_1}$ denoted by $\mathbb{G}_p^{\Gamma_1}$ is given in such a way that $\omega_p \in \mathcal{G}_p^{\Gamma_1}$ and an operation of addition '+' and multiplication '.' is define on $\mathbb{G}_p^{\Gamma_1}$ to show that $\mathbb{G}_p^{\Gamma_1}$ together with either addition '+' or multiplication '.' is an Abelian group. Also, the extended version $(\mathbb{G}_p^{\Gamma_1}, '+', '\cdot')$ is both a commutative ring with identity and a vector space. Furthermore, the range of the cycle $\omega_i \in \mathbb{G}_p^{\Gamma_1}$ is define and we show that the sum of all the ranges in the cycles is divisible by a prime number $p \geq 5$..

Keywords: Commutative ring, group, nonderanged permutation group, permutation group, vector space

INTRODUCTION

The concept of permutation may be traced to the effort of Lagrange in 1770 to find the algebraic solutions of polynomial equations. This concept can simply be described as a way in which numbers can be arranged or the rearrangement of elements of an ordered set mapped to itself in a one-to-one correspondence. However, a permutation group G according to (Beaumont & Peterson, 1955) is a set of permutations of G that forms a group under function composition and this type of group forms one of the oldest parts of group theory, (Dixon & Mortimer, 1996). In the study of group theory, its beauty and applications are in the algebraic properties; thus, this explains the vast interest of algebraic theorists in the properties of mathematical structures such as generalized groups, symmetry groups, permutation groups, etc. See (Ibrahim & Audu, 2007; Blass, 2011; Diaconescu, 2022).

Similarly, using some algebraic properties, (Adeniran et al., 2011) explore the properties of the generalized groups to show that some results that are true in classical groups are either generally true or only true in some special types of generalized groups; for instance, their studies showed that a Bol groupoid and a Bol quasi group can be constructed using a non Abelian generalized group. In the same vein, (Grigorchuk & Medynets, 2014) studied topological full group and noted that the structure is similar to a union of permutational wreath products of finite groups.

Over the years, studies on deranged permutation groups have been done, for instance, (Calkin et al., 2000) looked at the Lampert-Slater sequence but used a free mapping of $[\,n+1]$ which omits exactly k elements from its image and discovered that the sequence exhibits oscillatory behavior but little could be seen of non deranged permutation. However, Γ_1 nonderanged permutation group $\mathcal{G}_p^{\Gamma_1}$ was introduced by (Ibrahim et al.,2016), who constructed and studied the representation of the group and further noted that the group is an $\mathbb{F}G$-Module. Similarly, (Garba et al., 2017) showed that the Young tabulaux of the Γ_1-nonderanged permutation group $\mathcal{G}_p^{\Gamma_1}$ is nonstandard. Garba et al., (2018) studied some topological properties of Γ_1-nonderanged permutation group by constructing a topology on the structure and proved that the topology is bihomogeneous. Also, using the embracing sum, the ascent block of Γ_1 nonderanged permutation group $\mathcal{G}_p^{\Gamma_1}$ was studied by (Ibrahim et al., 2022). More recently, (Jega, 2022) constructed a new algebraic structure of permutation group with prime order that has fixed elements using modular arithmetic and permutation computation in studying some of the properties of the structure.

In this article, we extend the Γ_1- nonderanged permutation group $\mathcal{G}_p^{\Gamma_1}$ by allowing ω_p to be an element in $\mathcal{G}_p^{\Gamma_1}$ and denote the extended version by $\mathbb{G}_p^{\Gamma_1}$. An operation of addition '+' and multiplication '$\cdot$' are also defined on $\mathbb{G}_p^{\Gamma_1}$ and proved some properties of space $(\mathbb{G}_p^{\Gamma_1}, +, .)$. Meanwhile, for easy understanding of this article, we present some basic concepts, examples and definitions below.

Preliminary

Definition 1 (Group). A group consists of a set $\mathbb{G}$, together with a rule for combining any two elements a, b of $\mathbb{G}$ to form another element of $\mathbb{G}$, written $a * b$; this rule must satisfy certain axioms: (closure, identity, associativity and inverse) it a non-empty set $\mathbb{G}$ on which there is a binary operation $(a^* b) \to ab$ such that

i. If a and b belong to $\mathbb{G}$ then $a * b$ is also in $\mathbb{G}$ (closure),
ii. $a(b * c) = (a * b)c$ for all a, b, c in $\mathbb{G}$ (associativity),
iii. there is an element $1 \in \mathbb{G}$ such that $a * 1 = 1 * a = a$ for all $a \in \mathbb{G}$ (identity),
iv. if $a \in \mathbb{G}$, then there is an element $a^{-1} \in \mathbb{G}$ such that $a * a^{-1} = a^{-1} * a = 1$ (inverse).

Definition 2 (Permutation group). For a finite set X, let $|X|$ denote the number in elements of X. For any non-empty finite set X with $|X| = n$, the set of all bijective mappings on X to itself is denoted by S_n and is called the symmetric group on X. A permutation of a set X is a bijective function $\rho: X \to X$. It is a quantity or function that carries n indices or variables (where each can run from $1, \dots, N$).

Definition 3 (Γ_1 nonderanged permutation group $\mathcal{G}_p^{\Gamma_1}$). Let Γ_1 be a non-empty, totally ordered and finite subset of $\mathbb{N}$. Let p be a prime number greater than or equal to 5 such that
$\mathcal{G}_p^{\Gamma_1} = \{\omega_1, \dots, \omega_{p-1}\}$
where ω_i is a bijection on Γ_1 written in the form
$\omega_i = ((1)(1+i)_{mP}(1+2i)_{mP} \dots (1+(p-1)i)_{mP})$.

Then, $\mathcal{G}_p^{\Gamma_1}$ is said to be a Γ_1-nonderanged permutaions.

Remark 1. $\mathcal{G}_p^{\Gamma_1}$ together with a natural permutation composition is a group. Thus, $\mathcal{G}_p^{\Gamma_1}$ is called a Γ_1– nonderanged permutaion group.

Definition 4. The n^{th} successor in a cycle ω_i is given by
$a_n = (1 + (n-1)i)mP$
where $1 \le n \le p$ and $1 \le i \le p-1$.

Example 1. For $p = 5, \mathcal{G}_5^{\Gamma_1} = \{(12345), (13524), (14253), (15432)\}$, where $\omega_1 = (12345), \omega_2 = (13524), \omega_3 =$ (14253), $\omega_4 = (15432)$, each with length 5 .

Definition 5 (Vector space). A vector space (or linear space) over a field $\mathbb{F}$ is a nonempty set X with two binary operations
$+: X \times X \to X$ (vector addition), and
$\cdot: \mathbb{F} \times X \to X$ (scalar multiplication)
such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$, the following properties hold:

i. $x + y \in X$; (closure under +)
ii. $x + y = y + x$; (commutative under +)
iii. There exists a unique element in X, denoted by 0 , such that $x + 0 = 0 + x = x$ (identity under +)
iv. Associated with each $x \in X$ is a unique element in X, denoted by $-x$, such that $x + (-x) = -x + x = 0$; (inverse under +)
v. $(x + y) + z = x + (y + z)$; (associativity under +)
vi. $\alpha \cdot x \in X$ (closure under $\cdot$)
vii. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;
viii. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$;
ix. $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
x. $1 \cdot x = x$ for all $x \in X$.

Definition 6 (Ring). A nonempty set R with two binary operations + and $\cdot$ is said to be a ring if:
i) $(R, +)$ is an Abelian group, ii) $a, b \in R$ implies that $a \cdot b \in R$.
iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for $a, b, c \in R$.
iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ and
v) $(b + c) \cdot a = b \cdot a + c \cdot a$, for $a, b, c \in R$.
If in addition $a \cdot b = b \cdot a$, then, R is said to be a commutative ring.

Extended Γ_1-non deranged permutations

Definition 7 (Extended $\mathcal{G}_p$). Let Γ_1 be a non-empty, totally ordered and finite subset of $\mathbb{N}$. An extended Γ_1-nonderanged permutatutions denoted by $\mathbb{G}_p^{\Gamma_1}$ is given by
$\mathbb{G}_p^{\Gamma_1} = \{\omega_1, \omega_2, \dots, \omega_p\}$
where p is a prime number greater than or equal to 5 and that ω_i is a bijection on Γ_1 called cycle and written in the form
$\omega_i = ((1)(1+i)_{mP}(1+2i)_{mP} \dots (1+(p-1)i)_{mP})$.

Example 2. For $p = 5, \mathbb{G}_5^{\Gamma_1} = \{(12345), (13524), (14253), (15432), (11111)\}$. Where $\omega_1 = (12345), \omega_2 = (13524), \omega_3 = (14253), \omega_4 = (15432), \omega_5 = (1111)$, each with length 5 .

Definition 8. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1 - non deranged permutations and $\omega_i, \omega_j \in \mathbb{G}_p^{\Gamma_1}$. Then, the addition '+' and multiplication '.' in $\mathbb{G}_p^{\Gamma_1}$ are defined as
$\omega_i + \omega_j \quad := \omega_{(i+j)mP}$
$\omega_i \cdot \omega_j \quad := \omega_{(i\times j)mP}$
respectively.

Lemma 1. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1 - nonderanged permutations. Then, $\omega_{(i+j)mP} = \omega_i + \omega_j - \omega_p$, where $1 \le i, j \le p$ and p (prime ≥ 5).

Proof. Let $\omega_i, \omega_j, \omega_p \in \mathbb{G}_p^{\Gamma_1}$, by Definition (7), we have

$$\omega_i = ((1)(1+i)_{mP}(1+2i)_{mP} \dots (1+(p-1)i)_{mP}) \quad (1)$$
$$\omega_j = ((1)(1+j)_{mP}(1+2j)_{mP} \dots (1+(p-1)j)_{mP}) \quad (2)$$
$$\omega_p = ((1)(1+p)_{mP}(1+2p)_{mP} \dots (1+(p-1)p)_{mP}) = \big((1)(1)(1)(1)(1)\big) \quad (3)$$
$$\omega_{(i+j)mP} = ((1)(1+(i+j))_{mP}(1+2(i+j))_{mP} \dots (1+(p-1)(i+j))_{mP}) \quad (4)$$

Adding Equation (1) and Equation (2) we have

$$\omega_i + \omega_j = ((2)(2+i+j)_{mP}(2+2(i+j))_{mP} \dots (2+(p-1)(i+j))_{mP}) \quad (5)$$

Now, subtracting Equation (3) from Equation (5), we have

$$\omega_i + \omega_j - \omega_p = ((2)(2+i+j)_{mP}(2+2(i+j))_{mP} \dots (2+(p-1)(i+j))_{mP}) - ((1)(1)(1)(1)(1))$$
$$= ((1)(1+(i+j))_{mP}(1+2(i+j))_{mP} \dots (1+(p-1)(i+j))_{mP})$$

Example 3. For $p = 5, \mathbb{G}_5^{\Gamma_1} = \{(12345), (13524), (14253), (15432), (11111)\}$, where $\omega_1 = (12345), \omega_2 = (13524), \omega_3 = (14253), \omega_4 = (15432), \omega_5 = (11111)$. Adding any two cycles in $\mathbb{G}_p^{\Gamma_1}$, in this case $p = 5$, say $\omega_1 + \omega_2 = \omega_{1+2} = \omega_3$, which by Lemma (1) can be practically viewed as

$$\omega_3 = \omega_1 + \omega_2 - \omega_5$$
$$= (12345) + (13524) - (11111)$$
$$= (14758) = (14253) \text{ since we are in } \mathbb{G}_5^{\Gamma_1}$$
$$= \omega_3$$

Also, to multiply two cycles in $\mathbb{G}_p^{\Gamma_1}$. Let $\omega_2 \cdot \omega_3 = \omega_{(2\times3)mP} = \omega_{(6)mP} = \omega_1$. This can be viewed by using two line notation as below

$$\omega_2 \cdot \omega_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \omega_1$$

Also,

$$\omega_3 \cdot \omega_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \omega_1$$

Corollary 1. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1 - nonderanged permutations. Then, $\omega_i + \omega_j = \omega_i + \omega_j - \omega_p = \omega_{(i+j-p)mP}$, where $1 \leq i,j \leq p$.

Definition 9. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations and $\omega_i \in \mathbb{G}_p^{\Gamma_1}$. Then, the inverse of ω_i under the natural permutation composition '. ' denoted by ω_i^{-1} is given by

$$\omega_i^{-1} = \begin{cases} \omega_j, & \text{for i} \neq \text{p and } 1 \leq j < p \\ \omega_p, & \text{for i = p.} \end{cases}$$

where $\omega_i^{-1} \cdot \omega_j = \omega_1$.

Example 4. For $p = 5, \mathbb{G}_5^{\Gamma_1} = \{(12345), (13524), (14253), (15432), (11111)\}$, let $i = 2$. Then, to find ω_2^{-1}, we know that $\omega_2 = (13524)$ which in two-line notation can be written as $\omega_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$, and $\omega_2^{-1} = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} = \omega_3$.

Thus, $\omega_2 \cdot \omega_3 = \omega_{2\times 3} = \omega_1$ in mod 5 . Hence ω_3 is an inverse of ω_2 in $\mathbb{G}_5^{\Gamma_1}$.

And for $p = 7$, we have

$\mathbb{G}_7^{\Gamma_1} = \{(1234567), (1357246), (1473625), (1526374), (1642753), (1765432), (1111111)\}$, let $i = 2$ again where $\omega_2 = (1357246)$ which in two-line notation gives $\omega_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$, and

$\omega_2^{-1} = \begin{pmatrix} 1 & 3 & 5 & 7 & 2 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \end{pmatrix} = \omega_4$.

Hence, $\omega_2 \cdot \omega_4 = \omega_{2\times 4} = \omega_1$ in mode 7. Thus, ω_4 is an inverse of ω_2 in $\mathbb{G}_7^{\Gamma_1}$.

Lemma 2. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations. Then, $(\mathbb{G}_p, +)$ is an Abelian group.

Proof. Let $\omega_i, \omega_j, \omega_k \in \mathbb{G}_p^{\Gamma_1}$, where $1 \leq i,j,k \leq p$. By definition (8), $\omega_i + \omega_j = \omega_{(i+j)mP} \in \mathbb{G}_p^{\Gamma_1}$. Thus, $\mathbb{G}_p^{\Gamma_1}$ is closed under '+'. For associativity, $(\omega_i + \omega_j) + \omega_k = \omega_{(i+j)mP} + \omega_k = \omega_{((i+j)+k)mP} = \omega_{(i+(j+k))mP} = \omega_i + (\omega_j + \omega_k)$. For every $\omega_i \in \mathbb{G}_p^{\Gamma_1}, \omega_i + \omega_p = \omega_{(i+p)mP} = \omega_i$ which implies ω_p is an identity under '+'. Similarly, for every $\omega_i \in \mathbb{G}_p$ there exist $\omega_i' \in \mathbb{G}_p^{\Gamma_1}$ such that $\omega_i + \omega_i' = \omega_p$ which means that ω_i' is the inverse of ω_i under '+'. Additionally, $\omega_i + \omega_j = \omega_{(i+j)mP} = \omega_{(j+i)mP} = \omega_j + \omega_i$.

Lemma 3. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations. Then, $(\mathbb{G}_p, \cdot)$ is an Abelian group.

Proof. Let $\omega_i, \omega_j, \omega_k \in \mathbb{G}_p^{\Gamma_1}$, where $1 \leq i,j,k \leq p$. Then, we have

i. Closure : $\omega_i \cdot \omega_j = \omega_{(i\times j)mP} \in \mathbb{G}_p^{\Gamma_1}$
ii. Associativity : $\omega_i \cdot (\omega_j \cdot \omega_k) = \omega_i \cdot \omega_{(j\times k)mP}$
$$= \omega_{i\times(j\times k)mP} = \omega_{((i\times j)\times k)mP} = (\omega_i \cdot \omega_j) \cdot \omega_k$$
iii. Existence of identity : For any $\omega_i \in \mathbb{G}_p^{\Gamma_1} \exists \omega_1 \in \mathbb{G}_p^{\Gamma_1}$ such that $\omega_i \cdot \omega_1 = \omega_{(i\times 1)mP} = \omega_i$
iv. Existence of inverse : By Definition (9) for any $\omega_i \in \mathbb{G}_p^{\Gamma_1} \exists \omega_i^{-1} \in \mathbb{G}_p^{\Gamma_1}$ such that $\omega_i \cdot \omega_i^{-1} = \omega_1$
v. Commutativity : $\omega_i \cdot \omega_j = \omega_{(i\times j)mP} = \omega_{(j\times i)mP} = \omega_j \cdot \omega_i$

Proposition 1. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations. Then $(\mathbb{G}_p^{\Gamma_1}, +, \cdot)$ is a commutative ring with identity.

Proof. From Lemma (2) we know that $(\mathbb{G}_p^{\Gamma_1}, +)$ is an Abelian group. Let $\omega_i, \omega_j, \omega_k \in \mathbb{G}_p^{\Gamma_1}$. Then, $\omega_i \cdot \omega_j = \omega_{(i\times j)mP} = \omega_l \in \mathbb{G}_p^{\Gamma_1}$ where $l \in [1, p]$ which implies that $\mathbb{G}_p^{\Gamma_1}$ is closed under ' $'$. Associativity under ', follows from the fact that $(\omega_i \cdot \omega_j) \cdot \omega_k = \omega_{(i\times j)mP} \cdot \omega_k = \omega_{(i\times j\times k)mP} = \omega_{i\times(j\times k)mP} = \omega_i \cdot (\omega_j \cdot \omega_k)$. For left distributivity,

$$\begin{aligned} \omega_i \cdot (\omega_j + \omega_k) &= \omega_i(\omega_{(j+k)mP}) = \omega_i \cdot \omega_{(j+k)mP} \\ &= \omega_{i\times(j+k)mP} = \omega_{(i\times j+i\times k)mP} \\ &= \omega_{(i\times j)mP} + \omega_{(i\times k)mP} \\ &= \omega_i \cdot \omega_j + \omega_i \cdot \omega_k \end{aligned}$$

For right distributivity,

$$\begin{aligned} (\omega_i + \omega_j) \cdot \omega_k &= \omega_{(i+j)mP} \cdot \omega_k \\ &= \omega_{((i+j)\times k)mP} \\ &= \omega_{(i\times k+j\times k)mP} \\ &= \omega_{(i\times k)mP} + \omega_{(j\times k)mP} \\ &= \omega_i \cdot \omega_k + \omega_j \cdot \omega_k \end{aligned}$$

For commutativity, $\omega_i \cdot \omega_j = \omega_{(i\times j)mP} = \omega_{(j\times i)mP} = \omega_j \cdot \omega_i$. For identity, given any $\omega_i \in \mathbb{G}_p^{\Gamma_1}$ there exist $\omega_1 \in \mathbb{G}_p$ such that $\omega_1 \cdot \omega_i = \omega_{(1\times i)mP} = \omega_i = \omega_{(i\times 1)mP} = \omega_i \cdot \omega_1$

Theorem 1. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations and $\mathbb{F}_p$ be a finite fields of prime cardinality. Define operations of addition and multiplication as

$$+: \mathbb{G}_p^{\Gamma_1} \times \mathbb{G}_p^{\Gamma_1} \to \mathbb{G}_p^{\Gamma_1},\ (\omega_i, \omega_j) \mapsto \omega_{(i+j)mP}$$
$$\cdot: \mathbb{F}_p \times \mathbb{G}_p^{\Gamma_1} \to \mathbb{G}_p^{\Gamma_1},\ (\alpha, \omega_j) \mapsto \omega_{(\alpha\times j)mP}$$

Then, $(\mathbb{G}_p^{\Gamma_1}, +, \cdot)$ is a vector space over $\mathbb{F}_p$ with $1 \leq i,j \leq p$.

Proof. We know from Lemma (2) that $(\mathbb{G}_p^{\Gamma_1},+)$ is an Abelian group, hence the closure, associativity, existence of identity, existence of inverse and commutativity holds. Now, let $\alpha,\beta \in \mathbb{F}_p$, then $\alpha \cdot \omega_i = \omega_{(\alpha\times i)mP} \in \mathbb{G}_p^{\Gamma_1}$, this is because $(\alpha \times i)$ is reduced to mod$p$. Also,

$$\begin{aligned}
\alpha \cdot (\omega_i + \omega_j) &= \alpha(\omega_{(i+j)mP}) = \omega_{\alpha\times(i+j)mP} \\
&= \omega_{(\alpha\times i+\alpha\times j)mP} = \omega_{(\alpha\times i)mP} + \omega_{(\alpha\times j)mP} \\
&= \alpha\cdot\omega_i + \alpha\cdot\omega_j \\
(\alpha+\beta)\cdot\omega_i &= \omega_{((\alpha+\beta)\times i)mP} = \omega_{(\alpha\times i+\beta\times i)mP} \\
&= \omega_{(\alpha\times i)mP} + \omega_{(\beta\times i)mP} = \alpha\cdot\omega_i + \beta\cdot\omega_i \\
(\alpha\beta)\cdot\omega_i &= \alpha\omega_{(\beta\times i)mP} = \alpha(\beta\cdot\omega_i) \\
1\cdot\omega_i &= \omega_{(1\times i)} = \omega_i
\end{aligned}$$

Definition 10 (Range of a cycle of $\mathbb{G}_p^{\Gamma_1}$). Let $\omega_i \in \mathbb{G}_p^{\Gamma_1}$. The range of a cycle ω_i denoted by $\Delta_l^f(\omega_i)$ is defined as the difference between last and first successor in a cycle ω_i.

Lemma 4. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-non-deranged permutations. For any prime $p \geq 5$, there exist a cycle $\omega_p \in \mathbb{G}_p^{\Gamma_1}$ such that

$$\Delta_l^f(w_p) = p$$

Proof. Let $\omega_p \in \mathbb{G}_p^{\Gamma_1}$, then

$$\begin{aligned}
\Delta_l^f(\omega_p) &= \Delta_l^f(11111) \\
&= (1-1) \text{ by definition of } \Delta_l^f(\omega_i) \\
&= 0
\end{aligned}$$

But 0 is equivalent to p in mod p. Thus, $\Delta_l^f(\omega_p) = p$

Example 5. For $p = 5, \mathbb{G}_5^{\Gamma_1} = \{(12345),(13524),(14253),(15432),(11111)\}$. Where $\omega_1 = (12345), \omega_2 = (13524), \omega_3 = (14253), \omega_4 = (15432), \omega_5 = (11111)$. Thus,

$$\begin{aligned}
\Delta_l^f(\omega_1) &= 5-1 = 4 \\
\Delta_l^f(\omega_2) &= 4-1 = 3 \\
\Delta_l^f(\omega_3) &= 3-1 = 2 \\
\Delta_l^f(\omega_4) &= 2-1 = 1 \\
\Delta_l^f(\omega_5) &= 1-1 = 5
\end{aligned}$$

Proposition 2. Let $\mathbb{G}_p^{\Gamma_1}$ be an extetended Γ_1-nonderanged permutations. For any prime number $p \geq 5$ and $\omega_i \in \mathbb{G}_p^{\Gamma_1}$ with $1 \leq i \leq p$,

$$\sum_{i=1}^{p} \Delta_l^f(\omega_i)$$

is divisible by p.

Proof. Recall that $\omega_i = ((1)(1+i)_{mP}(1+2i)_{mP} \dots (1+(p-1)i)_{mP})$ and observes that $((1+(p-1)i)_{mP}) = (1+(p-i))$, for $1 \leq i \leq p$. Thus, we have

$$\begin{aligned}
\omega_1 &= ((1)(2)(3)\dots(1+(p-1))) \\
\omega_2 &= ((1)(3)(5)\dots(1+(p-2))) \\
\omega_3 &= ((1)(4)(7)_{mP}\dots(1+(p-3))) \\
&\vdots \\
\omega_{p-1} &= ((1)(1+(p-1))(1+2(p-1))_{mP}\dots(1+(p-(p-1)))) \\
&= ((1)(1+(p-1))(1+(p-2))\dots(2)) \\
\omega_p &= ((1)(1)(1)\dots(1))
\end{aligned}$$

By definition (10), we have

$\Delta_l^f(\omega_1) = p-1, \Delta_l^f(\omega_2) = p-2, \Delta_l^f(\omega_3) = p-3, \dots, \Delta_l^f(\omega_{p-1}) = 1$ and $\Delta_l^f(\omega_p) = p$ by Lemma (4). Now,

$$\begin{aligned}
\sum_{i=1}^{p} \Delta_l^f(\omega_i) &= p + (p-1) + (p-2) + (p-3) + \dots + 1 \\
&= \frac{1}{2}p(p+1).
\end{aligned}$$

Corollary 2. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1 - nonderanged permutations and ω_i be any cycle in $\mathbb{G}_p^{\Gamma_1}$. Then

$$\Delta_l^f(\omega_i) = \begin{cases} p-i, & \text{for } i \neq p, \\ p, & \text{for } i = p. \end{cases}$$

Proposition 3. Let $\mathbb{G}_p^{\Gamma_1}$ be an extended Γ_1-nonderanged permutations and let $\alpha: \mathbb{G}_p^{\Gamma_1} \times \mathbb{G}_p^{\Gamma_1} \to \mathbb{G}_p^{\Gamma_1}$ and $\beta: \mathbb{G}_p^{\Gamma_1} \to \mathbb{G}_p^{\Gamma_1}$ be the mappings defined by $\alpha(\omega_i,\omega_j) = \omega_i\omega_j$ and $\beta(\omega_i) = \omega_i^{-1} \forall \omega_i,\omega_j \in \mathbb{G}_p^{\Gamma_1}$ respectively. Then, $\alpha(\mathbb{G}_p^{\Gamma_1},\mathbb{G}_p^{\Gamma_1}) \subset \mathbb{G}_p^{\Gamma_1}$ and $\beta(\mathbb{G}_p^{\Gamma_1}) \subset \mathbb{G}_p^{\Gamma_1}$.

Proof. Let $\omega_i,\omega_j \in \mathbb{G}_p^{\Gamma_1}$. Then,

$$\alpha(\mathbb{G}_p^{\Gamma_1}, \mathbb{G}_p^{\Gamma_1}) := \alpha(\omega_i, \omega_j)$$
$$= \omega_i \omega_j$$
$$= \omega_{(i+j)mP} \in \mathbb{G}_p^{\Gamma_1}$$

Thus, $(\mathbb{G}_p^{\Gamma_1}, \mathbb{G}_p^{\Gamma_1}) \subset \mathbb{G}_p^{\Gamma_1}$. Also, $\beta(\mathbb{G}_p^{\Gamma_1}) = \beta(\omega_i) = \omega_i^{-1} \in \mathbb{G}_p^{\Gamma_1}$ since by Definition (9) $\omega_i^{-1} = \omega_k \in \mathbb{G}_p^{\Gamma_1}$ for some $k \in [1, p)$ and $\omega_i^{-1} = \omega_p \in \mathbb{G}_p^{\Gamma_1}$ for $i = p$. Thus, $\beta(\mathbb{G}_p^{\Gamma_1}) \subset \mathbb{G}_p^{\Gamma_1}$.

CONCLUSION

In this paper, we give an extension of Γ_1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}$ and defined an operations of addition '+' and multiplication '·' on an extended $\mathcal{G}_p^{\Gamma_1}$ which we denoted by $\mathbb{G}_p^{\Gamma_1}$. In furtherance, we show that the pairs $(\mathbb{G}_p^{\Gamma_1}, +)$ and $(\mathbb{G}_p^{\Gamma_1}, \cdot)$ are all Abelian groups and that the triplets $(\mathbb{G}_p^{\Gamma_1}, +, \cdot)$ was shown to be both a commutative ring with identity and a vector space. In addition, the range of a cycle ω_i in $\mathbb{G}_p^{\Gamma_1}$ was defined with examples and show that the summation of all the ranges of cycles in $\mathbb{G}_p^{\Gamma_1}$ is divisible by a prime number ($p \geq 5$).

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