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CONVERGENCE TEST FOR THE EXTENDED 3 - POINT SUPER CLASS OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR INTEGRATING STIFF IVP

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ABSTRACT

In this work, a new scheme is generated from the extended 3–point super class of block backward differentiation formula for integrating stiff IVP and the proposed method is subjected to convergence test. The proposed scheme is found to be zero stable, consistent and of order 5. Thus, possess all the required criteria for convergence. The scheme can approximate the values of three points at a time per integration step. The scheme maintained the same technique of co-opting a stability control parameter (ρ) in the formula and by adjusting its value within the interval $(-1, 1)$, more A-stabled schemes can be generated. However, this research considers $\rho = -\frac{1}{9}$ and arrived at zero and A– Stabled method, capable of solving any stiff IVPs. Hence, the proposed convergent scheme can be used for integrating stiff IVPs and archives accuracy of scale error and less executional time.

Keywords: Block Backward Differentiation Formula, Convergence, IVPs, Stiff, Zero stable

INTRODUCTION

Block numerical method can generate more approximate solution values at a time per integration step, this phenomena makes it an easier to converge faster than single step scheme, which generate only one solution value per iteration. Backward differentiation formula was first discovered by Curtiss & Hirschfield (1952), and then it is extended by (Cash, 1980 &2000). The implicit block BDF method was proposed by(Ibrahim *et al.*, 2007), Super class aspect of BBDF formula by(Sueiman *et al.*,2013), subsequent development of super class BBDF by (Musa & Unwala, 2019), the diagonally implicit aspect of BBDF formula (Zawawi *et al*, 2012), other extension and developments can be found in (Abdullahi *et al.*, 2023), (Sagir & Abdullahi, 2023a), (Fatokun *et al.*,2005). These schemes among others possessed various degree of accuracy of the scaled error in one way or the other when it comes to solutions of stiff IVPs. A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small (Sulaiman *et al.*, 2013). Due to the preferences of seeking approximate solutions to most of the stiff problems, numerical schemes are been developed continuously with various capacities to handle current realities of stiff IVPs. Most of the methods stated are zero stable, A- stable or both, and recorded an effective accuracy of the scaled errors and of executional time.

This research aimed at using an implicit super class formula, extended 3 - point super class of backward differentiation formula for solving first order stiff IVPs developed by (Musa & Unwala, 2019) to generate another scheme with different value of the free parameter, $\rho = -\frac{1}{9}$and go further to test a convergence criteria for the proposed scheme.

MATERIALS AND METHODS

In this section, we consider the extended 3 - point super class of backward differentiation formula for solving first order stiff IVPs developed by (Musa & Unwala, 2019) of the form

$$\left.\begin{aligned}
y_{n+1} &= -\frac{1\ 6\rho-1}{103\rho+1} y_{n-2} - \frac{113\rho+3}{4\ 3\rho+1} y_{n-1} + \frac{3(2\rho-1)}{3\rho+1} y_n + \frac{12\rho-3}{23\rho+1} y_{n+2} - \frac{3\ \rho-1}{203\rho+1} y_{n+3} - \frac{3}{3\rho+1} h f_{n+1} - \frac{3}{3p+1} h\rho f_{n-1} \\
y_{n+2} &= \frac{3\ \rho-1}{513+3\rho} y_{n-2} - \frac{2(3\rho-2)}{13+3\rho} y_{n-1} - \frac{4(\rho+3)}{13+3\rho} y_n + \frac{12(2+\rho)}{13+3\rho} y_{n+1} + \frac{3\ \rho-6}{513+3\rho} y_{n+3} + \frac{12}{13+\rho} h f_{n+2} + \frac{12}{13+p} \rho h f_n \\
y_{n+2} &= \frac{3\ \rho-1}{513+3\rho} y_{n-2} - \frac{2(3\rho-2)}{13+3\rho} y_{n-1} - \frac{4(\rho+3)}{13+3\rho} y_n + \frac{12(2+\rho)}{13+3\rho} y_{n+1} + \frac{3\ \rho-6}{513+3\rho} y_{n+3} + \frac{12}{13+\rho} h f_{n+2} + \frac{12}{13+p} \rho h f_n
\end{aligned}\right\} \quad (1)$$

From (1) we consider $\rho = -\frac{1}{9}$ throughout the work.

Theorem (1): Henrici (1962) stated the following conditions for convergence of Linear Multi-Step Method (LMM):

i. Necessary condition for convergence of the Linear Multi-step Method (1) is that the modulus of none of the root of the associated polynomial $\gamma(\xi)$ exceeds one, and that the roots of modulus one is simple. The condition, thus imposed on $\gamma(\xi)$ is called the condition of zero stability.
ii. A necessary condition for convergence of the Linear Multi-step Method (1) is that the order of the associated difference operator be at least one. The condition that the order $\gamma \geq 1$, is called the condition of consistency.

To investigate the convergence of the method (1), the method need to meet conditions I and II in the stated theorem.

Condition of Zero Stability

The stability of the method (1) can be obtains by applying the standard test equation of the form

$y' = \Lambda y \Lambda\ is\ a\ complex\ number,\ Re(\Lambda) < 0$ (2)

$$\begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} & \frac{1}{4}\frac{13+3p}{3p+1} & -\frac{3(2p-1)}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+\mathrm{p}} & \frac{2(3p-2)}{13+3p} & \frac{4(p+3)}{13+3p} \\ \frac{2(p-6)}{3p+137} & -\frac{15(p-5)}{3p+137} & \frac{20(3p-10)}{3p+137} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & -\frac{1}{2}\frac{2p-3}{3p+1} & \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+\mathrm{p}} & 1 & -\frac{3}{5}\frac{p-6}{3p+137} \\ -\frac{20(p-15)}{3p+137} & -\frac{30(p+10)}{3p+137} & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = h \begin{bmatrix} 0 & \frac{3}{3p+1} & 0 \\ 0 & 0 & -\frac{12}{13+p} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{12}{3p+1} & 0 \\ -\frac{60}{3p+137} & 0 & \frac{60}{3p+137} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \quad (3)$$

The stability polynomial of (1) is obtained by evaluating from (3) using the relation

$$det\left[(\mathrm{A}_0 - \bar{h}\mathrm{B}_1)t - (\mathrm{A}_1 + \bar{h}\mathrm{B}_0)\right] = 0 \quad (4)$$

where

$$\mathrm{A}_0 = \begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} & \frac{1}{4}\frac{13+3p}{3p+1} & -\frac{3(2p-1)}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+\mathrm{p}} & \frac{2(3p-2)}{13+3p} & \frac{4(p+3)}{13+3p} \\ \frac{2(p-6)}{3p+137} & -\frac{15(p-5)}{3p+137} & \frac{20(3p-10)}{3p+137} \end{bmatrix} \mathrm{A}_1 = \begin{bmatrix} 1 & -\frac{1}{2}\frac{2p-3}{3p+1} & \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+\mathrm{p}} & 1 & -\frac{3}{5}\frac{p-6}{3p+137} \\ -\frac{20(p-15)}{3p+137} & -\frac{30(p+10)}{3p+137} & 1 \end{bmatrix}$$

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & \frac{3}{3p+1} & 0 \\ 0 & 0 & -\frac{12}{13+p} \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{12}{3p+1} & 0 \\ -\frac{60}{3p+137} & 0 & \frac{60}{3p+137} \end{bmatrix}$$

Hence, from (4) with $\rho = -\frac{1}{9}$ we have

$$R(t,\bar{h}) = -\frac{1074666933}{1257254257}t^3 - \frac{368762757}{26754416}t^3\bar{h} + \frac{3343095}{6688604}t^3\overline{h^2} + \frac{58263839007}{247478348}t^2\overline{h^2} + \frac{58263839007}{43247478348}t^2 - \frac{89250}{128627}t^3\overline{h^3} - \frac{297613352}{61869587}t^2\bar{h} - \frac{25020464}{24759199}t + \frac{45696}{4759199}t\bar{h} + \frac{142767}{249683} = 0 \quad (5)$$

By substituting $\bar{h} = 0$ in (4),we obtained

$$R(t,0) = -\frac{402210}{249683}t^3 + \frac{706617}{249683}t^2 + \frac{63882}{35669}t + \frac{142767}{249683} = 0 \quad (6)$$

Solving (6) for t gives the roots as $t = 1$, $t = 0.455458367$ and $t = -0.027686857$ Therefore by definitions (1), the method is zero Stable.

Condition of Consistency

A necessary condition for convergence of the Linear Multi-step Method (1) is that the order of the associated difference operator be at least one. The condition that the order $\gamma \geq 1$, is called the condition of consistency.

Order of the Method

In this section, we derive the order of the methods (1). Now, define the method (1) in the general matrix form as follows

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} D_j^* Y_{m+j-1} \quad (7)$$

$$\begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} & \frac{1}{4}\frac{13+3p}{3p+1} & -\frac{3(2p-1)}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} & \frac{2(3p-2)}{13+3p} & \frac{4(p+3)}{13+3p} \\ \frac{2(p-6)}{3p+137} & -\frac{15(p-5)}{3p+137} & \frac{20(3p-10)}{3p+137} \end{bmatrix}\begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & -\frac{1}{2}\frac{2p-3}{3p+1} & \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} & 1 & -\frac{3}{5}\frac{p-6}{3p+137} \\ -\frac{20(p-15)}{3p+137} & -\frac{30(p+10)}{3p+137} & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = h\begin{bmatrix} 0 & \frac{3}{3p+1} & 0 \\ 0 & 0 & -\frac{12}{13+p} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{12}{3p+1} & 0 \\ -\frac{60}{3p+137} & 0 & \frac{60}{3p+137} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \quad (8)$$

Where C_0^*, C_1^*, D_0^* and D_1^* are square matrices defined by

$$C^* = \begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} & \frac{1}{4}\frac{13+3p}{3p+1} & -\frac{3(2p-1)}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} & \frac{2(3p-2)}{13+3p} & \frac{4(p+3)}{13+3p} \\ \frac{2(p-6)}{3p+137} & -\frac{15(p-5)}{3p+137} & \frac{20(3p-10)}{3p+137} \end{bmatrix}, C_1^* = \begin{bmatrix} 1 & -\frac{1}{2}\frac{2p-3}{3p+1} & \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} & 1 & -\frac{3}{5}\frac{p-6}{3p+137} \\ -\frac{20(p-15)}{3p+137} & -\frac{30(p+10)}{3p+137} & 1 \end{bmatrix}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{3}{3p+1} & 0 \\ 0 & 0 & -\frac{12}{13+p} \\ 0 & 0 & 0 \end{bmatrix}, \quad D_1^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{12}{3p+1} & 0 \\ -\frac{60}{3p+137} & 0 & \frac{60}{3p+137} \end{bmatrix}, \quad C_0 = \begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} \\ \frac{2(p-6)}{3p+137} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix}, \quad C_2 = \begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix}, C_4 = \begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix},$$

$$C_5 = \begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}, D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix}, \quad D_5 = \begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}$$

And Y_{m-1}, Y_m, F_{m-1} and F_m are column vectors defined by

$$Y_m=\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix},\ Y_{m-1}=\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_{n}\end{bmatrix},\ F_{m-1}=\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_{n}\end{bmatrix},\ F_m=\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}$$

Definition 1 : (Order of the method) the order of the block method (1) and its associated linear operator are given by

$$L[y(x);h]=\sum_{j=0}^{5}\left[C_j y(x+jh)\right]-h\sum_{j=0}^{5}\left[D_j y'(x+jh)\right] \tag{9}$$

where p is unique integer such that

$E_q=0,\ q=0,1,\dots\ p$ and $E_{p+1}\neq 0$, where the E_q are constant matrice (Suleiman *et al*., 2013).

Now using the definition (1) above, considering the value of the free parameter, $\rho=-\frac{1}{9}$ throughout the research and using the Maple software (Version 19) the following are obtained

$$E_0=\sum_{j=0}^{5}C_J=\begin{bmatrix}\frac{1}{10}\frac{6p-1}{3p+1}\\-\frac{3}{5}\frac{p-1}{13+p}\\\frac{2(p-6)}{3p+137}\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\frac{13+3p}{3p+1}\\\frac{2(3p-2)}{13+3p}\\-\frac{15(p-5)}{3p+137}\end{bmatrix}+\begin{bmatrix}-\frac{3(2p-1)}{3p+1}\\\frac{4(p+3)}{13+3p}\\\frac{20(3p-10)}{3p+137}\end{bmatrix}+\begin{bmatrix}1\\-\frac{3}{5}\frac{p-1}{13+p}\\-\frac{20(p-15)}{3p+137}\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}\frac{2p-3}{3p+1}\\1\\-\frac{30(p+10)}{3p+137}\end{bmatrix}+\begin{bmatrix}\frac{3}{20}\frac{p-1}{3p+1}\\-\frac{3}{5}\frac{p-6}{3p+137}\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$E_1=\sum_{j=0}^{5}\left[jC_j-2D_j\right]=\begin{bmatrix}\frac{1}{4}\frac{13+3p}{3p+1}\\\frac{2(3p-2)}{13+3p}\\-\frac{15(p-5)}{3p+137}\end{bmatrix}+2\begin{bmatrix}-\frac{3(2p-1)}{3p+1}\\\frac{4(p+3)}{13+3p}\\\frac{20(3p-10)}{3p+137}\end{bmatrix}+3\begin{bmatrix}1\\-\frac{3}{5}\frac{p-1}{13+p}\\-\frac{20(p-15)}{3p+137}\end{bmatrix}+4\begin{bmatrix}-\frac{1}{2}\frac{2p-3}{3p+1}\\1\\-\frac{30(p+10)}{3p+137}\end{bmatrix}+5$$

$$\begin{bmatrix}\frac{3}{20}\frac{p-1}{3p+1}\\-\frac{3}{5}\frac{p-6}{3p+137}\\1\end{bmatrix}-\left[\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{3}{3p+1}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-\frac{12}{13+p}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\-\frac{60}{3p+137}\end{bmatrix}+\begin{bmatrix}0\\\frac{12}{3p+1}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\frac{60}{3p+137}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$E_2 = \sum_{j=0}^{5}\left[\frac{1}{2!}j^2C_j - 2jD_j\right] = \frac{1}{2!}\left[\begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + 2^2\begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + 3^2\begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix}\right.$$

$$\left. + 4^2\begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix} + 5^2\begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}\right] - \frac{1}{1!}\left[\begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix}\right.$$

$$\left. + 4\begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_3 = \sum_{j=0}^{5}\left[\frac{1}{3!}j^3C_j - 2\frac{1}{2!}j^2D_j\right]$$

$$= \frac{1}{3!}\left[\begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + 2^3\begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + 3^3\begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix} + 4^3\begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix}\right.$$

$$\left. + 5^3\begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}\right]$$

$$- \frac{1}{2!}\left[\begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix} + 2^2\begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix} + 3^2\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix} + 4^2\begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix} + 5^2\begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_4 = \sum_{j=0}^{5}\left[\frac{1}{4!}j^4C_j - 2\frac{1}{3!}j^3D_j\right]$$

$$= \frac{1}{4!}\left[\begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + 2^4\begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + 3^4\begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix} + 4^4\begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix} + 5^4\begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}\right]$$

$$-\frac{1}{3!}\left[\begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix} + 2^3\begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix} + 3^3\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix} + 4^3\begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix} + 5^3\begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_5 = \sum_{j=0}^{5}\left[\frac{1}{5!}j^5 C_j - 2\frac{1}{4!}j^4 D_j\right]$$

$$= \frac{1}{5!}\left[\begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + 2^5\begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + 3^5\begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+\mathrm{p}} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix} + 4^5\begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix} + 5^5\begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}\right]$$

$$-\frac{1}{4!}\left[\begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix} + 2^4\begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix} + 3^4\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix} + 4^4\begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix} + 5^4\begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_6 = \sum_{j=0}^{5}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right]$$

$$= \frac{1}{6!}\left[\begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + 2^6\begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + 3^6\begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix} + 4^6\begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix} + 5^6\begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix}\right]$$

$$-\frac{1}{5!}\left[\begin{bmatrix} \frac{3}{3p+1} \\ 0 \\ 0 \end{bmatrix} + 2^5\begin{bmatrix} 0 \\ -\frac{12}{13+p} \\ 0 \end{bmatrix} + 3^5\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3p+137} \end{bmatrix} + 4^5\begin{bmatrix} 0 \\ \frac{12}{3p+1} \\ 0 \end{bmatrix} + 5^5\begin{bmatrix} 0 \\ 0 \\ \frac{60}{3p+137} \end{bmatrix}\right]$$

$$= \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the method is of order 5, with error constant as $E_6 = \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix}$ (10)

Hence, the order of the proposed method (1) is 5, the condition $\gamma \geq 1$ is met. However, we need to show that $\sum_{j=0}^{5} C_j = 0$ and $\sum_{j=0}^{5} jC_j = \sum_{j=0}^{5} D_j$

Now

$$\sum_{j=0}^{5} C_J = \begin{bmatrix} \frac{1}{10}\frac{6p-1}{3p+1} \\ -\frac{3}{5}\frac{p-1}{13+p} \\ \frac{2(p-6)}{3p+137} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\frac{13+3p}{3p+1} \\ \frac{2(3p-2)}{13+3p} \\ -\frac{15(p-5)}{3p+137} \end{bmatrix} + \begin{bmatrix} -\frac{3(2p-1)}{3p+1} \\ \frac{4(p+3)}{13+3p} \\ \frac{20(3p-10)}{3p+137} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{3}{5}\frac{p-1}{13+p} \\ -\frac{20(p-15)}{3p+137} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\frac{2p-3}{3p+1} \\ 1 \\ -\frac{30(p+10)}{3p+137} \end{bmatrix} + \begin{bmatrix} \frac{3}{20}\frac{p-1}{3p+1} \\ -\frac{3}{5}\frac{p-6}{3p+137} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (11)$$

And

$$\sum_{j=0}^{5} jC_j = 0\cdot C_0 + 1\cdot C_1 + 2\cdot C_2 + 3\cdot C_3 + 4\cdot C_4 + 5\cdot C_5$$

$$0\cdot\begin{bmatrix}\frac{1}{10}\frac{6p-1}{3p+1}\\ -\frac{3}{5}\frac{p-1}{13+p}\\ \frac{2(p-6)}{3p+137}\end{bmatrix}+1\cdot\begin{bmatrix}\frac{1}{4}\frac{13+3p}{3p+1}\\ \frac{2(3p-2)}{13+3p}\\ -\frac{15(p-5)}{3p+137}\end{bmatrix}+2\cdot\begin{bmatrix}-\frac{3(2p-1)}{3p+1}\\ \frac{4(p+3)}{13+3p}\\ \frac{20(3p-10)}{3p+137}\end{bmatrix}+3\cdot\begin{bmatrix}1\\ -\frac{3}{5}\frac{p-1}{13+p}\\ -\frac{20(p-15)}{3p+137}\end{bmatrix}+4\cdot\begin{bmatrix}-\frac{1}{2}\frac{2p-3}{3p+1}\\ 1\\ -\frac{30(p+10)}{3p+137}\end{bmatrix}+5\cdot\begin{bmatrix}\frac{3}{20}\frac{p-1}{3p+1}\\ -\frac{3}{5}\frac{p-6}{3p+137}\\ 1\end{bmatrix}=\begin{bmatrix}\frac{4p-17}{3p+1}\\ \frac{2(p-1)}{13+3p}\\ -\frac{5(p-46)}{3p+137}\end{bmatrix} \quad (12)$$

Also

$$\sum_{j=0}^{5} D_j=\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{3}{3p+1}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-\frac{12}{13+p}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\-\frac{60}{3p+137}\end{bmatrix}+\begin{bmatrix}0\\ \frac{12}{3p+1}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\ \frac{60}{3p+137}\end{bmatrix}=\begin{bmatrix}\frac{4p-17}{3p+1}\\ \frac{2(p-1)}{13+3p}\\ -\frac{5(p-46)}{3p+137}\end{bmatrix} \quad (13)$$

Therefore, $\sum_{j=0}^{5} jC_j=\sum_{j=0}^{5} D_j$. Thus, conditions (2) of theorem (1) are also met; the method (10) is consistent. Hence, the method (1) is Convergent by the theorem 1

CONCLUSION

The necessary conditions for the convergence for the extended 3 - point super class of backward differentiation formula for solving first order stiff IVPs highlighted in theorem (1) are satisfied with different value of the parameter, $\rho=-\frac{1}{9}$. The propose scheme found to be of order 5, zero stable and consistent. The convergent scheme is recommended for the solution of first order stiff IVPs.

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