FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 7 No. 5, October, 2023, pp 228 - 233
DOI: https://doi.org/10.33003/fjs-2023-0705-2016

NUMERICAL SOLUTION TO OPTIMAL CONTROL PROBLEMS USING COLLOCATION METHOD VIA PONTRYAGIN'S PRINCIPLE

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ABSTRACT

In this study, Lucas polynomial approximate solution is considered to develop a collocation technique for solving optimal control problems with implementation in block using forward backward sweep method. The collocation block method developed is stable and convergent. The method is implemented using MATLAB code, and the examples show that forward-backward sweep methods with the collocation method is an efficient technique for solving optimal control problems as compared with some existing methods.

Keywords: Block method, Optimal control problem, Pontryagin's principle

INTRODUCTION

Optimal control is a process of determining the inputs to a dynamical system that minimize or maximize a specific performance index at the same time satisfying any constraints on the motion of the system (Adamu, 2023). Optimal Control Problems (OCPs) are used to model many classes of phenomena, such as population dynamics problem, continuum mechanics of materials with memory, economic problems, the spread of epidemics, non-local problems of diffusion and heat conduction problem (Maleknejad & Ebrahimzadeh, 2014).

OC are applied in fields of ordinary differential equations, partial differential equations, discrete equations, stochastic differential equations, integro difference equations, combination of discrete and continuous systems, to solve problems of physical systems, aerospace, economics and management, biology and medicine (Rodrigues *et al.*, 2014). In this study, we consider OCPs that optimizes the performance index

$$J[x(\cdot),u(\cdot)] = \int_{t_0}^{t_f} f(x,x(t),u(t))dt \qquad (1)$$

subject to

$$\dot{x} = g(t,x(t),u(t)), x(t_0) = x_0,\ x(t_f) \qquad (2)$$

Numerical methods for solving equations (1) and (2) have enable the simulation of highly complex real world scenarios (Rodrigues *et al.*, 2014). Pontryagin's maximum principles which gives the necessary and sufficient condition for the solution of optimal control problems has been considered by many researchers (Adamu, 2023; Garret, 2015). The conditions are derived from the Hamilton, H given as

$$H(t,x,\lambda,u) = f(t,x,u) + \lambda g(t,x,u) \qquad (3)$$

where λ denotes the adjoint and is dependent of t, x and u.

$$\frac{\partial H}{\partial u} = 0 \text{ at } u^* \quad \Rightarrow f_u + \lambda g_u = 0 \quad \text{(optimality condition)}$$

$$\dot{\lambda}(t) = \frac{\partial H}{\partial x} = -(f_x + \lambda g_x) \qquad \text{(Adjoint condition)}.$$

$$\left.\begin{array}{r}\lambda(t_f) = 0\\ \lambda(t_f) = \dot{\varphi}\left(x(t_f)\right) \text{ if with pay off term.}\end{array}\right\}$$

(Transversality condition)

Notable among the forward-backward sweep based method in literature are: classical Runge Kutta method (Garret, 2015), Eurler's method and trapezoidal method (Lenhart & Workman, 2007). Garret (2015) applied forward-backward sweep method using Runge-Kutta 4 routine to solve optimal control problems via Pontryagin's principle. Garret (2015) compare the acuracy of FBS methods with other methods used to solved optimal control problems with initial value problems using classical Runge Kutta method via Pontryagin's principle and reported that FBS methods are better.

Runge Kutta collocation methods have been efficient in solving ordinary differential equation, it has the traditional advantages of being self-starting and does not require development of separate predictors or starting values, hence Runge Kutta methods are cheaper to implement than the traditional linear multistep method. Some of the authors that worked on Runge Kutta collocation methods include: (Garret, 2015), (Landis, 2005), (Butcher, 2008) among others. (Alkali *et al.*, 2023) and (Aizenofe *et al.*, 2021) construct collocation methods for the solution of ordinary differential equations of initial value problems.

A new collocation method is developed in this study for the numerical solution of optimal control problems modelled in ordinary differential equation. This method is an improvement to the CRKM used by (Rodrigues *et al.*, 2014) and (Garret, 2015) for the solution of optimal control problems.

The optimal control technique, optimized the given performance index and if there is change in the state, only the code needed to be adjusted in Forward Backward Sweep. Therefore, the method developed in this research is faster, more computationally stable, possess better rate of convergence and economical to implement. An efficient MATLAB code is developed which is easy and less expensive to implement.

MATERIALS AND METHODS

Development of the block method

We approximate the exact solution $y(x)$ within the integration interval $[t_0, t_f]$ using the partition $[t_0, t_f] = [t_0 = t_1 <\ldots< t_n < t_{n+1} <\ldots< t_N = f_f]$ by the polynomial

$y(t) = \sum_{n=0}^{k} a_n L(x)\,,\; a \le x \le b$ (4)

where a_n are unknown parameters to be determined, and $L_n(x)$ indicates the Lucas polynomials. Lucas polynomials are defined recursively as follows (Gümgüm *et al.*, 2018)
$L_{n+1}(x) = xL_{n-1}(x) + L_0(x), n \ge 1$

where
$L_0(x) = 2$ and $L_1(x) = x$.
Evaluate (4) at y_n and its derivatives at $x = \{x_{n+i} | i = 0,1,\ldots,k\}$, impose the following conditions on (4) give
$y(x_{n+i}) = y_{n+i},\quad i = 0,1,\ldots,r$
$y'(x_{n+i}) = f_{n+i},\quad i = 0,1,\ldots,s$

gives a system of equation
$VA = Y$ (5)
where
$Y = [y_0\ f_n\ f_{n+1} \cdots f_{n+s}]$

$A = [a_0\ a_1 \ldots a_{s+1}]$

$$V = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^k \\ 0 & 1 & 2x_n & \cdots & kx_n^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2x_{n+s} & \cdots & kx_{n+s}^{k-1} \\ 0 & 0 & 2 & \cdots & k(k-1)x_n^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 2 & \cdots & k(k-1)x_{n+s}^{k-2} \end{bmatrix},$$

Solving (5) for the unknown parameters by applying Crammers' rule and then substitute the result into the approximate solution gives an implicit linear multistep method in the form
$$\sum_{j=0}^{k} \alpha_j(t) X_{n+j} = h\sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^2 \sum_{j=1}^{k} \gamma_j(t) g_{n+j} \quad (6)$$

where $\alpha_j(t)$ and $\beta_j(t)$ are polynomial (Musa *et al.*, 2010).
We then resolve (6) into block method in the form
$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + B^{(0)}hF_m + B^{(1)}hF_{m+1} + \gamma^{(1)}h^2 G_{m+1} \quad (7)$$

where $A^{(1)}$ is $s \times s$ identity matrix $A^{(0)}, B^{(0)}, B^{(1)}$ are $s \times s$ matrices
$Y_{m+1} = [y_{n+1}\ y_{n+2} \cdots y_{n+s}]^T, Y_m = [y_{n-1}\ y_{n-2} \cdots y_n]^T,$

$G_m = [g_{n-1}\ g_{n-2} \cdots g_n]^T,\ G_{m+1} = [g_{n+1}\ g_{n+2}\ \cdots\ g_{n+r}]^T,$

$F_{m+1} = [f_{n+1}\ f_{n+2} \cdots f_{n+s}]^T, F_m = [f_{n-1}\ f_{n-2} \cdots f_n]^T$

Conversion of Block Method to Runge Kutta form
Transforming (7) into the Runge-Kutta type of the form
$Y_{m+1} = Y_m + h\sum \gamma_i K_i,$ (8)

$K_i = f(x_n + C_i h, Ym + i)$

$Y_{m+i} = A^{(0)}u_m + B^{(0)}hfm + B^{(1)}hf_{m+i}, i = 0,1,\ldots$
where
$C_i = \sum_{i=0}^{k} \beta_i$ (9)
Writing (8) in Butcher's table gives

$$\left[\begin{array}{c|c} C & A \\ \hline & b^T \end{array}\right] = \left[\begin{array}{c|cccc} C_1 & a_{11} & a_{12} & \cdots & a_{1k} \\ C_2 & a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ C_k & a_{k1} & a_{k2} & \cdots & a_{kk} \\ \hline & b_1 & b_2 & \cdots & b_k \end{array}\right]$$

Specification of the Method
We considered Lucas polynomial approximate solution of the form

$y(x) = \sum_{n=0}^{6} a_n L(x)$ (10)

with first and second derivative given as
$y'(x) = \sum_{n=1}^{6} a_n L'(x)$ (11)

$y''(x) = \sum_{n=2}^{6} a_n L''(x)$ (12)

The collocation points are determined using
$$x_i = \frac{2n-1}{n^2+1}, n = 1,3,5$$

Interpolating (10) at point x_n, and collocating (11) at $\left[x_n, x_{n+\frac{1}{10}}, x_{n+\frac{5}{10}}, x_{n+\frac{9}{10}}, x_{n+1}\right]$ and (12) at points x_{n+1} to give

$$\begin{bmatrix} 8 & 9x_n & 14x_n^2 & 16x_n^3 & 7x_n^4 & x_n^5 & x_n^6 \\ 0 & 9 & 28x_n & 48x_n^2 & 28x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 9 & 28x_{n+\frac{1}{10}} & 48x^2_{n+\frac{1}{10}} & 28x^3_{n+\frac{1}{10}} & 5x^4_{n+\frac{1}{10}} & 6x^5_{n+\frac{1}{10}} \\ 0 & 9 & 28x_{n+\frac{5}{10}} & 48x^2_{n+\frac{5}{10}} & 28x^3_{n+\frac{5}{10}} & 5x^4_{n+\frac{5}{10}} & 6x^5_{n+\frac{5}{10}} \\ 0 & 9 & 28x_{n+\frac{9}{10}} & 48x^2_{n+\frac{9}{10}} & 28x^3_{n+\frac{9}{10}} & 5x^4_{n+\frac{9}{10}} & 6x^5_{n+\frac{9}{10}} \\ 0 & 9 & 28x_{n+1} & 48x_{n+1}^2 & 28x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 \\ 0 & 0 & 28 & 96x_{n+1} & 84x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{10}} \\ f_{n+\frac{5}{10}} \\ f_{n+\frac{9}{10}} \\ f_{n+1} \\ g_{n+1} \end{bmatrix}$$

Solving the system using Crammer's rule and then substituting into the approximate solution to give the continuous nonlinear scheme, and evaluating at the continuous scheme at $\left[x_{n+\frac{1}{10}}, x_{n+\frac{5}{10}}, x_{n+\frac{9}{10}}\right]$ to give the discrete schemes

$$y_{n+\frac{1}{10}} = y_n + \frac{22787}{540000}hf_n + \frac{3521}{57600}hf_{n+\frac{1}{10}} - \frac{311}{48000}hf_{n+\frac{5}{10}} + \frac{4163}{172800}hf_{n+\frac{9}{10}} - \frac{3769}{180000}hf_{n+1} + \frac{83}{60000}h^2 g_{n+1}$$

$$y_{n+\frac{5}{10}} = y_n - \frac{1}{32}hf_n + \frac{19625}{62208}hf_{n+\frac{1}{10}} + \frac{37}{128}hf_{n+\frac{5}{10}} - \frac{125}{256}hf_{n+\frac{9}{10}} + \frac{3227}{7776}hf_{n+1} - \frac{23}{864}h^2 g_{n+1}$$

$$y_{n+\frac{9}{10}} = y_n - \frac{369}{20000}hf_n + \frac{1851}{6400}hf_{n+\frac{1}{10}} + \frac{36693}{80000}hf_{n+\frac{5}{10}} + \frac{1611}{6400}hf_{n+\frac{9}{10}} - \frac{1623}{20000}hf_{n+1} + \frac{27}{20000}h^2 g_{n+1}$$

$$y_{n+1} = y_n - \frac{1}{54}hf_n + \frac{125}{432}hf_{n+\frac{1}{10}} + \frac{11}{24}hf_{n+\frac{5}{10}} + \frac{125}{432}hf_{n+\frac{9}{10}} - \frac{1}{54}hf_{n+1} + 0h^2 g_{n+1}$$

Writing the discrete schemes in block, we have

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2\gamma^{(1)}G_{m+1} \tag{13}$$

where

$$Y_{m+1} = \left[y_{n+\frac{1}{10}}\ y_{n+\frac{5}{10}}\ y_{n+\frac{9}{10}}\ y_{n+1}\right]^T, Y_m = [y_{n-1}\ \ y_{n-2}\ y_{n-3}\ y_n]^T, F_m = [f_{n-1}\ f_{n-2}\ f_{n-3}\ \ f_n]^T,$$

$$F_{m+1} = \left[f_{n+\frac{1}{10}}\ f_{n+\frac{5}{10}}\ f_{n+\frac{9}{10}}\ f_{n+1}\right]^T,\ G_{m+1} = [g_{n-1}\ g_{n-2}\ g_{n-3}\ g_{n+1}]^T,$$

$$A^{(1)} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, A^{(0)} = \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}, B^{(0)} = \begin{bmatrix}0&0&0&\frac{22787}{540000}\\0&0&0&-\frac{1}{32}\\0&0&0&-\frac{369}{20000}\\0&0&0&-\frac{1}{54}\end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix}\frac{3521}{57600}&-\frac{311}{48000}&\frac{4163}{172800}&-\frac{3769}{180000}\\\frac{19625}{62208}&\frac{37}{128}&-\frac{125}{256}&\frac{3227}{7776}\\\frac{1851}{6400}&\frac{36693}{80000}&\frac{1611}{6400}&-\frac{1623}{20000}\\\frac{125}{432}&\frac{11}{24}&\frac{125}{432}&-\frac{1}{54}\end{bmatrix} \text{ and } \gamma^{(1)} = \begin{bmatrix}0&0&0&\frac{83}{60000}\\0&0&0&-\frac{23}{864}\\0&0&0&\frac{27}{20000}\\0&0&0&0\end{bmatrix}$$

Representing the predictor (4.14) and the corrector (4.16) in table to give

$$\left[\begin{array}{c|ccccc|ccccc}0&0&0&0&0&0&0&0&0&0&0\\\frac{1}{10}&\frac{22787}{540000}&\frac{3521}{57600}&-\frac{311}{48000}&\frac{4163}{172800}&-\frac{3769}{180000}&0&0&0&0&\frac{83}{60000}\\\frac{5}{10}&-\frac{1}{32}&\frac{19625}{62208}&\frac{37}{128}&-\frac{125}{256}&\frac{3227}{7776}&0&0&0&0&-\frac{23}{864}\\\frac{9}{10}&-\frac{369}{20000}&\frac{1851}{6400}&\frac{36693}{80000}&\frac{1611}{6400}&-\frac{1623}{20000}&0&0&0&0&\frac{27}{20000}\\1&-\frac{1}{54}&\frac{125}{432}&\frac{11}{24}&\frac{125}{432}&-\frac{1}{54}&0&0&0&0&0\\\hline &-\frac{1}{54}&\frac{125}{432}&\frac{11}{24}&\frac{125}{432}&-\frac{1}{54}&&&&&\end{array}\right]$$

Stability Properties of the block Method

Order of the block method (Kida et al., 2022)

Evaluating equations (4.11), (4.12a), (4.12b) and (4.13) in a Taylor series about x_n gives the order of the block method $p = [5,6,6,6]^T$ with error constant

$$\text{Error Constant} = \left[-\frac{1}{80000000}h^6, \frac{367}{322560000}h^7, \frac{68067}{112000000000}h^7, \frac{37}{60480000}h^7\right]^T$$

Zero stability of the block methods (Kida et al., 2022)

Since the roots $z_s, s = 1,2,3,\ldots n$ of the first characteristics polynomial $\rho(z)$ of block method defined by

$\rho(\lambda) = det\left[A^{(1)}\lambda - A^{(0)}\right] = 0$

are $\lambda = [0,0,0,1]^T$. Hence the block method is zero stable.

Consistency of the block method (Kida et al., 2022)

Since the block method is of order $p > 1$, therefore, the block method is consistent.

Convergence of the block method (Kida et al., 2022)

The method (4.14) is convergent since it is consistent and zero-stable.

Region of absolute stability

The region of absolute stability of the block method is shown in Figure 1:

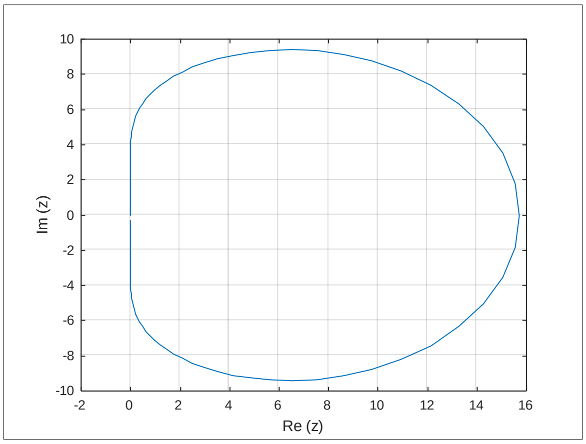

Figure 1: Region of Absolute Stability of the block method

Numerical Experiments

The following optimal control problems are considered to test the accuracy of the developed method.

Table 1: Notations

Abbreviation	Meaning
CBM	Collocation Block Method
CRKM	Point of Evaluation

Example 1 *(Garret, 2015,* Alkali *et al., 2019)*

$$minJ = \int_0^1 x(t) + u(t)dt$$

subject to $\begin{cases} x'(t) = 1 - u(t), \\ x(0) = 1, \end{cases}$

Solution

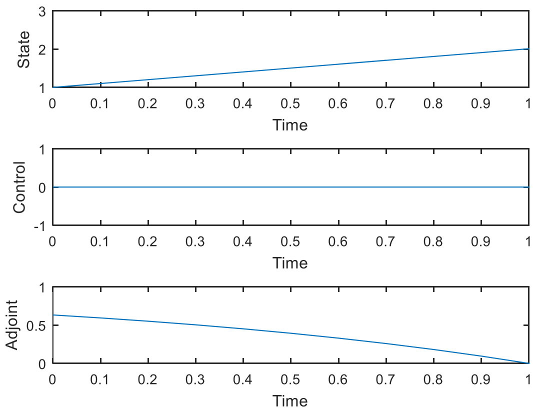

Figure 2: FBS for CRKM Example 1

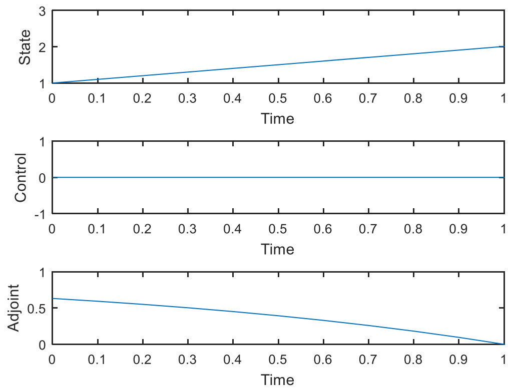

Figure 3: FBS for CBM Example 1

Table 2: Results for Example 1

t	State	Control	Lambda
	CRKM		
1	1.7510e-00	4.9951e-01	0.0000
	Alkali et al. (2019)		
1	1.7510e-00	4.9951e-01	0.0000
	CBM		
1	1.7510e-01	3.5144e-02	0.0000

We solve this problem with N = 100: The results are shown in Fig. 2, Fig. 3 and Table I. The results show that this method gives better approximation than Runge-Kutta method of order four and that of (Alkali *et al*., 2019).

Example 2 *(Garret, 2015,* Alkali *et al., 2019)*

$$\min_u J = \frac{1}{2}\int_0^1 x(t)^2 + u(t)^2 dt$$

subject to $\begin{cases} x'(t) = -x(t) + u(t) \\ x(0) = 1 \end{cases}$

with the optimal solution

$$x^*(t) = \frac{\sqrt{2}\cosh(\sqrt{2}(t-1)) - \sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})},$$

$$u^*(t) = -\frac{\sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}$$

Solution

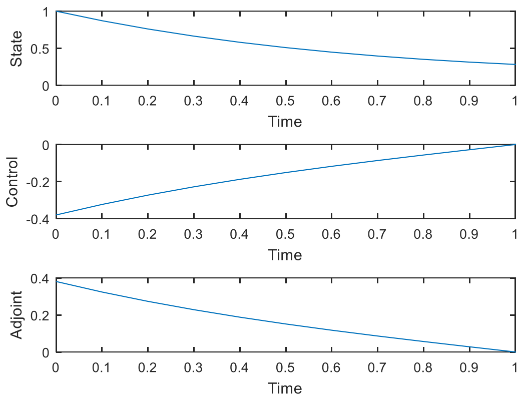

Figure 4: FBS for CRKM Example 2

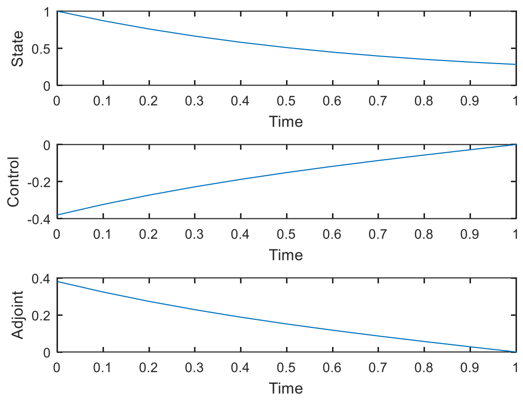

Figure 5: FBS for CBM Example2

Table 3: Results for Example 1

t	State	Control	Lambda
	CRKM		
1	2.8744e-01	0.0000	0.0000
	Alkali et al. (2019)		
1	2.8177e-01	0.0000	0.0000
	CBM		
1	1.6823e-02	0.0000	0.0000

We solve this problem with N = 100: The results are shown in Fig. 4, Fig. 5 and Table 2. The results show that this method gives better approximation than Runge-Kutta method of order four and that of method developed by *(*Alkali *et al., 2019).*

Example 3 *(Garret, 2015)*

$$\min_u J = \int_0^1 u(t)^2 dt$$

subject to $\begin{cases} x'(t) = x(t) + u(t), \\ x(0) = 1, x(1) \text{ free} \end{cases}$

with the optimal solution

$x^*(t) \equiv e^t$,

$u^*(t) \equiv 0$

Solution

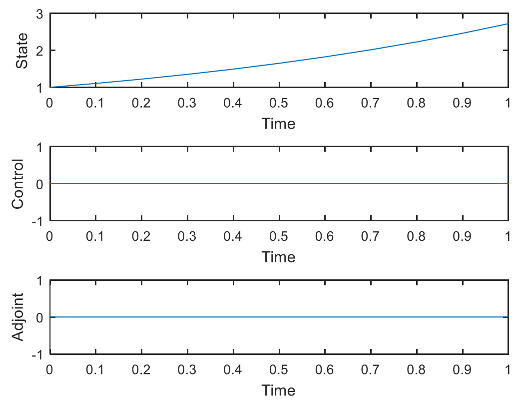

Figure 6: FBS for CRKM Example 3

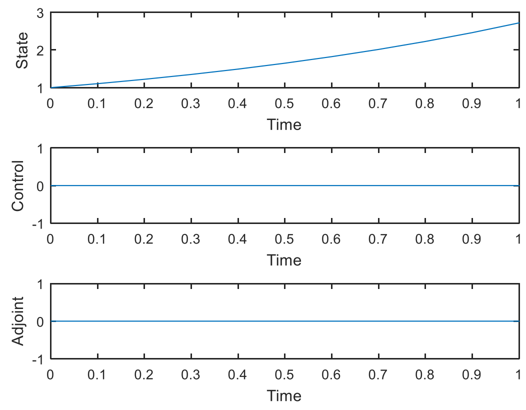

Figure 7: FBS for CBM Example 3

Table 4: Results for Example 1

t	State	Control	Lambda
	CRKM		
1	5.8692e-01	0.0000	0.0000
	CBM		
1	6.5732e-02	0.0000	0.0000

We solve this problem with N = 100: The results are shown in Fig. 6, Fig. 7 and Table 3. The results show that this method gives better approximation than Runge-Kutta method of order four.

CONCLUSION

A technique for the solving optimal control system is developed and implemented in block form. The basic properties of the method are investigated and found to be zero stable, consistent and convergent. The approximate solution of the control and state functions are obtained by forward backward sweep methods and solving the necessary conditions derived from Pontryagin's minimum principles. The absolute stabilities property of the new method is investigated and were found to be absolutely stable within the region of absolute stability. A MATLAB code is written for the implementation of the new method.

Finally, the accuracy of the method is tested on some numerical examples and compare the results with the results obtained using classical Runge Kutta method and the block method. The results obtained by the new method with implementation in block form, are found to be accurate.

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