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MAPLE SIMULATION CODES FOR STABILITY ANALYSIS OF VARIABLE STEP SUPER CLASS OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR INTEGRATING A SYSTEM OF FIRST ORDER STIFF IVPS

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ABSTRACT

Strength of numerical scheme is rated by the properties it possessed and in turn the kind of problems it can handle. Zero stable method can effectively handle ODEs problem. While, an A – stable method can solve stiff ODEs problem. Analyzing stability of block methods are been carried out using various software. This work aimed at using simplified Maple simulation code to critically analyze avariable step size multi-block backward differentiation formula for the solution stiff initial value problems of ordinary differential equations. The Graphical comparisons of the simulated result obtained is made using Matlab to depict the performing schemes.

Keywords: A - Stability, Simulation Code, Maple, Stiff IVPs, Zero Stability

INTRODUCTION

A Numerical method is a differential equation involving a number of consecutive approximations from which it will be possible to compute the solutions, sequentially. Backward Differentiation formula (BDF) is a family of implicit method for the numerical integration of ordinary differential equations. Stiff ordinary differential equations are equations where certain implicit methods, in particular block backward differentiation formulas (BBDF), perform better, usually better than explicit ones (Curtiss & Hirschfelder, 1952). The formula undergoes different development and modifications. The following scholar contribute tremendously with regard to BDF and BBDF (Cash, 1980); (Ibrahim *et al.*,2007); (Sulaiman *et al.*,2013a & 2013b); (Musa & Unwala, 2019),(Sagir & Abdullahi, 2022),(Soomro *et al*., 2022), (Nasarudin *et al.*,2020), (Abdullahi *et al*, 2022, & 2023. A system of stiff ordinary differential equations represent a couple of physical systems varying with very different times scales: That is they are systems having some components varying much more rapidly than others. Most of the methods stated are zero stable, A- stable or both, and displays different degree of accuracy of the scale error and executional time.

MATERIAL AND METHODS

In this section, we are considering Maple code for the critical analysis of the steps adopted in achieving zero and A-stable criteria of a 2-point multi - block super class of BBDF developed by Abdullahi *et al* (2023) of the form:

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{2r+7}y_{n-1} + \frac{5r+13}{2(2r+7)}y_n - \frac{r+1}{2(2r+7)} + \frac{2(r+2)}{2r+7}hf_{n+1} + \frac{2(r+2)}{2r+7}hf_n \\ y_{n+2} &= \frac{2}{9r+17}y_{n-1} - \frac{3(r+1)}{9r+17}y_n + \frac{6(2r+3)}{9r+17}y_{n+1} + \frac{4(r+2)}{9r+17}hf_{n+2} + \frac{4(r+2)}{9r+17}hf_{n+1} \end{aligned}\right\} \quad (1)$$

Hence, (1) is called a new multi-block super class of BBDF for integrating first order stiff IVPs with a variable mesh size strategy. From the proposed scheme, different stable methods can be obtain by appropriate changes in the mesh size ratio r

Table 1: Variable step size ratios with the stable methods obtained

Step Size Ratio (r)	Approximate Points	Formulae (VSSMBBDF)
$r = 1$	y_{n+1}	$y_{n+1} = \frac{1}{9}y_{n-1} + y_n - \frac{1}{9}y_{n+2} + \frac{2}{3}hf_{n+1} + \frac{2}{3}hf_n$
	y_{n+2}	$y_{n+2} = \frac{1}{13}y_{n-1} - \frac{3}{13}y_n + \frac{15}{13}y_{n+1} + \frac{6}{13}hf_{n+2} + \frac{6}{13}hf_{n+1}$
$r = 2$	y_{n+1}	$y_{n+1} = \frac{1}{11}y_{n-1} + \frac{22}{22}y_n - \frac{3}{22}y_{n+2} + \frac{8}{11}hf_{n+1} + \frac{8}{11}hf_n$
	y_{n+2}	$y_{n+2} = \frac{2}{35}y_{n-1} - \frac{9}{35}y_n + \frac{6}{5}y_{n+1} + \frac{16}{35}hf_{n+2} + \frac{16}{35}hf_{n+1}$
$r = \frac{1}{2}$	y_{n+1}	$y_{n+1} = \frac{1}{8}y_{n-1} + \frac{31}{32}y_n - \frac{3}{32}y_{n+2} + \frac{5}{8}hf_{n+1} + \frac{5}{8}hf_n$
	y_{n+2}	$y_{n+2} = \frac{4}{43}y_{n-1} - \frac{9}{43}y_n + \frac{48}{43}y_{n+1} + \frac{20}{43}hf_{n+2} + \frac{20}{43}hf_{n+1}$

Maple Code for Analyzing Zero - Stability of the Methods

Definition 1 (Zero Stability): A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus higher than 1 and that any root with modulus 1 is simple. (Sulaiman *et al*,2013)

In the method (1) and if $r = 1$.The constant coefficient matrix can be found as

Step I:
A := Matrix([[1 - 26/15*h, 7/75 + 26/75*h], [-93/53 + 6/53*h, 1 - 30/53*h]]);

Output $:A = \begin{bmatrix} 1-2h & \frac{2}{3} \\ -\frac{18}{11} & 1-\frac{6h}{11} \end{bmatrix}$

Step 2:
B := Matrix([[-53/75, 9/5], [11/53, -51/53]]);

Output B $= \begin{bmatrix} -\frac{53}{75} & \frac{9}{5} \\ \frac{11}{53} & -\frac{51}{53} \end{bmatrix}$

To find the first characteristic polynomial, using the coefficients matrices, we use
Step 3:
LinearAlgebra[Determinant](A*t - B);

Output: $$\frac{1542}{1325}t^2 - \frac{6764}{3975}t^2h - \frac{1948}{1325}t + \frac{1248}{1325}t^2h^2 - \frac{7124}{3975}th + \frac{406}{1325} \quad (2)$$

Step 4: subs(h = 0, 1542/1325*t^2 - 6764/3975*t^2*h - 1948/1325*t + 1248/1325*t^2*h^2 - 7124/3975*t*h + 406/1325);

Output: $\frac{1542}{1325}t^2 + \frac{406}{1325} - \frac{1948}{1325}t$

Step 5: solve(%, t);

Output: $1, \frac{203}{771}$

According to definition (1), the method (1) is zero stable.

If r = 2 the constant coefficient matrix is given as

Step 1:

A:= Matrix([[1 - 155/67*h, 33/134 + 31/67*h], [-328/201 + 8/67*h, 1 - 40/67*h]]);

Output:

$$A = \begin{bmatrix} 1-\frac{155}{67}h & \frac{33}{134}+\frac{31}{67}h \\ -\frac{328}{201}+\frac{8}{67}h & 1-\frac{40}{67}h \end{bmatrix}$$

Step 2:

B: = Matrix([[-53/268, 387/268], [11/201, -46/67]]);

Output B $= \begin{bmatrix} -\frac{53}{268} & \frac{387}{268} \\ \frac{11}{201} & -\frac{46}{67} \end{bmatrix}$

Step 3:

LinearAlgebra[Determinant](A*t - B);

Output: $$\frac{6293}{4489}\,t^2 - \frac{29423}{13467}\,t^2\,h - \frac{26191}{17956}\,t + \frac{5952}{4489}\,t^2\,h^2 - \frac{20317}{13467}\,t\,h + \frac{1019}{17956} \tag{3}$$

Step 4: subs(h = 0, 6293/4489*t^2 - 29423/13467*t^2*h - 26191/17956*t + 5952/4489*t^2*h^2 - 20317/13467*t*h + 1019/17956);

Output: $$\frac{6293}{4489}\,t^2 + \frac{1019}{17956} - \frac{26191}{17956}\,t$$

Step 5: solve(%, t);

Output: $$1, \frac{1019}{25172}$$

Also according to definition (1), the method (1) is zero stable.

If r $= \frac{1}{2}$ the constant Coefficient matrix is given as

Step I:

A: = Matrix([[1 - 3845/2416*h, 19/302 + 3845/2416*h/5], [-34/19 + 17/152*h, 1 - 85/152*h]]);

Output:

$$A = \begin{bmatrix} 1 - \frac{3845\,h}{2416} & \frac{19}{302} + \frac{769\,h}{2416} \\ -\frac{34}{19} + \frac{17\,h}{152} & 1 - \frac{85\,h}{152} \end{bmatrix}$$

Step 2:

B: = Matrix ([[-1431/1510, 1518/755], [27/95, -102/95]]);

Output $$B = \begin{bmatrix} -\frac{1431}{1510} & \frac{1518}{755} \\ \frac{27}{95} & -\frac{102}{95} \end{bmatrix}$$

Step 3:
LinearAlgebra[Determinant](A*t - B);

Output: $$\frac{168}{151}\,t^2 - \frac{36451}{22952}\,t^2\,h - \frac{22359}{14345}\,t + \frac{39219}{45904}\,t^2\,h^2 - \frac{44145}{22952}\,t\,h + \frac{6399}{14345} \tag{4}$$

Step 4: subs(h = 0, 168/151*t^2 - 36451/22952*t^2*h - 22359/14345*t + 39219/45904*t^2*h^2 - 44145/22952*t*h + 6399/14345);

Output: $$\frac{168}{151}\,t^2 + \frac{6399}{14345} - \frac{22359}{14345}\,t$$

Step 5: solve(%, t);

Output: $$1, \frac{2133}{5320}$$

Also according to definition (1), the method (1) is zero stable.

Table 1: A Zero Stable method across different choice of r

Step Size Ratio (r)	Roots of the proposed methods
$r = 1$	$t = 1, -0.0909090909$
$r = 2$	$t = 1, -0.0714285714$
$r = \frac{1}{2}$	$t = 1, -0.1052631579$

Maple Code Analyzing A - Stability Region of the Methods

Definition 2 Stability): A linear multistep method is said to be an A-stable method if its region of stability encloses the entire negative half-plane. (Sulaiman *et al*, 2013)

For the method (1), the using its characteristic polynomial (2), we have

Step 1:

```
subs(t = exp(phi*I), 1542/1325*t^2 - 6764/3975*t^2*h - 1948/1325*t + 1248/1325*t^2*h^2 - 7124/3975*t*h + 406/1325);
```

Output: $$\frac{1542\left(e^{I\phi}\right)^2}{1325} - \frac{6764\left(e^{I\phi}\right)^2 h}{3975} - \frac{1948\, e^{I\phi}}{1325} + \frac{1248\left(e^{I\phi}\right)^2 h^2}{1325} - \frac{7124\, e^{I\phi} h}{3975} + \frac{406}{1325}$$

Step 2:

```
solve(%, h);
```

Output: $$\frac{\frac{1691\, e^{I\phi}}{1872} + \frac{137}{144} + \frac{\sqrt{-1470455\left(e^{I\phi}\right)^2 + 11493326\, e^{I\phi} + 2031913}}{1872}}{e^{I\phi}},$$

$$\frac{\frac{1691\, e^{I\phi}}{1872} + \frac{137}{144} - \frac{\sqrt{-1470455\left(e^{I\phi}\right)^2 + 11493326\, e^{I\phi} + 2031913}}{1872}}{e^{I\phi}}$$

Step 3:

```
p := ((1691*exp(phi*I))/1872 + 137/144 + sqrt(-1470455*exp(phi*I)^2 + 11493326*exp(phi*I) +
2031913)/1872)/exp(phi*I);
```

Output: $$p := \frac{\frac{1691\, e^{I\phi}}{1872} + \frac{137}{144} + \frac{\sqrt{-1470455\left(e^{I\phi}\right)^2 + 11493326\, e^{I\phi} + 2031913}}{1872}}{e^{I\phi}};$$

Step 4:

```
q := ((1691*exp(phi*I))/1872 + 137/144 - sqrt(-1470455*exp(phi*I)^2 + 11493326*exp(phi*I) +
2031913)/1872)/exp(phi*I);
```

Output: $$q := \frac{\frac{1691\, e^{I\phi}}{1872} + \frac{137}{144} - \frac{\sqrt{-1470455\left(e^{I\phi}\right)^2 + 11493326\, e^{I\phi} + 2031913}}{1872}}{e^{I\phi}};$$

Step 5:

```
>with(plots);
```

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]

Step 6:

```
>complexplot(p,phi=0..2*Pi, numpoints=1000, colour=red);
```

Output:

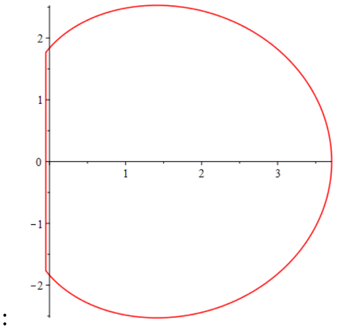


Figure 1: Absolute region for the p - point

Step 7:
complexplot(q, phi = 0 .. 2*Pi, numpoints = 1000, colour = red);
Output:

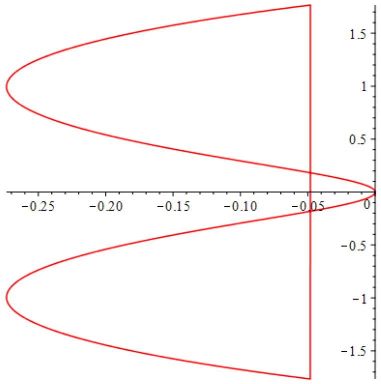


Figure 2: Absolute region for the q - point

Step 8:
complexplot([p, q], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

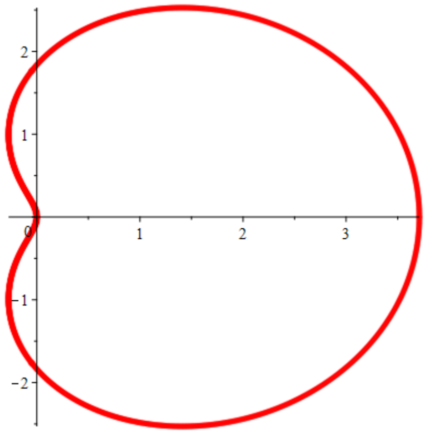
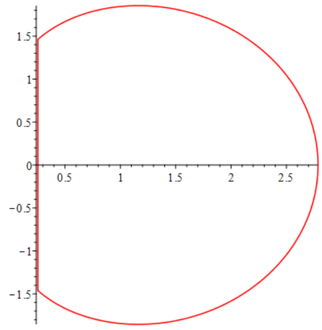


Figure 3: A – Stability region for $r = 1$ in (1)

From figure (3), the definition (2) is satisfied, the method (1) is A- stable when $r = 1$
Using similar procedure for method (3) and method (4) respectively, we have

>with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]

Step 6:
>complexplot(r,phi=0..2*Pi, numpoints=1000, colour=red);
Output:



:
Figure 4: Absolute region for the r - point

Step 7:
complexplot(s, phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

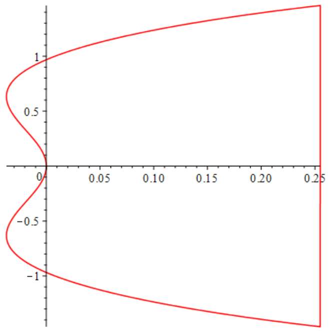


Figure 5: Absolute region for the s - point

Step 8:
complexplot([r, s], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

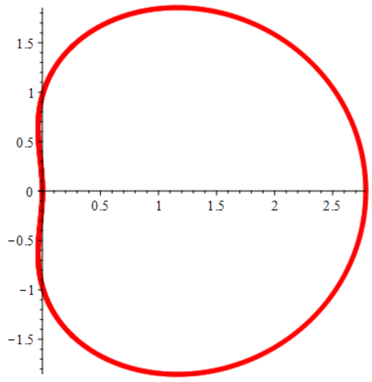
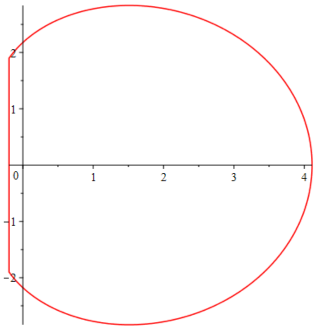


Figure 6: A – Stability region for r = 2 in (1)
From figure (6), the definition (2) is satisfied, the method (1) is A- stable when r = 2

>with(plots);
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot]

Step 6:
>complexplot(u,phi=0..2*Pi, numpoints=1000, colour=red);

Output:



:

Figure 7: Absolute region for the u - point

Step 7:
complexplot(v, phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

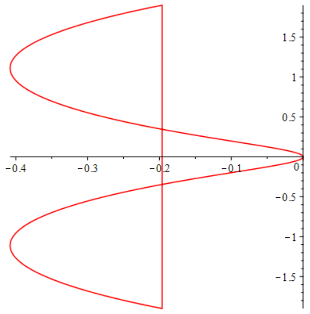


Figure 8: Absolute region for the v - point

Step 8:
complexplot([u, v], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

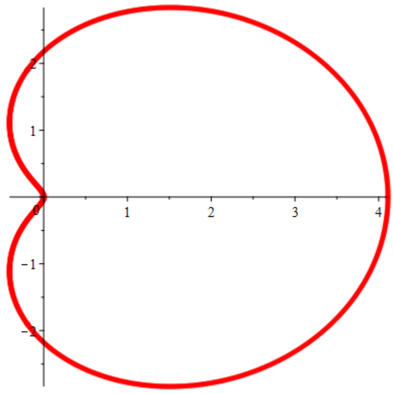


Figure 9: A – Stability region for $r = \frac{1}{2}$ in (1)

From figure (9), the definition (2) is satisfied, the method (1) is A- stable when $r = \frac{1}{2}$

Step 9:

complexplot([p, q, r, s, u, v], phi = 0 .. 2*Pi, numpoints = 1000, colour = red);

Output:

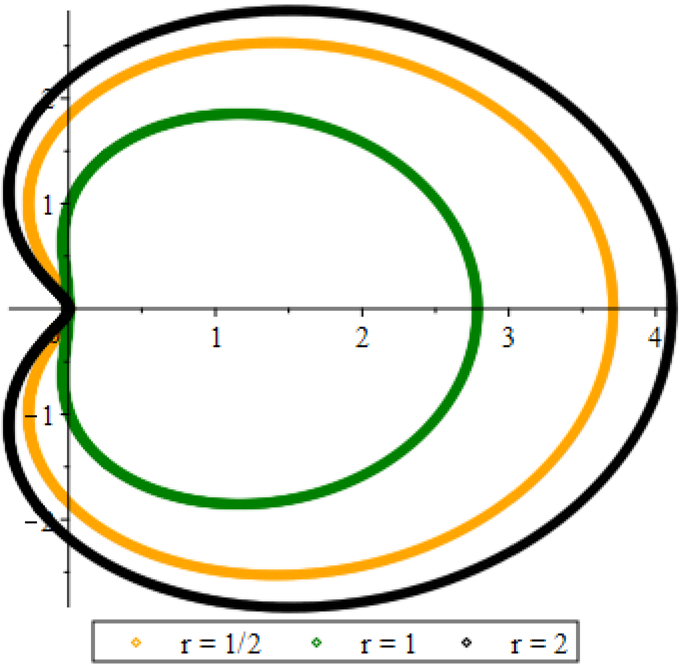


Figure 10: Combine plot of all the absolute stability regions in figure 3, 6 and 9.

CONCLUSION

A new 2 point multi – block super class of BDF for integrating system of first order stiff IVPs is considered in this work for a critical stability analysis. A simplified Maple algorithm is adopted to analyzes how to achieve zero and A – stability criteria, which remained necessary properties for optimal performance of a numerical scheme, particularly in handling stiff system of IVPs are studied with simplified code. The scheme considered in the work is variable step size, which has a variable r in the formula that can have different step sizes ratios. In this work, $r = 1$, $r = 2$ & $r = \frac{1}{2}$ are adapted in generating the methods and all its stability criteria.

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