MAPLE SIMULATION CODES FOR STABILITY ANALYSIS OF VARIABLE STEP SUPER CLASS OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR INTEGRATING A SYSTEM OF FIRST ORDER STIFF IVPs

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ABSTRACT
Strength of numerical scheme is rated by the properties it possessed and in turn the kind of problems it can handle. Zero stable method can effectively handle ODEs problem. While, an A – stable method can solve stiff ODEs problem. Analyzing stability of block methods are been carried out using various software. This work aimed at using simplified Maple simulation code to critically analyze a variable step size multi-block backward differentiation formula for the solution stiff initial value problems of ordinary differential equations. The Graphical comparisons of the simulated result obtained is made using Matlab to depict the performing schemes.

Keywords: A - Stability, Simulation Code, Maple, Stiff IVPs, Zero Stability

INTRODUCTION
A Numerical method is a differential equation involving a number of consecutive approximations from which it will be possible to compute the solutions, sequentially. Backward Differentiation formula (BDF) is a family of implicit method for the numerical integration of ordinary differential equations. Stiff ordinary differential equations are equations where certain implicit methods, in particular block backward differentiation formulas (BBDF), perform better, usually better than explicit ones (Curtiss & Hirschfelder, 1952). The formula undergoes different development and modifications.

MATERIAL AND METHODS
In this section, we are considering Maple code for the critical analysis of the steps adopted in achieving zero and A-stable criteria of a 2-point multi - block super class of BBDF developed by Abdullahi et al (2023) of the form:

\[
\begin{align*}
\frac{y_{n+1} - y_n}{\Delta t} &= \frac{5r+13}{2(2r+7)}y_n - \frac{r+1}{2(2r+7)}y_{n+1} + \frac{2(r+2)}{2r+7}h_{n+1} + \frac{2(r+2)}{2r+7}h_n \\
\frac{y_{n+2} - y_{n+1}}{\Delta t} &= \frac{3r+11}{2(2r+7)}y_{n+1} - \frac{6r+19}{2(2r+7)}y_n + \frac{4(r+2)}{2r+7}h_{n+1} + \frac{6(r+2)}{2r+7}h_n \\
\end{align*}
\]

Hence, (1) is called a new multi-block super class of BBDF for integrating first order stiff IVPs with a variable mesh size strategy. From the proposed scheme, different stable methods can be obtain by appropriate changes in the mesh size ratio \(r\).

<p>| Table 1: Variable step size ratios with the stable methods obtained |
|---|---|---|</p>
<table>
<thead>
<tr>
<th>Step Size Ratio ((r))</th>
<th>Approximate Points</th>
<th>Formulae (VSSMBBDF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r = 1)</td>
<td>(y_{n+1})</td>
<td>(y_{n+1} = \frac{1}{3}y_{n-1} + y_n - \frac{1}{3}y_{n+2} + \frac{2}{3}h_{n+1} + \frac{2}{3}h_n)</td>
</tr>
<tr>
<td>(y_{n+2})</td>
<td>(y_{n+2} = \frac{1}{13}y_{n-1} - \frac{3}{13}y_n + \frac{5}{13}y_{n+1} + \frac{6}{13}h_{n+1} + \frac{6}{13}h_n)</td>
<td></td>
</tr>
<tr>
<td>(r = 2)</td>
<td>(y_{n+1})</td>
<td>(y_{n+1} = \frac{1}{35}y_{n-1} + \frac{22}{35}y_n - \frac{3}{35}y_{n+1} + \frac{8}{35}h_{n+1} + \frac{8}{35}h_n)</td>
</tr>
<tr>
<td>(y_{n+2})</td>
<td>(y_{n+2} = \frac{2}{35}y_{n-1} - \frac{6}{35}y_n + \frac{16}{35}y_{n+1} + \frac{16}{35}h_{n+1} + \frac{16}{35}h_n)</td>
<td></td>
</tr>
<tr>
<td>(r = \frac{1}{2})</td>
<td>(y_{n+1})</td>
<td>(y_{n+1} = \frac{1}{2}y_{n-1} + \frac{9}{32}y_n - \frac{3}{32}y_{n+2} + \frac{5}{8}h_{n+1} + \frac{5}{8}h_n)</td>
</tr>
<tr>
<td>(y_{n+2})</td>
<td>(y_{n+2} = \frac{4}{43}y_{n-1} - \frac{9}{43}y_n + \frac{48}{43}y_{n+1} + \frac{20}{43}h_{n+1} + \frac{20}{43}h_n)</td>
<td></td>
</tr>
</tbody>
</table>

Maple Code for Analyzing Zero - Stability of the Methods

Definition 1 (Zero Stability): A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus higher than 1 and that any root with modulus 1 is simple. (Sulaiman et al, 2013)

In the method (1) and if \(r = 1\). The constant coefficient matrix can be found as
Step 1:
A := Matrix([[1 - 26/15*h, 7/75 + 26/75*h], [-93/53 + 6/53*h, 1 - 30/53*h]]);
Output: \[ A = \begin{bmatrix}
1 - 2/3h & 2/3 \\
-18/11 & 1 - 6/11h
\end{bmatrix} \]

Step 2:
B := Matrix([[-53/75, 9/5], [11/53, -51/53]]);
Output: \[ B = \begin{bmatrix}
-53/75 & 9/5 \\
11/53 & -51/53
\end{bmatrix} \]

To find the first characteristic polynomial, using the coefficients matrices, we use

Step 3:
LinearAlgebra[Determinant](A*t - B);
Output: \[ \left(\frac{1542}{1325}t^2 - \frac{6764}{3975}t^2h - \frac{1948}{1325}t + \frac{1248}{1325}t^2h^2 - \frac{7124}{3975}th + \frac{406}{1325}\right) \]

Step 4: subs(h = 0, 1542/1325*t^2 - 6764/3975*t^2*h - 1948/1325*t + 1248/1325*t^2*h^2 - 7124/3975*t*h + 406/1325);
Output: \[ \frac{1542}{1325}t^2 + \frac{406}{1325}t \]

Step 5: solve(%, t);
Output: \[ t = \frac{203}{371} \]

According to definition (1), the method (1) is zero stable.

If \( r = 2 \) the constant coefficient matrix is given as

Step 1:
A := Matrix([[1 - 155/67*h, 33/134 + 31/67*h], [-328/201 + 8/67*h, 1 - 40/67*h]]);
Output:
\[ A = \begin{bmatrix}
1 - 155/67h & 33/134 + 31/67h \\
-328/201 + 8/67h & 1 - 40/67h
\end{bmatrix} \]

Step 2:
Output: \[ B = \begin{bmatrix}
-53/268 & 387/268 \\
11/201 & -46/67
\end{bmatrix} \]

Step 3:
LinearAlgebra[Determinant](A*t - B);
MAPLE SIMULATION CODES FOR STA

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Output: \[
\frac{6293}{4489} t^2 - \frac{29423}{13467} t \frac{h^2}{2} - \frac{26191}{17956} t + \frac{5952}{4489} \frac{h^2}{2} - \frac{20317}{13467} t h + \frac{1019}{17956}
\]

Step 4: `subs(h = 0, 6293/4489*t^2 - 29423/13467*t^2*h - 26191/17956*t + 5952/4489*t^2*h^2 - 20317/13467*t*h + 1019/17956);`

Output: \[
\frac{6293}{4489} t^2 + \frac{1019}{17956} - \frac{26191}{17956} t
\]

Step 5: solve(%, t);

Output: \[
1, \frac{1019}{25172}
\]

Also according to definition (1), the method (1) is zero stable.

If \( r = \frac{1}{2} \) the constant Coefficient matrix is given as

Step 1:

\[
A := \text{Matrix}(\begin{bmatrix}
1 - \frac{3845}{2416} h & 19/302 + \frac{3845}{2416} h/5 \\
-34/19 + 17/152 h & 1 - \frac{85}{152} h
\end{bmatrix})
\]

Output:

\[
A = \begin{bmatrix}
1 - \frac{3845 h}{2416} & \frac{19}{302} + \frac{769 h}{2416} \\
-\frac{34}{19} + \frac{17 h}{152} & 1 - \frac{85 h}{152}
\end{bmatrix}
\]

Step 2:

\[
B := \text{Matrix}(\begin{bmatrix}
-\frac{1431}{1510} & \frac{1518}{755} \\
\frac{27}{95} & -\frac{102}{95}
\end{bmatrix})
\]

Output B = \[
\begin{bmatrix}
-\frac{1431}{1510} & \frac{1518}{755} \\
\frac{27}{95} & -\frac{102}{95}
\end{bmatrix}
\]

Step 3:

LinearAlgebra[Determinant](A*t - B);

Output: \[
\frac{168}{151} t^2 - \frac{36451}{22952} t h^2 + \frac{22359}{14345} t + \frac{39219}{45904} t^2 h^2 - \frac{44145}{22952} t h + \frac{6399}{14345}
\]

Step 4: `subs(h = 0, 168/151*t^2 - 36451/22952*t^2*h - 22359/14345*t + 39219/45904*t^2*h^2 - 44145/22952*t*h + 6399/14345);`

Output: \[
\frac{168}{151} t^2 + \frac{6399}{14345} - \frac{22359}{14345} t
\]

Step 5: solve(%, t);

Output: \[
1, \frac{2133}{5320}
\]

Also according to definition (1), the method (1) is zero stable.
Table 1: A Zero Stable method across different choice of $r$

<table>
<thead>
<tr>
<th>Step Size Ratio ($r$)</th>
<th>Roots of the proposed methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 1$</td>
<td>$t = 1, -0.0909090909$</td>
</tr>
<tr>
<td>$r = 2$</td>
<td>$t = 1, -0.0714285714$</td>
</tr>
<tr>
<td>$r = \frac{1}{2}$</td>
<td>$t = 1, -0.1052631579$</td>
</tr>
</tbody>
</table>

Maple Code Analyzing A - Stability Region of the Methods

**Definition 2 Stability**: A linear multistep method is said to be an A-stable method if its region of stability encloses the entire negative half-plane. (Sulaiman et al, 2013)

For the method (1), using its characteristic polynomial (2), we have:

**Step 1**:

\[ \text{subs}(t = \exp(\phi i), 1542/1325 \cdot t^2 - 6764/3975 \cdot t^2 \cdot h - 1948/1325 \cdot t + 1248/1325 \cdot t^2 \cdot h^2 - 7124/3975 \cdot t \cdot h + 406/1325); \]

Output:
\[ \frac{1542 \left(e^{i\phi}\right)^2}{1325} - \frac{6764 \left(e^{i\phi}\right)^2 \cdot h}{3975} - \frac{1948 \cdot e^{i\phi}}{1325} + \frac{1248 \left(e^{i\phi}\right)^2 \cdot h^2}{3975} - \frac{7124 \cdot e^{i\phi} \cdot h}{3975} + \frac{406}{1325}; \]

**Step 2**:

\[ \text{solve}(% \cdot h); \]

Output:
\[ \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} + \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}, \]

\[ \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} - \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}; \]

**Step 3**:

\[ p := \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} + \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}; \]

Output:
\[ p := \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} + \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}; \]

**Step 4**:

\[ q := \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} - \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}; \]

Output:
\[ q := \frac{1691 \cdot e^{i\phi}}{1872} + \frac{137}{144} - \frac{\sqrt{-1470455 \left(e^{i\phi}\right)^2 + 11493326 \cdot e^{i\phi} + 2031913}}{1872}; \]

**Step 5**:

> with(plots);

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeadplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedrplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]
Output:

Figure 1: Absolute region for the $p$ - point

Step 7:
complexplot(q, phi = 0 .. 2*Pi, numpoints = 1000, colour = red);
Output:

Figure 2: Absolute region for the $q$ - point

Step 8:
complexplot([p, q], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);
Output:

Figure 3: A – Stability region for $r = 1$ in (1)

From figure (3), the definition (2) is satisfied, the method (1) is A - stable when $r = 1$
Using similar procedure for method (3) and method (4) respectively, we have

>with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedrplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]
Step 6:
\[ \text{with}(	ext{plots}); \]
\[ >\text{complexplot}(r,\phi=0..2\pi, \text{numpoints}=1000, \text{colour}=\text{red}); \]
Output:

![Figure 4: Absolute region for the r - point](image)

Step 7:
\[ \text{complexplot}(s, \phi=0..2\pi, \text{numpoints}=1000, \text{colour}=\text{red}); \]
Output:

![Figure 5: Absolute region for the s - point](image)

Step 8:
\[ \text{complexplot([r, s], } \phi=0..2\pi, \text{numpoints}=1000, \text{colour}=\text{red}); \]
Output:

![Figure 6: A – Stability region for } r = 2 \text{ in (1)}](image)

From figure (6), the definition (2) is satisfied, the method (1) is A- stable when \( r = 2 \)
Output:

Figure 7: Absolute region for the u-point

Step 7:

`complexplot(v, phi = 0 .. 2*Pi, numpoints = 1000, colour = red);`

Output:

Figure 8: Absolute region for the v-point

Step 8:

`complexplot([u, v], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);`

Output:

Figure 9: A-Stability region for $r = \frac{1}{2}$ in (1)

From figure (9), the definition (2) is satisfied, the method (1) is A-stable when $r = \frac{1}{2}$

Step 9:

`complexplot([p, q, r, s, u, v], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);`
Output:

Figure 10: Combine plot of all the absolute stability regions in figure 3, 6 and 9.

CONCLUSION
A new 2 point multi – block super class of BDF for integrating system of first order stiff IVPs is considered in this work for a critical stability analysis. A simplified Maple algorithm is adopted to analyzes how to achieve zero and A – stability criteria, which remained necessary properties for optimal performance of a numerical scheme, particularly in handling stiff system of IVPs are studied with simplified code. The scheme considered in the work is variable step size, which has a variable in the formula that can have different step sizes ratios. In this work, \( r = 1 \), \( r = 2 \) & \( r = \frac{1}{2} \) are adapted in generating the methods and all its stability criteria.

REFERENCE


H. Musa, A.M. Unwala (2019); Extended 3 point super class of block backward differentiation formula for solving first order stiff initial value problems. Abacus (Mathematics Science Series) Vol. 44, No 1, Aug. 2019.


A.A. Nasarudin, Z.B. Ibrahim, H. Rosali, On the integration of stiff ODEs using block backward differentiation formulas of order six. Symmetry 12(6), 952 (2020)