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A THIRD REFINEMENT OF JACOBI METHOD FOR SOLUTIONS TO SYSTEM OF LINEAR EQUATIONS

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ABSTRACT

Solving linear systems of equations stands as one of the fundamental challenges in linear algebra, given their prevalence across various fields. The demand for an efficient and rapid method capable of addressing diverse linear systems remains evident. In scenarios involving large and sparse systems, iterative techniques come into play to deliver solutions. This research paper contributes by introducing a refinement to the existing Jacobi method, referred to as the "Third Refinement of Jacobi Method." This novel iterative approach exhibits its validity when applied to coefficient matrices exhibiting characteristics such as symmetry, positive definiteness, strict diagonal dominance, and M -matrix properties. Importantly, the proposed method significantly reduces the spectral radius, thereby curtailing the number of iterations and substantially enhancing the rate of convergence. Numerical experiments were conducted to assess its performance against the original Jacobi method, the second refinement of Jacobi, and the Gauss-Seidel method. The outcomes underscore the "Third Refinement of Jacobi" method's potential to enhance the efficiency of linear system solving, thereby making it a valuable addition to the toolkit of numerical methodologies in scientific and engineering domains.

Keywords: Linear system, iteration process, third refinement, Jacobi method, coefficient matrix, rapid convergence

INTRODUCTION

Numerous problems in Engineering and the Sciences, as well as applications of mathematics to the social sciences and quantitative studies of business, statistics, and economics, involve systems of linear equations (Audu *et al.*, 2021a). To comprehend the resolution of physical problems, it is occasionally necessary to employ algorithms that converge swiftly in their solution (Audu et al., 2021b). Processes such as weather forecasting, image processing, and simulation to anticipate aerodynamics performance include a large number of simultaneous equations solved using numerical methods, and time is a crucial aspect in the practical implementation of the results. Iterative approaches are preferable and mostly unaffected by rounding errors for large sets of linear equations (Audu, 2022). Jacobi and Gauss- Seidel Methods are well-known classical numerical iterative techniques. Agboola and Nehad (2022) explored the examination of transient distribution in Markov chains through the utilization of matrix scaling and powering techniques, specifically focusing on small state spaces. By implementing the concept of Lasker & Behera (2014) and employing successive refinement methodology, the convergence rate of the Jacobi approaches can be accelerated. As a required condition for convergence, the convergence rate is dependent on the spectral radius of the iteration matrix of the desired Jacobi refinement technique; thus, the SR approach is extremely sensitive to the spectral radius of any stationary iteration approach. The Jacobi iterative method is a fundamental technique that is widely applied in various fields (Chalermwut *et al.*, 2023). Islam (2023) applied Jacobi to solve linear systems using theory, data, and high-performance computing, while Agboola *et al.* (2023) solved the stationary distribution of a markov chain using the Jacobi method. Meanwhile, Zhen *et al.* (2023) found a new way of solving fuzzy linear systems using the Jacobi method, and Huang and Jia (2023) did an approximate orthogonal tensor diagonalization of Jacobi convergence.

Despite the advancements made in iterative methods for solving systems of linear equations, there is a conspicuous void in the exploration of a third refinement of the Jacobi method. Existing research predominantly focuses on the original Jacobi method and its well-known variations. This research gap underscores the need for a comprehensive investigation into the potential benefits, convergence properties, and practical applications of this third refinement, thereby contributing to a deeper understanding of iterative techniques for linear equation solving. The research problem addressed in this study is to enhance the efficiency and applicability of iterative methods for solving systems of linear equations, particularly focusing on the development and analysis of the "Third-Refinement of Jacobi" (TRJ) method. The study is justified by the need to enhance the efficiency and applicability of iterative methods for solving linear equations, which have broad applications in science and engineering. The novelty of the research lies in the introduction of a third refinement to the Jacobi method, a step beyond existing variations. This unexplored refinement promises to enhance the method's efficiency, convergence, and applicability in solving systems of linear equations, opening new avenues for improving iterative techniques in this domain.

The aim of this research study is to further refine the Jacobi method for solving systems of linear equations, building upon previous advancements in numerical linear algebra. The Jacobi method is a widely used iterative technique for approximating solutions to linear systems, known for its simplicity and applicability to various fields, including engineering and computer science. In this third refinement, we seek to enhance the method's convergence properties, computational efficiency, and applicability to larger and more complex systems. By doing so, our study contributes to the ongoing efforts to improve the accuracy and speed of numerical methods for solving linear equations, thereby advancing the field of numerical linear algebra, and facilitating more efficient problem-solving in diverse scientific and engineering applications. This research focuses on refining the existing Jacobi methods to obtain faster convergence when applied to linear systems.

MATERIALS AND METHODS

Development of the Third Refinement of Jacobi (TRJ) Method

Consider large and spare linear system of the form

$$Js = f \quad (1)$$

where $J \in \mathbb{R}^{n\times n}$ denotes the coefficient matrix, $f \in \mathbb{R}^n$ express the values on the right-hand side and $s \in \mathbb{R}^n$ is the variables whose values need to be ascertained. If J has diagonal elements that are non-vanishing, then J can be decomposed as:

$$J = D - P - Q \quad (2)$$

Alternatively, combination of (1) and (2) becomes:

$$s = D^{-1}(P+Q)s + D^{-1}f \quad (3)$$

And setting (3) into an iteration process gives the classical Jacobi iteration approach:

$$s_J^{(k+1)} = D^{-1}(P+Q)s^{(k)} + D^{-1}f \quad (4)$$

Assuming (2) is equivalent to $D - J = P + Q$, then Refinement of Jacobi Method (RJ) is described from the expression (Dafchahi, 2008).

$$s^{(k+1)} = \tilde{s}^{(k+1)} + D^{-1}\left(f - J\tilde{s}^{(k+1)}\right) \quad (5)$$

Insertion of (4) into $\tilde{s}^{(k+1)}$ with further simplification gives the RJ in (6)

$$s_{RJ}^{(k+1)} = [D^{-1}(P+Q)]^2 s^{(k)} + [I + D^{-1}(P+Q)]D^{-1}f \quad (6)$$

Similarly, putting (6) into $\tilde{s}^{(k+1)}$ in (5) and further algebraic manipulation gives the Second-Refinement of Jacobi (SRJ), (Eneyew *et al.*, 2019);

$$s_{SRJ}^{(k+1)} = [D^{-1}(P+Q)]^3 s^{(k)} + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2\right]D^{-1}f \quad (7)$$

The proposed method is a modification of Jacobi second-refinement approach. With the general format of the refinement of Jacobi method $s^{(k+1)} = \tilde{s}^{(k+1)} + D^{-1}\left(f - J\tilde{s}^{(k+1)}\right)$, replacing (7) into $\tilde{s}^{(k+1)}$to get

$$s^{(k+1)} = [D^{-1}(P+Q)]^3 s^{(k)} + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2\right]D^{-1}f + D^{-1}\left(f - J\left\{[D^{-1}(P+Q)]^3 s^{(k)} + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2\right]D^{-1}f\right\}\right) \quad (8)$$

Next, it is further simplified into the Third-Refinement of Jacobi method (TRJ), represented as thus;

$$s_{TRJ}^{(k+1)} = [D^{-1}(P+Q)]^4 s^{(k)} + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3\right]D^{-1}f \quad (9)$$

The iteration matrix of TRJ is denoted as $[D^{-1}(P+Q)]^4$ and its spectral radius that indicate convergence is of the form $\rho([D^{-1}(P+Q)]^4)$. The method converges if its spectral radius is less than one.

Convergence of the Proposed TRJ

In conducting the convergence analysis, ideas from similar theorems by (Salkuyeh, 2007; Vatti & Tesfaye, 2011; Genanew, 2016; Vatti, 2016; Saha & Chakrabarty, 2020 and Eneyew *et al*, 2020) were considered.

Theorem 1: If J is strictly diagonally dominant (SDD) matrix, then the Third-refinement of Jacobi method (TRJ) converges for any selection of the preliminary estimate $s^{(0)}$.

Proof: Let S be the factual solution of linear system in (1). Whenever J is SDD matrix, then $\tilde{s}^{(k+1)} \to S$. The proposed Jacobi third refinement iteration method can be written as; $s^{(k+1)} = \tilde{s}^{(k+1)} + D^{-1}\left(f - J\tilde{s}^{(k+1)}\right) \Rightarrow s^{(k+1)} - S = \tilde{s}^{(k+1)} - S + D^{-1}\left(f - J\tilde{s}^{(k+1)}\right)$. Hence, after normalizing, it gives;

$$\left\|s^{(k+1)} - S\right\| = \left\|\tilde{s}^{(k+1)} - S + D^{-1}\left(f - J\tilde{s}^{(k+1)}\right)\right\| \le \left\|s^{(k+1)} - S\right\| + \left\|D^{-1}\left(f - J\tilde{s}^{(k+1)}\right)\right\|$$

$$\left\|s^{(k+1)} - S\right\| \le \left\|\tilde{s}^{(k+1)} - S\right\| + \|D^{-1}\|\left\|f - J\tilde{s}^{(k+1)}\right\| \to \|S - S\| + \|D^{-1}\|\|f - JS\|$$

$$= 0 + \|D^{-1}\|\|f - f\| \qquad (since \quad Js = f) \quad (10)$$

$$\therefore 0 + 0 = 0$$

So, $\tilde{s}^{(k+1)} \to S$ implying $\rho([D^{-1}(P+Q)]^4) = \left[\rho\left(D^{-1}(P+Q)\right)\right]^4 < 1$, hence TRJ converges.

Theorem 2: If J is a symmetric positive definite matrix, then the TRJ method is convergent for any arbitrary preliminary estimate $s^{(0)}$.

Proof: The assumption is proved using consistency and spectral radius of the iteration matrix. Firstly, we verify the consistency of the proposed method (TRJ) with Jacobi method. For the fact that$\|D^{-1}(P+Q)\| < 1 \Rightarrow \rho\left(D^{-1}(P+Q)\right) < 1$, The Jacobi method can be written as $S = [I - D^{-1}(P+Q)]^{-1}D^{-1}f$, since $\tilde{s}^{(k+1)} \to S$, from (7), implies that;

$$S = \left[I - \left(D^{-1}(P+Q)\right)^4\right]^{-1}\left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3\right]D^{-1}f$$

By expansion, we get

$$= \left[I + \left(D^{-1}(P+Q)\right)^4 + \left(D^{-1}(P+Q)\right)^8 + \ldots\right] \times \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3\right]D^{-1}f$$

$$= \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3 + \left(D^{-1}(P+Q)\right)^4 + \ldots\right]D^{-1}f$$

$$= [I - D^{-1}(P+Q)]^{-1}D^{-1}f$$

Hence, on observation, $[I - D^{-1}(P+Q)]^{-1}D^{-1}f$ is consistent to J, RJ and SRJ. Therefore,

$$s^{(k+1)} = [D^{-1}(P+Q)]^4 s^{(k)} + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3\right]D^{-1}f$$

$$= [D^{-1}(P+Q)]^4 s^{(k-1)} + \left(D^{-1}(P+Q)\right)^4 + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3 + \left(D^{-1}(P+Q)\right)^5 + \left(D^{-1}(P+Q)\right)^6 + \left(D^{-1}(P+Q)\right)^7\right]D^{-1}f$$

$$= [D^{-1}(P+Q)]^{12} s^{(k-2)} + \left(D^{-1}(P+Q)\right)^4 + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3 + \left(D^{-1}(P+Q)\right)^5 + \ldots + \left(D^{-1}(P+Q)\right)^{11}\right]D^{-1}f$$

$$\Rightarrow \quad [D^{-1}(P+Q)]^{4k+4} s^{(0)} + \left(D^{-1}(P+Q)\right)^4 + \left[I + D^{-1}(P+Q) + \left(D^{-1}(P+Q)\right)^2 + \left(D^{-1}(P+Q)\right)^3 + \left(D^{-1}(P+Q)\right)^5 + \ldots + \left(D^{-1}(P+Q)\right)^{4k+3}\right]D^{-1}f$$

Since the coefficient matrix J is characterized as SPD, thus,

$$\lim_{k\to\infty}[D^{-1}(P+Q)]^{4k+4}=0$$

$$\lim_{k\to\infty}S^{(k+1)}=\lim_{k\to\infty}[D^{-1}(P+Q)]^{4k+4}+\lim_{k\to\infty}\sum_{m=0}^{4k+3}\left(D^{-1}(P+Q)\right)^{m}D^{-1}f$$

$$=0+[I-D^{-1}(P+Q)]^{-1}D^{-1}f \;=[I-D^{-1}(P+Q)]^{-1}D^{-1}f\to S$$

$$\Rightarrow\rho\left(\left(D^{-1}(P+Q)\right)^{4}\right)=\left[\rho\left(D^{-1}(P+Q)\right)\right]^{4}<1$$

Thus, the proposed refinement iteration method (TRJ) is convergent.

Theorem 3: The Third-refinement of Jacobi method converges more rapidly than Jacobi method and its initial refinements (RJ and SRJ) when Jacobi is convergent.

Proof: Equivalently, equations (4), (6), (7) and (9) may be expressed as, $s^{(k+1)}=V^2s^{(k)}+B$, $s^{(k+1)}=V^3s^{(k)}+C$ and $s^{(k+1)}=V^4s^{(k)}+E$,

where, $V=D^{-1}(P+Q)$, $A=D^{-1}f$, $B=[I+D^{-1}(P+Q)]D^{-1}f$

$C=\left[I+D^{-1}(P+Q)+\left(D^{-1}(P+Q)\right)^2\right]D^{-1}f$ and $E=\left[I+D^{-1}(P+Q)+\left(D^{-1}(P+Q)\right)^2+\left(D^{-1}(P+Q)\right)^3\right]D^{-1}f$.

For the refinement of Jacobi scheme:

$$s^{(k+1)}=V^2s^{(k)}+B\Rightarrow s^{(k+1)}-S=V^2s^{(k)}-S+B\Rightarrow s^{(k+1)}-S=V^2\left(s^{(k)}-S\right)$$

$$\therefore\left\|s^{(k+1)}-S\right\|=\left\|V^2(s^k-S)\right\|\le\|V^2\|\left\|s^{(k+1)}-S\right\|\le\|V^4\|\left\|s^{(k-1)}-S\right\|\le\ldots\le\left\|V^{2k}\right\|\left\|s^{(1)}-S\right\|$$

$$\Rightarrow\left\|s^{(k-1)}-S\right\|\le\left\|V^{2k}\right\|\left\|s^{(1)}-S\right\|\le\|V\|^{2k}\left\|s^{(1)}-S\right\|$$

Also, for second-refinement of Jacobi scheme:

$$s^{(k+1)}=V^3s^{(k)}+C\quad\Rightarrow\quad s^{(k+1)}-S=V^3s^{(k)}-S+C$$

$$\left\|s^{(k+1)}-S\right\|=\left\|V^3(s^k-S)\right\|\le\|V^3\|\left\|s^{(k+1)}-S\right\|\le\|V^6\|\left\|s^{(k-1)}-S\right\|\le\ldots\le\left\|V^{3k}\right\|\left\|s^{(1)}-S\right\|$$

$$\left\|s^{(k-1)}-S\right\|\le\left\|V^{3k}\right\|\left\|s^{(1)}-S\right\|\le\|V\|^{3k}\left\|s^{(1)}-S\right\|$$

Now, we consider the proposed third-refinement of Jacobi scheme:

$$s^{(k+1)}=V^4s^{(k)}+E\Rightarrow s^{(k+1)}-S=V^4s^{(k)}-S+E=s^{(k+1)}-S=V^4\left(s^{(k)}-S\right)$$

$$\therefore\left\|s^{(k+1)}-S\right\|=\left\|V^4(s^k-S)\right\|\le\|V^4\|\left\|s^{(k+1)}-S\right\|\le\|V^8\|\left\|s^{(k-1)}-S\right\|\le\ldots\le\left\|V^{4k}\right\|\left\|s^{(1)}-S\right\|$$

$$\left\|s^{(k-1)}-S\right\|\le\left\|V^{4k}\right\|\left\|s^{(1)}-S\right\|\le\|V\|^{4k}\left\|s^{(1)}-S\right\|$$

Coefficient of the above inequalities clearly shows that $\|V\|^{4k}\le\|V\|^{3k}\le\|V\|^{2k}\le\|V\|^{k}$ Since $\|V\|^{k}<1$. Therefore, TRJ is convergent.

Theorem 4: If J is an M -matrix, then the third-refinement of Jacobi method converges for any initial guess $s^{(0)}$.

Proof: Since TRJ is consistent with Jacobi method. Therefore, we can illustrate convergence of proposed TRJ by means of the spectral radius of the iterative matrix. If J is an M -matrix, then the spectral radius becomes $\rho\left(D^{-1}(P+Q)\right)<1\Rightarrow\rho\left[\left(D^{-1}(P+Q)\right)^4\right]=\left[\rho\left(D^{-1}(P+Q)\right)\right]^4<1$. Since the spectral radius of Jacobi method is less than 1, it signifies that TRJ is convergent.

RESULTS AND DISCUSSION

Numerical Applications and Results

Test 1: Consider the linear system represented as $Js=f$

$$\begin{pmatrix}4&-1&0&0&0&0&0&0&0\\-1&4&-1&0&-1&0&0&0&0\\0&-1&4&0&-1&0&0&0&0\\0&-1&0&4&-1&0&-1&0&0\\0&0&-1&-1&4&-1&0&-1&0\\0&0&0&0&-1&4&0&0&-1\\0&0&0&-1&0&0&4&-1&0\\0&0&0&0&-1&0&-1&4&-1\\0&0&0&0&0&-1&0&-1&4\end{pmatrix}\begin{pmatrix}s_1\\s_2\\s_3\\s_4\\s_5\\s_6\\s_7\\s_8\\s_9\end{pmatrix}=\begin{pmatrix}20.0\\20.0\\0.00\\20.0\\0.00\\0.00\\30.0\\10.0\\10.0\end{pmatrix}$$

Test 2: Consider the linear system $(Js=f)$in the form

$$\begin{pmatrix}1&-0.25&0&0&0&0&0&0&0\\-0.25&1&-0.25&0&-0.25&0&0&0&0\\0&-0.25&1&0&-0.25&0&0&0&0\\0&-0.25&-0.25&1&-0.25&0&-0.25&0&0\\0&0&-0.25&-0.25&1&-0.25&0&-0.25&0\\0&0&0&0&-0.25&1&0&0&-0.25\\0&0&0&-0.25&0&0&1&-0.25&0\\0&0&0&0&-0.25&0&-0.25&1&-0.25\\0&0&0&0&0&-0.25&0&-0.25&1\end{pmatrix}\begin{pmatrix}s_1\\s_2\\s_3\\s_4\\s_5\\s_6\\s_7\\s_8\\s_9\end{pmatrix}=\begin{pmatrix}5.00\\5.00\\0.00\\5.00\\0.00\\0.00\\7.50\\2.50\\2.50\end{pmatrix}$$

Test 3: Consider the linear system of the form

$$\begin{pmatrix} 4.2 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 4.2 & 0 & -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 4.2 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 4.2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 4.2 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & -1 & 4.2 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & -1 & 4.2 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & -1 & 4.2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{pmatrix} = \begin{pmatrix} 6.20 \\ 5.40 \\ -9.20 \\ 0.00 \\ 6.20 \\ 1.20 \\ -13.4 \\ 4.20 \end{pmatrix}$$

The numerical applications (Test 1, Test 2, and Test 3) were computed using maple 2015 software and the results are shown in the following Tables.

Table 1 Convergence Comparison for Test 1

Method	Spectral Radius	Iteration Number	CPU Time(sec)	Convergence Rate
RJ	0.46651	17	16.350	0.33114
SRJ	0.31863	13	10.961	0.49671
TRJ	0.21764	9	7.256	0.66226
GS	0.45144	16	13.632	0.34539

Table 2 Convergence Comparison for Test 2

Method	Spectral Radius	Iteration Number	CPU Time(sec)	Convergence Rate
RJ	0.49787	20	17.856	0.30288
SRJ	0.35129	12	9.850	0.45433
TRJ	0.24787	9	6.890	0.60578
GS	0.47283	18	15.600	0.32529

Table 3 Convergence Comparison for Test 3

Method	Spectral Radius	Iteration Number	CPU Time(sec)	Convergence Rate
RJ	0.90703	116	30.900	0.04238
SRJ	0.86384	77	18.790	0.06357
TRJ	0.82270	58	15.210	0.08476
GS	0.89530	88	22.450	0.04803

Table 4: Comparison of Numerical Estimates for Test 1

n	$s_1^{(k+1)}$	$s_2^{(k+1)}$	$s_3^{(k+1)}$	$s_4^{(k+1)}$	$s_5^{(k+1)}$	$s_6^{(k+1)}$	$s_7^{(k+1)}$	$s_8^{(k+1)}$	$s_9^{(k+1)}$
RJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.2500	6.2500	1.2500	8.1250	1.8750	0.6250	9.3750	5.0000	3.1250
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
16	7.4262	9.7049	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
17	7.4262	9.7050	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
SRJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.5625	7.3438	2.0312	9.375	3.7500	1.2500	10.781	6.0938	3.9062
2	7.1436	8.9453	3.5352	11.509	6.0205	2.5806	12.158	8.0542	4.9634
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	7.4262	9.7050	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
13	7.4262	9.7050	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
TRJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.8359	8.0859	2.7734	10.469	4.6875	1.9141	11.367	7.1094	4.3359
2	7.2937	9.3494	3.8986	11.996	6.6385	2.8946	12.517	8.4923	5.2579
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	7.4262	9.7049	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
9	7.4262	9.7050	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124
GS									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	5.0000	6.2500	1.5625	6.5625	2.0312	0.5078	9.1406	5.2930	3.9502
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	7.4262	9.7049	4.2196	12.426	7.1738	3.1716	12.826	8.8781	5.5124
16	7.4262	9.7049	4.2197	12.426	7.1739	3.1716	12.826	8.8781	5.5124

Table 5: Comparison of Numerical Estimates for Test 2

n	$s_1^{(k+1)}$	$s_2^{(k+1)}$	$s_3^{(k+1)}$	$s_4^{(k+1)}$	$s_5^{(k+1)}$	$s_6^{(k+1)}$	$s_7^{(k+1)}$	$s_8^{(k+1)}$	$s_9^{(k+1)}$
RJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.2500	6.2500	1.2500	8.1250	1.8750	0.6250	9.3750	5.0000	3.1250
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	7.4687	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090
20	7.4688	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090
SRJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.5625	7.3438	2.0312	9.6875	3.7500	1.2500	10.781	6.0938	3.9062
2	7.1484	8.9819	3.5718	12.380	6.2231	2.6123	12.351	8.1177	4.9780
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
11	7.4687	9.8750	4.3812	13.783	7.6500	3.3147	13.226	9.1212	5.6090
12	7.4688	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090
TRJ									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	6.8359	8.0859	2.7734	10.977	4.7656	1.9141	11.445	7.1094	4.3359
2	7.3090	9.4293	3.9766	13.083	6.9429	2.9633	12.793	8.6194	5.2971
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	7.4687	9.8749	4.3812	13.783	7.6500	3.3147	13.226	9.1212	5.6090
9	7.4688	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090
GS									
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	5.0000	6.2500	1.5625	6.9531	2.1289	0.53223	9.2383	5.3418	3.9685
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	7.4687	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090
18	7.4688	9.8750	4.3813	13.783	7.6501	3.3148	13.226	9.1213	5.6090

Table 6: Comparison of Numerical Estimates for Test 3

n	$s_1^{(k+1)}$	$s_2^{(k+1)}$	$s_3^{(k+1)}$	$s_4^{(k+1)}$	$s_5^{(k+1)}$	$s_6^{(k+1)}$	$s_7^{(k+1)}$	$s_8^{(k+1)}$
RJ								
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.4331	2.2268	-1.1134	-0.9070	0.4331	1.2268	-2.1134	0.0929
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
115	0.9999	2.0000	-1.0000	0.0000	0.9999	0.9999	-2.0000	0.9999
116	1.0000	2.0000	-1.0000	0.0000	1.0000	1.0000	-2.0000	1.0000
SRJ								
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.5141	1.2981	-1.1620	0.0539	0.5141	0.2981	-2.1620	1.0540
2	0.7668	1.6036	-1.3265	-0.1632	0.7668	0.6036	-2.3265	0.8368
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
76	0.9999	2.0000	-1.0000	0.0000	0.9999	0.9999	-2.0000	0.9999
77	1.0000	2.0000	-1.0000	0.0000	1.0000	1.0000	-2.0000	1.0000
TRJ								
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.9485	1.7943	-1.5656	-0.4114	0.9486	0.7943	-2.5656	0.5887
2	0.6933	1.7250	-1.2009	-0.2327	0.6933	0.7250	-2.2009	0.7673
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
48	0.9999	2.0000	-1.0000	0.0000	0.9999	0.9999	-2.0000	0.9999
49	1.0000	2.0000	-1.0000	0.0000	1.0000	1.0000	-2.0000	1.0000
GS								
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	1.4762	1.6372	-1.4492	0.0448	1.1418	0.9197	-1.9584	0.7975
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
87	0.9999	2.0000	-1.0000	0.0000	1.0000	1.0000	-2.0000	1.0000
88	1.0000	2.0000	-1.0000	0.0000	1.0000	1.0000	-2.0000	1.0000

Table 1 presents a clear depiction of the performance of the proposed method (TRJ), revealing its convergence at the 9th iteration with an impressive convergence rate of 0.66226. This notably outperforms the conventional RJ method by a margin of 8 steps, surpasses SRJ by 4 steps, and trails the GS method by just a single step. Moreover, Table 2 reinforces these findings, demonstrating TRJ's convergence at the 9th iteration with a convergence rate of 0.60578. Impressively, this result is one step better than half of RJ, exceeds SRJ by 3 steps, and matches the GS method at exactly half the number of steps. Lastly, Table 3 showcases TRJ's ability to minimize the spectral radius, drastically reducing the iteration steps to just half of what is required by RJ, a substantial 19-step advantage over SRJ, and an impressive 30-steps improvement over the GS method. Tables 4-6 displays the iterates of the numerical experiments for Test1, Test 2 and Test 3 respectively. These findings underscore the significant contributions of TRJ in enhancing the efficiency and effectiveness of iterative methods for solving linear systems.

CONCLUSION

In this research paper, we have introduced an enhanced iterative approach known as the "Third-refinement of Jacobi" method. This novel iterative technique exhibits validity when applied to iteration matrices that fall under the categories of strictly diagonally dominant matrices, symmetric positive definite matrices, or -matrices. Through rigorous convergence analysis, we have established the method's convergent nature. Additionally, empirical numerical testing has substantiated its superiority, demonstrating faster convergence and a substantially smaller spectral radius in comparison to alternative methods. These findings collectively establish the suitability of our proposed method for efficiently solving systems of linear equations.

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REFERENCES

Agboola, S. O., Ayinde, S. A., Ibikunle, O. & Obaromi, A. D. (2023). Application of Jacobi and Gauss-Seidel Numerical Iteratives Solution Methods for the Stationary Distribution of Markov Chain. *Dutse Journal of Pure and Applied Sciences, 9*(1a): 127-138.

Agboola, S. O. & Nehad, A. S. (2022). On the Application of Matrix Scaling and Powering Methods of Small State Spaces for Solving Transient Distribution in Markov Chain. *FUDMA Journal of Sciences,6*(1), 135-140.

Audu, K. J., Yahaya, Y. A., Adeboye, K. R. & Abubakar, U. Y. (2021a). Extended Accelerated Over Relaxation (EAOR) Method for Solution of a Large and Sparse Linear System. *Journal of Science, Technology, Mathematics and Education*, *17*(1), 228-236.

Audu, K. J. (2022). Extended accelerated overrelaxation iteration techniques in solving heat distribution problems. *Songklanakarin Journal of Science and Technology, 4*(5), 1232–1237.

Audu, K. J., Yahaya, Y. A., Adeboye, K. R. & Abubakar, U. Y. (2021b). Refinement of Extended Accelerated Over Relaxation method for solution of linear systems. *Nigerian Annals of Pure and Applied Sciences, 4*(1), 51-61.

Dafchahi, F. N. (2008). A new refinement of Jacobi method for solution of linear system equations $Ax = b$. *International Journal of Contemporary Mathematics and Science, l3*(17), 819-827.

Eneyew, T. K., Awgichew, G., Haile, E., & Gashaye, D. A. (2019). Second refinement of Jacobi method for solving linear system of equations. *International Journal of Computing Science and Applied Mathematics, 5*(2), 41-47.

Eneyew, T. K., Awgichew, G., Haile, E., & Gashaye, D. A. (2020). Second refinement of Gauss-Seidel iteration method for solving linear system of equations. *Ethiopia Journal of Science and Technology, 13*(1), 1-15.

Genanew, G. G. (2016). Refined iterative method for solving system of linear equations. *American Journal of Computational and Applied Mathematics, 6*(3), 144-147.

Huang, J. & Jia, Z. (2023). A Cross-product free Jacobi-Davidson Type Method for computing a partial Generalized Singular Value Decomposition of a Large Matrix pair. *Journal of Scientific Computing*, 94, 3-10.

Huang, Z., Chen, Z., Zhang, S., Wang, S., & Wang, K. (2023). A New Method Based on Jacobi Iteration for Fuzzy Linear Systems. *Thai Journal of Mathematics, 21*(1), 29-37.

Islam, M. S. (2023). Accelerating the Jacobi Iteration for Solving Linear Systems of Equations using Theory, Data and High Performance Computing. PhD Thesis, Massachusetts Institute of Technology. Published.

Laskar, A. H. & Behera, S. (2014). Refinement of iterative methods for the solution of system of linear equations $Ax = b$. *IOSR Journal of Mathematics*, 10, 70-73.

Saha, M. & Chakrabarty, J. (2020). Convergence of generalized Jacobi, Gauss-Seidel and SOR methods for linear systems. *International Journal of Applied Computational Mathematics*, 77, 1-6.

Salkuyeh, D. K. (2007). Generalized Jacobi and Gauss-Seidel method for solving linear system of equations. *Numerical Mathematics. Journal of Chinese University (English Series)*, 16, 164-170.

Vatti, V. B. K. (2016). *Numerical Analysis Iterative Methods*. I. K International Publishing House, Pvt, Limited, New Delhi, India.

Vatti, V. B. K., & Tesfaye, E. K. (2011). A refinement of Gauss-Seidel method for solving of linear system of equations. *International Journal of Contemporary Mathematics and Sciences*, 63, 117 –127.