FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 7 No. 4, August, 2023, pp 1 - 13
DOI: https://doi.org/10.33003/fjs-2023-0704-1888

MATHEMATICAL MODELING OF CORRUPTION DYNAMICS: EXAMINING THE REINTEGRATION OF FORMERLY CORRUPT INDIVIDUALS

***[1]Rwat, Solomon Isa, [2]Emmanuel, Sabastine, [1]Danat, Nanle Tanko, [1]Tsok, Samuel Hwere**

[1]Department of Mathematics, Plateau State University Bokkos, Nigeria.
[2]Department of Mathematics, Federal university Lokoja, Kogi State, Nigeria.

*Corresponding authors' email: rwatsolomonisa@gmail.com

ABSTRACT

Corruption is a global menace that undermines the foundations of societies, including the rule of law, fairness, human rights, democracy, and economic growth. This research aims to comprehensively understand the dynamics of corruption and explore strategies for its prevention and control. Specifically, it focuses on the re-integration of individuals who have recovered from corrupt practices back into the population. By evaluating the effectiveness of rehabilitation programs, the potential for relapse, and the influence of societal support systems, the study seeks to determine whether the re-integration of formerly corrupt individuals contributes to a reduction in corruption or reintroduces corrupt practices. The research employs mathematical models and analyses to investigate the transmission of corruption within the population. It examines the stability properties of uncontrolled corruption models and explores the effectiveness of different combinations of corruption prevention measures. By studying these factors, the research aims to gain insights into the underlying dynamics of corruption and identify strategies that can effectively mitigate its prevalence. The findings of this research will contribute to a deeper understanding of corruption dynamics and provide valuable insights for designing intervention programs. By informing policies and strategies, this research aims to combat corruption and foster a society that upholds integrity and ethical practices. The goal is to create a framework that supports the eradication of corruption and the promotion of transparency and accountability in all aspects of society.

Keywords: Corruption, Dynamics, Prevention, Reintegration, Strategies

INTRODUCTION

Corruption entails the exploitation of authority or power by individuals occupying public or private offices for personal gain. It takes various forms, including public corruption, private corruption, pervasive corruption, and arbitrary corruption, as described in the literature (Binuyo and Akinsola, 2020). Corruption can originate from either the supply side or the demand side (Alemneh, 2020). This pervasive problem poses serious threats to the rule of law, democracy, human rights, fairness, social justice, economic growth, and the proper functioning of market economies (Legesse and Shiferaw, 2018; Nathan and Jackob, 2019). Although many nations have implemented anti-corruption regulations and strategies, corruption continues to plague society, particularly in emerging countries (Lemecha, 2018; Binuyo and Akinsola, 2020).

To understand the dynamics of corruption transmission and develop effective intervention programs, mathematical models are used, but there is a lack of quantitative studies on corruption in current literature. One study proposed a deterministic model that analyzed corruption within a population, identifying the endemic equilibrium point, corruption-free equilibrium point, and basic reproduction number (BRN) (Egudam *et al*., 2017). Numerical simulations demonstrated that while corruption cannot be eliminated, it can be reduced to a tolerable level. Another mathematical model examined corruption dynamics, calculating the fundamental reproduction number, as well as corruption-free and endemic equilibrium points (Legesse and Shiferaw, 2018).

In a study by (Danford *et al.*, 2020), a mathematical model was constructed to examine corruption, considering the influence of anti-corruption campaigns and in-prison counseling in raising awareness about the issue. The existence of distinct corruption-free and endemic equilibrium points was explored, and the fundamental reproduction number was calculated. (Nathan and K. O. Jackob, 2020) created differential equation-based models to represent the growth or decay of corruption. Additionally, (Castillo Chavez and Song, 2004) investigated the use of game theory to combat corruption. (Binuyo and Akinsola, 2020; Li *et al*., 2016) proposed a corruption prevention model that demonstrated how corruption can be completely prevented if the rate of dismissal is equal to the rate of corruption, focusing on individuals exploiting positions of authority.

The aim of this research is to examine the effect of recycling individuals who have recovered from corrupt practices back into the population and to assess its impact on the overall levels of corruption. It seeks to understand whether the reintegration of formerly corrupt individuals contributes to the reduction of corruption or if it poses a risk of reintroducing corrupt practices within society. By investigating factors such as the success rate of rehabilitation programs, the likelihood of relapse into corrupt behaviors, and the influence of societal attitudes and support systems, this research aims to provide insights into the dynamics of reintegrating recovered individuals and to inform strategies for effectively managing corruption in the population.

The essay is structured as follows: Section 2 provides a description of the model and its underlying assumptions. Mathematical analysis, including bifurcation and model sensitivity, is performed in Section 3. Section 4 discusses the implications of recycling corrupt individuals. Numerical simulations are presented in Section 5, while Section 6 offers a discussion and conclusion.

MATERIALS AND METHODS

The Mathematical Model

In this paper, we present a deterministic model for the dynamics of corruption using five ordinary differential equations. The model divides the total population into five distinct compartments, representing different subpopulations: susceptible (S), exposed (E), corrupt (C), prison (P), and recovered (R). Throughout the study, the total population (N) is assumed to remain constant, with a natural death rate (μ). The recruitment rate is denoted as (Λ), while the contact rate is represented by (β). The rate at which exposed individuals develop clinical symptoms is ξ, γ is the rate of recovery. It is assumed in this paper that corrupt individual when caught will be imprison at the rate (ζ) and the rate at which the prisoner gets out of prison to join the recovered class is (θ).It is possible that recovered individuals acquire certain level of immunological memory for a certain duration (Nathan *et al.*, 2021; Rwat and Atinah, 2023) and we assume that recovered individuals are losing immunological memory at a rate of $(1-\delta)\beta$ where $0<\delta<1$ .The flowchart of this SECPRE model is shown in Figure 1.

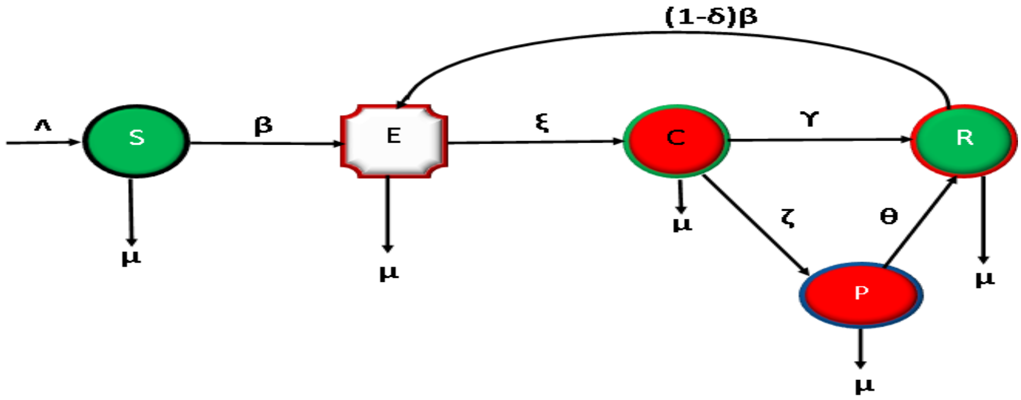

Figure 1: The Flow Diagram of the Model

The set of ordinary differential equations that describe the system is as follows:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta SC - \mu S \\
\frac{dE}{dt} &= \beta SC + (1-\delta)\beta CR - (\mu+\xi)E \\
\frac{dC}{dt} &= \xi E - (\mu+\gamma+\zeta)C \\
\frac{dP}{dt} &= \zeta C - (\mu+\theta)P \\
\frac{dR}{dt} &= \gamma C + \theta P - (\mu + (1-\delta)\beta C)R
\end{aligned}\right\} \tag{1}$$

Table1: The Model Parameters and Their Description

S/no	parameter	Description
1	Λ	Recruitment rate
2	μ	Natural death rate
3	β	Exposure rate of corruption
4	θ	Rate at which imprison individuals are release
5	ζ	Rate at which corrupt population move to prison
6	δ	Scaling factor for recycling corrupted individuals
7	$(1-\delta)\beta$	Recycling corruption rate
8	ξ	Corruption rate of exposed individual
9	γ	Recovery rate of corrupt population

Table2: The Model Variables and Their Description

S/no	variables	Description
1	S	Susceptible population
2	E	Exposed population
3	C	Corrupted population
4	P	Imprison population
5	R	Recovered population

With the initial conditions

$$S(0)=S_0\geq 0,,E(0)=E_0\geq 0,C(0)=C_0\geq 0,P(0)=P_0\geq 0R(0)=R_0\geq 0$$

All the parameters of system (1) are assumed to be positive for all time $t>0$.

Basic Properties of the Secre Model

We discuss in this section the properties of the proposed model

The Invariant Region

Theorem 1: The initial conditions of the solutions of system (1) are contained in the region $B \in \Re_{+}^{5}$, defined by

$$B = \{(S, E, C, P, R) \in \Re^5 : 0 \leq N \leq \frac{\Lambda}{\mu}\}$$

Proof: Summation of all equations of model system (1) gives

$$N = S + E + C + P + R$$

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dC}{dt} + \frac{dP}{dt} + \frac{dR}{dt}$$

$$\frac{dN}{dt} = \Lambda - \beta SC - \mu S + \beta SC + (1-\delta)\beta CR - (\mu+\xi)E + \xi E - (\mu+\gamma+\zeta)C + \zeta C - (\mu+\theta)P + \gamma C + \theta P - (\mu + (1-\delta)\beta C)R$$

$$\frac{dN}{dt} = \Lambda - \mu S - \mu E - \mu C - \mu P - \mu R$$

$$\frac{dN}{dt} = \Lambda - \mu(S + E + C + P + R)$$

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

So if we differentiate N with respect to time and substitute the values of $\frac{dS}{dt}, \frac{dE}{dt}, \frac{dC}{dt}, \frac{dP}{dt}$ and $\frac{dR}{dt}$ and simplify, we get

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

Solve this inequality gives

$$B = \{(S, E, C, P, R) \in \Re^5 : 0 \leq N \leq \frac{\Lambda}{\mu}\}$$

Positivity of the Solution

Theorem 2: Let D= $\{(S, E, C, P, R) \in \Re^5 : S(0) = S_0 > 0, E(0) = E_0 > 0, C(0) = C_0 > 0, P(0) = P_0 > 0, R(0) = R_0 > 0\}$; then the solutions of $\{S, E, C, P, R\}$ are all positive for t≥0

Proof: From the first equation of the system of differential equation (1), we get

$$\frac{dS}{dt} \geq -\mu S$$

Solving this equation and applying the initial condition yields

$$S(t) \geq S_0 e^{-\mu t} \geq 0$$

Similarly, from the second, third, fourth and five equations we get

$$E(t) \geq E_0 e^{-(\mu+\xi)t} \geq 0, \quad C(t) \geq C_0 e^{-(\mu+\gamma+\zeta)t} \geq 0, \quad P(t) \geq P_0 e^{-(\mu+\theta)t} \geq 0, \quad R(t) \geq R_0 e^{-\mu t} \geq 0$$

By concluding the proof of the theorem, it is established that the solution of the model remains positive for all future time periods.

Corruption Free Equilibrium Point

The corruption-free equilibrium (*CFE*) state, E_0, of the system model (1) is obtained by setting the right-hand side of the equation equal to zero and let *E* = *C* =P = *R* = 0, we solve for S and we get:

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right) \qquad (2)$$

Basic Reproduction Number $\Re_{cor}$

To calculate the basic reproduction number, we applied the next generation matrix method (Rwat and Atinah, 2023) to the system, utilizing the matrix F to represent the newly corrupt terms and the matrix H to represent the transition terms. Therefore, we have the following expression:

$$\frac{dE}{dt} = \beta SC + (1-\delta)\beta CR - (\mu+\xi)E$$

$$\frac{dC}{dt} = \xi E - (\mu+\gamma+\zeta)C$$

$$\frac{dP}{dt} = \zeta C - (\mu+\theta)P$$

$$\mathcal{F} = \begin{pmatrix} \beta SC + (1-\delta)\beta CR \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} (\mu+\xi)E \\ -\xi E + (\mu+\gamma+\zeta)C \\ -\zeta C + (\mu+\theta)P \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} \frac{d\mathcal{F}_1}{dE} & \frac{d\mathcal{F}_1}{dC} & \frac{d\mathcal{F}_1}{dP} \\ \frac{d\mathcal{F}_2}{dE} & \frac{d\mathcal{F}_2}{dC} & \frac{d\mathcal{F}_2}{dP} \\ \frac{d\mathcal{F}_3}{dE} & \frac{d\mathcal{F}_3}{dC} & \frac{d\mathcal{F}_3}{dP} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \frac{d\mathcal{H}_1}{dE} & \frac{d\mathcal{H}_1}{dC} & \frac{d\mathcal{H}_1}{dP} \\ \frac{d\mathcal{H}_2}{dE} & \frac{d\mathcal{H}_2}{dC} & \frac{d\mathcal{H}_2}{dP} \\ \frac{d\mathcal{H}_3}{dE} & \frac{d\mathcal{H}_3}{dC} & \frac{d\mathcal{H}_3}{dP} \end{pmatrix}$$

Where,

$$\mathcal{F}_1 = \beta SC + (1-\delta)\beta CR$$

$\mathcal{F}_2 = 0$

$\mathcal{F}_2 = 0$

and

$\mathcal{H}_1 = (\mu + \xi)E$

$\mathcal{H}_2 = -\xi E + (\mu + \gamma + \zeta)C$

$\mathcal{H}_3 = -\zeta C + (\mu + \theta)P$

Therefore, our next generation matrices are

$$\mathcal{F} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} (\mu+\xi) & 0 & 0 \\ -\xi & (\mu+\gamma+\zeta) & 0 \\ 0 & -\zeta & (\mu+\theta) \end{pmatrix}$$

The inverse of $\mathcal{H}$ is

$$\mathcal{H}^{-1} = \begin{pmatrix} \frac{1}{(\mu+\xi)} & 0 & 0 \\ \frac{\xi}{(\mu+\xi)(\gamma+\mu+\zeta)} & \frac{1}{(\gamma+\mu+\zeta)} & 0 \\ \frac{\xi\zeta}{(\mu+\theta)(\mu+\xi)(\gamma+\mu+\zeta)} & \frac{\zeta}{(\mu+\theta)(\gamma+\mu+\zeta)} & \frac{1}{(\mu+\theta)} \end{pmatrix}$$

$$\mathcal{F}\mathcal{H}^{-1} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\mu+\xi)} & 0 & 0 \\ \frac{\xi}{(\mu+\xi)(\gamma+\mu+\zeta)} & \frac{1}{(\gamma+\mu+\zeta)} & 0 \\ \frac{\xi\zeta}{(\mu+\theta)(\mu+\xi)(\gamma+\mu+\zeta)} & \frac{\zeta}{(\mu+\theta)(\gamma+\mu+\zeta)} & \frac{1}{(\mu+\theta)} \end{pmatrix}$$

$$\mathcal{F}\mathcal{H}^{-1} = \begin{pmatrix} \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)} & \frac{\beta\Lambda}{\mu(\gamma+\mu+\zeta)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

With characteristic equation

$$\lambda^3 - \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}\lambda^2$$

With solutions $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}$

$$|\mathcal{F}\mathcal{H}^{-1} - \lambda I| = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}$$

Hence the basic reproduction number is

$$\Re_{cor} = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)} \quad (3)$$

Sensitivity Analysis of the Parameters of the Basic Reproduction Number

Considering the uncertainty associated with certain parameter values, conducting a sensitivity analysis becomes valuable to assess the impact of these parameters on the basic reproduction number. This analysis allows us to identify which control measure, among the constant controls, results in the most significant decrease in the reproduction number and thus proves to be the most effective in curbing the progression of corruption. To achieve this, we calculate the normalized forward sensitivity index of the reproduction number concerning these variables. This index evaluates the correlation between the relative change of a variable and corresponding parameter variations. Following the methodology proposed in (Emmanuel *et al*., 2023; Rwat and Atinah 2023) the forward normalized sensitivity index can be calculated using the following formula:

$$\Delta_{\theta}^{\Re_{cor}} = \frac{d\Re_{cor}}{d\theta} X \frac{\theta}{\Re_{cor}} \quad (4)$$

With respect to the fundamental reproduction number, we compute the sensitivity indices for each parameter, and the results are displayed in table 2 below:

Table 3: Sensitivity Indices of the Parameter of the Basic Reproduction Number

PARAMETER	RELATIONSHIP WITH OTHER PARAMETERS	INDICES
ξ	$\frac{\mu}{(\mu+\xi)} > 0$	9. 4845×10^{-2}
β	$1 > 0$	1.00000
Λ	$1 > 0$	1.00000
γ	$\frac{-\gamma}{(\gamma+\mu+\zeta)} < 0$	-0.41929

ζ	$\frac{-\zeta}{(\gamma+\mu+\zeta)} < 0$	-0.48917
μ	$\frac{-[(\mu+\xi)(\gamma+\mu+\zeta)+\mu(\gamma+\mu+\zeta)+\mu(\mu+\xi)]}{(\mu+\xi)(\gamma+\mu+\zeta)} < 0$	-1. 1864

Interpretation of the Sensitivity Indices of the Parameters

If the values of the parameters in table 2 above with positive sensitivity indices are increased while the values of the other parameters are held constant, the spread of the disease is significantly impacted, especially by parameter β, π and α. This is so because as their levels rise, the average number of secondary infections also rises. Their values must be reduced in order to eradicate the diseases. The sensitivity analysis shows that even though the corruption-induced death rate d and the natural mortality rate μ have negative sensitivity indices, raising them to control the disease is not physiologically logical. Another potentially sensitive parameter with a negative sensitivity index that must be enhanced for the disease to be effectively controlled is the recovery rate of corruption. A bar chart depicting the parameter sensitivity indices is shown in Figure 2 below. In order to eradicate infections, parameters with their bars pointing upward must be decreased and those with their bars pointing downward must be increased.

Notwithstanding the fact that the bars on μ and γ are pointing downward, it is not physiologically tenable to claim that we should increase the rates of natural death and corruption-related mortality in order to eliminate the disease because our goal in doing so is to save life, not to take it away.

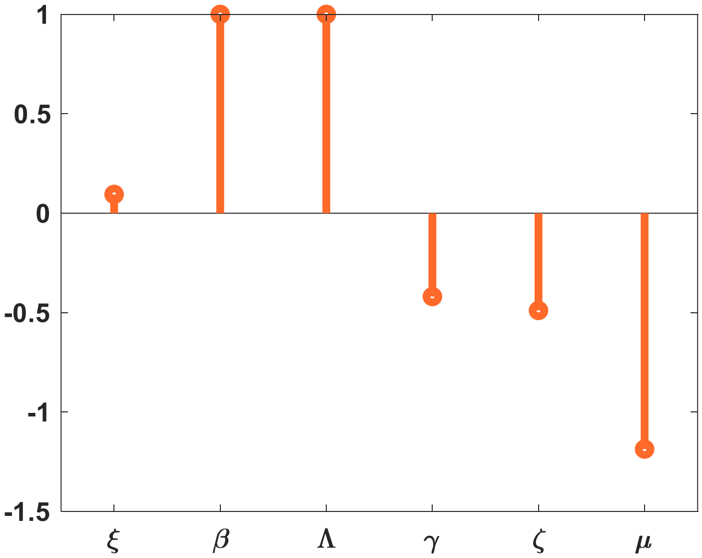

Figure 2: Depicts the Sensitivity Indices in a Stem Conversation Visually.

Local Stability Analysis of the Corruption-Free Equilibrium Point

Using Routh Hurwitz criteria (Emam, 2022), we established the following theorem.

Theorem 3: The corruption-free equilibrium point is locally asymptotically stable if $\Re_{cor}$<1 and unstable if $\Re_{cor}$>1.

Proof To prove local stability of corruption free equilibrium, we obtained the Jacobian matrix of the system (1) at the corrupt-free equilibrium$\boldsymbol{E_0}$

$$\mathrm{J} = \begin{pmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial C} & \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial C} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial C} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial R} \\ \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial E} & \frac{\partial f_4}{\partial C} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial R} \\ \frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial E} & \frac{\partial f_5}{\partial C} & \frac{\partial f_5}{\partial P} & \frac{\partial f_5}{\partial R} \end{pmatrix} \tag{5}$$

The Jacobian matrix of the system of equation (1) at the corruption-free equilibrium point becomes.

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\zeta) & 0 & 0 \\ 0 & 0 & \zeta & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{pmatrix}$$

With characteristic equation

$$(\lambda+\mu)^2(\lambda+\theta+\mu)(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\zeta)-\Lambda\beta\xi)=0$$

Which means
$(\lambda+\mu)^2=0$
$(\lambda+\mu+\theta)=0$
$(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\zeta)-\Lambda\beta\xi)=0$
Solving for λ we get
$\lambda_1=-\mu$
$\lambda_2=-\mu$
$\lambda_3=-(\mu+\theta)$
The second other two eigenvalues are the roots of the quadratic equation.
$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda+\mu(\mu+\xi)(\gamma+\mu+\zeta)-\Lambda\beta\xi=0$
Which can simplify to
$$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda+\mu(\mu+\xi)(\gamma+\mu+\zeta)\left(1-\frac{\Lambda\beta\xi}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}\right)=0$$
$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda+\mu(\mu+\xi)(\gamma+\mu+\zeta)(1-\Re_{cor})=0$
Which can be written as
$$M_0\lambda^2+M_1\lambda+M_2=0 \tag{6}$$
Where
$M_0=\mu$
$M_1=\mu(\zeta+\gamma+2\mu+\xi)$
$M_2=\mu(\mu+\xi)(\gamma+\mu+\zeta)(1-\Re_{cor})$
Using Routh Hurwitz criteria, the eigenvalues of the matrix all have negative real parts, and so the system of equations with the above characteristic equation is locally asymptotically stable.

Global Stability Analysis of the Corruption-Free Equilibrium Point
We shall investigate the global stability of the system using the techniques implemented by Castillo-Chavez and Song. (Castillo-Chavez and Song, 2004) Let re-write system (1) as follows
$\frac{dX}{dt}=$ F(X,Z),
$\frac{dZ}{dt}=\ \mathrm{G(X,Z)},\mathrm{G(X,0)}=0$
Where X represent the in-corrupted population, that is $X=\{S,R\}$ while Z represent infected population, that is Z = $\{E,C,P\}$. The corruption free equilibrium point of the model is denoted by U = (X*,0).
The point U= (X*,0) is globally asymptotically stable equilibrium for the model provided that $\Re_{cor}<1$, which is locally asymptotically stable, and the following conditions must be met:
$(\mathbb{H}_1)$: For $\frac{dX}{dt}=$ F(X,0), X* is globally asymptotically stable.
$(\mathbb{H}_2)$: $\mathrm{G(X,Z)}=\mathrm{AZ}-\widetilde{\mathrm{G}}(\mathrm{X,Z}),\widetilde{\mathrm{G}}(\mathrm{X,Z})\geq 0$ for $(\mathrm{X,Z})\in\Omega$
If model (1) meets the given two criteria, then the following theorem holds
Theorem 4: The corruption-free equilibrium point U=(X*,0) is globally asymptotically stable provided $R_{cor}<1$ and conditions $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$ are satisfied.
Proof: from the system we can get F(X,Z) and G(X, Z):

$$\mathrm{F(X,Z)}=\begin{pmatrix}\Lambda-\beta SC-\mu S\\ \gamma C+\theta P-(\mu+(1-\delta)\beta C)R\end{pmatrix}$$
$$\mathrm{G(X,Z)}=\begin{pmatrix}\beta SC+(1-\delta)\beta CR-(\mu+\xi)E\\ \xi E-(\mu+\gamma+\zeta)C\\ \zeta C-(\mu+\theta)P\end{pmatrix}$$
At $E=C=P=R=0$
$$\frac{dX}{dt}=\mathrm{F(X,0)}=\begin{pmatrix}\Lambda-\mu S\\ 0\end{pmatrix} \tag{7}$$
From the above system we see that X*=($\frac{\Lambda}{\mu}$, 0) is globally asymptotic point. This can be verified from the solutions, namely $S(\mathrm{t})=\frac{\Lambda}{\mu}+(S(0)-\frac{\Lambda}{\mu})e^{-\mu t}$. As t→∞, the solution $S(\infty)\rightarrow\frac{\Lambda}{\mu}$
Which implies the global convergence of (2) in Ω and this satisfies condition $\mathbb{H}_1$. Now from $\mathbb{H}_2$ we have that $\mathrm{G(X,Z)}=\mathrm{AZ}-\widetilde{\mathrm{G}}(\mathrm{X,Z}),\widetilde{\mathrm{G}}(\mathrm{X,Z})\geq 0$ for $(\mathrm{X,Z})\in\Omega$
Therefore, $\widetilde{\mathrm{G}}(\mathrm{X,Z})=\mathrm{AZ}-\mathrm{G(X,Z)}$.
Where A is an nxn matrix, Z is a column vector and G(X, Z) is a column vector formed from the corruption equations.
The first partial derivative of G(X, Z) with respect to $E, CandP$ computed at the corruption free equilibrium point gives matrix A.
$$A=\begin{pmatrix}-(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0\\ \xi & -(\mu+\gamma+\zeta) & 0\\ 0 & \zeta & -(\mu+\theta)\end{pmatrix} \tag{8}$$

$$\mathrm{AZ} = \begin{pmatrix} -(\mu+\xi)E + \frac{\beta\Lambda}{\mu}C \\ \xi E - (\mu+\gamma+\zeta)C \\ \zeta C - (\mu+\theta)P \end{pmatrix}$$

From the expression $\widetilde{\mathrm{G}}(\mathrm{X},\mathrm{Z}) = \mathrm{AZ} - \mathrm{G}(\mathrm{X},\mathrm{Z})$, we have

$$\widetilde{\mathrm{G}}(\mathrm{X},\mathrm{Z}) = \begin{pmatrix} -(\mu+\xi)E + \frac{\beta\Lambda}{\mu}C \\ \xi E - (\mu+\gamma+\zeta)C \\ \zeta C - (\mu+\theta)P \end{pmatrix} - \begin{pmatrix} \beta SC + (1-\delta)\beta CR - (\mu+\xi)E \\ \xi E - (\mu+\gamma+\zeta)C \\ \zeta C - (\mu+\theta)P \end{pmatrix}$$

$$\widetilde{\mathrm{G}}(\mathrm{X},\mathrm{Z}) = \begin{pmatrix} -(1-\delta)\beta R \\ 0 \end{pmatrix}$$

From $\widetilde{\mathrm{G}}(\mathrm{X},\mathrm{Z})$,we can see that $\tilde{G}_1(\mathrm{X},\mathrm{Z}) \le 0, \tilde{G}_2(\mathrm{X},\mathrm{Z}) = 0$,which leads to $\widetilde{\mathrm{G}}(\mathrm{X},\mathrm{Z}) \le 0$. That means the second condition $(\mathbb{H}_2)$ is not satisfied. So, the system of equation may not be globally asymptotically stable when $\mathfrak{R}_{\mathrm{cor}} < 1$□

Endemic Equilibrium Point

We compute the endemic equilibrium points in terms of the force of corruption using the model system equation (1) which gives

From the first equation, we got

$$\mathrm{S}^* = \frac{\Lambda}{\beta C^* + \mu}$$

From the third equation, we got

$$E = \frac{(\gamma+\mu+\zeta)C}{\xi}$$

$$\mathrm{C}^* = C$$

From the fourth equation we got

$$P = \frac{\zeta C}{(\mu+\theta)}$$

From the five equations, we got

$$R = \frac{\gamma(\mu+\theta)C + \theta\zeta C}{(\mu+\theta)(\mu+(1-\delta)\beta C)}$$

Hence the endemic equilibrium point is

$$E_* = \left(\frac{\Lambda}{\beta C^* + \mu}, \frac{(\gamma+\mu+\zeta)C}{\xi}, \mathrm{C}^*, \frac{\zeta C}{(\mu+\theta)}, \frac{\gamma(\mu+\theta)C + \theta\zeta C}{(\mu+\theta)(\mu+(1-\delta)\beta C)}\right)$$

When the values of S, E, C, H and R are substituted into equation 2 we get

$$\beta SC + (1-\delta)\beta CR - (\mu+\xi)E = 0$$

$$\frac{\Lambda\beta C}{\beta C^* + \mu} + \frac{(1-\delta)\beta C(\gamma(\mu+\theta)C + \theta\zeta C)}{(\mu+\theta)(\mu+(1-\delta)\beta C)} - \frac{(\mu+\xi)(\gamma+\mu+\zeta)C}{\xi} = 0$$

$$\frac{\Lambda\beta}{\beta C^* + \mu} + \frac{(1-\delta)\beta(\gamma(\mu+\theta) + \theta\zeta)C}{(\mu+\theta)(\mu+(1-\delta)\beta C)} - \frac{(\mu+\xi)(\gamma+\mu+\zeta)}{\xi} = 0$$

$$\Lambda\beta\xi(\mu+\theta)(\mu+(1-\delta)\beta C) + \xi(\beta C^* + \mu)(1-\delta)\beta(\gamma(\mu+\theta) + \theta\zeta)C - (\mu+\xi)(\gamma+\mu+\zeta)(\beta C^* + \mu)(\mu+\theta)(\mu+(1-\delta)\beta C) = 0$$

After simplification we got

$$\mu\beta^2(\delta-1)\big(\zeta(\theta+\mu+\xi) + (\theta+\mu)(\gamma+\mu+\xi)\big)C^2 + \Big(-\big(\beta\mu - \beta\mu(\delta-1)\big)(\theta+\mu)(\mu+\xi)(\zeta+\gamma+\mu) - \Lambda\beta^2\xi(\delta-1)(\theta+\mu) - \beta\mu\xi(\gamma(\theta+\mu) + \zeta\theta)(\delta-1)\Big)C - \mu^2(\theta+\mu)(\mu+\xi)(\zeta+\gamma+\mu)(1-\mathfrak{R}_{cor}) = 0$$

$$B_0C^2 + B_1C + B_2 = 0 \tag{9}$$

Where

$$B_0 = \mu\beta^2(\delta-1)\big(\zeta(\theta+\mu+\xi) + (\theta+\mu)(\gamma+\mu+\xi)\big)$$

$$B_1 = \Big(-\big(\beta\mu - \beta\mu(\delta-1)\big)(\theta+\mu)(\mu+\xi)(\zeta+\gamma+\mu) - \Lambda\beta^2\xi(\delta-1)(\theta+\mu) - \beta\mu\xi(\gamma(\theta+\mu) + \zeta\theta)(\delta-1)\Big)$$

$$B_2 = -\mu^2(\theta+\mu)(\mu+\xi)(\zeta+\gamma+\mu)(1-\mathfrak{R}_{cor})$$

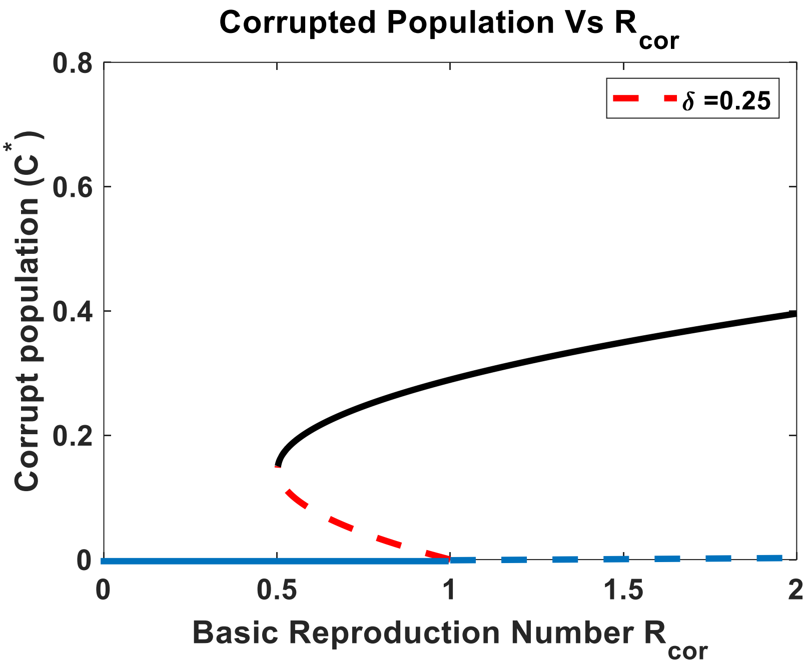

Figure 3: Plot of Corrupt Population Against R_{cor}

Determination of Backward or Forward Bifurcation

To explore the possibility of backward or forward bifurcation of the model system (1) we use the center manifold theory. The backward or forward bifurcation phenomena will help us determine the local stability of the system at the endemic equilibrium point. Now from the basic reproduction number

$$\Re_{cor} = \frac{\beta\xi\Lambda}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}$$

We let β=β^* be the bifurcation parameter and consider the value of $\Re_{cor}$ at 1, then, we make β the subject of the formula to obtain

$$\beta^* = \frac{\mu(\mu+\xi)(\gamma+\mu+\zeta)}{\xi\Lambda}$$

Let rename the variables as $S=x_1$,$E=x_2$,$C=x_3$, $P=x_4$and R=x_5.

Also let use the vector X= $(x_1, x_2, x_3, x_4, x_5)^T$

formulated as $\frac{dX}{dt}$ = G(x),

Where G = $(g_1, g_2, g_3, g_4, g_5)^T$

Now the corruption-free equilibrium becomes $X_0 = \left(x_1 = \frac{\pi}{\mu}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0\right)$

Substitute these values of $S=x_1$,$E=x_2$,$C=x_3$,$P=x_4$and R=x_5 into (1) we get

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= \pi - \beta x_3 x_1 - \mu x_1 \\
\frac{dx_2}{dt} &= \beta x_3 x_1 + (1-\delta)\beta x_3 x_5 - (\alpha+\xi)x_2 \\
\frac{dx_3}{dt} &= \xi x_2 - (\gamma+\mu+\zeta)x_3 \\
\frac{dx_4}{dt} &= \zeta x_3 - (\mu+\theta)x_4 \\
\frac{dx_5}{dt} &= \gamma x_3 + \theta x_4 - (\mu + (1-\delta)\beta x_3)x_5
\end{aligned}\right\} \quad (10)$$

and its Jacobian at the corruption-free equilibrium point is

$$J(E^0_{cor}) = \begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\zeta) & 0 & 0 \\ 0 & 0 & \zeta & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{pmatrix}$$

With characteristic equation

$$(\lambda+\mu)^2(\lambda+\theta+\mu)(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\zeta) - \Lambda\beta\xi) = 0$$

Which means
$(\lambda+\mu)^2=0$
$(\lambda+\mu+\theta)=0$
$(\mu(\lambda+\mu+\xi)(\lambda+\gamma+\mu+\zeta)-\Lambda\beta\xi)=0$

Solving for λ we get
$\lambda_1=-\mu$
$\lambda_2=-\mu$
$\lambda_3=-(\mu+\theta)$

The second other two eigenvalues are the roots of the quadratic equation
$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda+\mu(\mu+\xi)(\gamma+\mu+\zeta)-\Lambda\beta\xi=0$

Which can simplify to
$$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda+\mu(\mu+\xi)(\gamma+\mu+\zeta)\left(1-\frac{\Lambda\beta\xi}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}\right)=0$$
$\mathrm{m}_0\lambda^2+\mathrm{m}_1\lambda+\mathrm{m}_2=0$

Where
$\mathrm{m}_0=\mu$
$\mathrm{m}_1=\mu(\zeta+\gamma+2\mu+\xi)$
$$\mathrm{m}_2=\mu(\mu+\xi)(\gamma+\mu+\zeta)\left(1-\frac{\Lambda\beta\xi}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}\right)$$

Substitute the value of β we get
$$\mathrm{m}_2=\mu(\mu+\xi)(\gamma+\mu+\zeta)\left(1-\frac{\Lambda\beta\xi}{\mu(\mu+\xi)(\gamma+\mu+\zeta)}\right)$$
$\mathrm{m}_2=\mu(\mu+\xi)(\gamma+\mu+\zeta)(1-1)$
$\mathrm{m}_2=0$

Therefore, the equation becomes
$\mathrm{m}_0\lambda^2+\mathrm{m}_1\lambda=0$

That is
$\mu\lambda^2+\mu(\zeta+\gamma+2\mu+\xi)\lambda=0$
$\lambda\big(\lambda+\mu(\zeta+\gamma+2\mu+\xi)\big)=0$
Which gives
$\lambda_4=0,\quad \lambda_5=-\mu(\zeta+\gamma+2\mu+\xi)$
Now λ_3 is a simple eigenvalue, so that allow us to use the center manifold theorem. So, we proceed with the computation as follows: The right eigenvector, $\mathcal{W}=\ (\mathcal{W}_1,\mathcal{W}_2\ ,\mathcal{W}_3,\mathcal{W}_4,\mathcal{W}_5)^T$, associated with this simple zero eigenvalue can be obtained from$J(E_{cor}^0)\mathcal{W}=0$

$$\begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -(\mu+\xi) & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \xi & -(\mu+\gamma+\zeta) & 0 & 0 \\ 0 & 0 & \zeta & -(\mu+\theta) & 0 \\ 0 & 0 & \gamma & \theta & -\mu \end{pmatrix}\begin{pmatrix}\mathcal{W}_1\\ \mathcal{W}_2\\ \mathcal{W}_3\\ \mathcal{W}_4\\ \mathcal{W}_5\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$
Transform the matrix into system of equation we get
$$-\mu\mathcal{W}_1-\frac{\beta\Lambda}{\mu}\mathcal{W}_3=0$$
$$-(\mu+\xi)\mathcal{W}_2+\frac{\beta\Lambda}{\mu}\mathcal{W}_3=0$$
$\xi\mathcal{W}_2-(\gamma+\mu+\zeta)\mathcal{W}_3=0$
$\zeta\mathcal{W}_3-(\mu+\theta)\mathcal{W}_4=0$
$\gamma\mathcal{W}_3+\theta\mathcal{W}_4-\mu\mathcal{W}_5=0$
Solve $\mathcal{W}_1$,$\mathcal{W}_2$,$\mathcal{W}_3$,$\mathcal{W}_4$,$\mathcal{W}_5$ we get $\mathcal{W}_3=\mathcal{W}_3>0,\ w_2=\frac{\beta\Lambda\mathcal{W}_3}{\mu(\mu+\xi)},\ \mathcal{W}_4=\frac{\zeta\mathcal{W}_3}{\mu+\theta},\ \mathcal{W}_1=-\frac{\beta\Lambda\mathcal{W}_3}{\mu^2},w_5=\frac{\gamma(\mu+\theta)\mathcal{W}_3+\zeta\theta\mathcal{W}_3}{\mu(\mu+\theta)}$
Therefore, $\mathcal{W}=\ (-\frac{\beta\Lambda\mathcal{W}_3}{\mu^2},\frac{\beta\Lambda\mathcal{W}_3}{\mu(\mu+\xi)}\ ,w_3\frac{\zeta\mathcal{W}_3}{\mu+\theta},\frac{\gamma(\mu+\theta)\mathcal{W}_3+\zeta\theta\mathcal{W}_3}{\mu(\mu+\theta)})^T$

The transpose of the matrix gives

$$\begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\xi) & \xi & 0 & 0 \\ -\frac{\beta\Lambda}{\mu} & \frac{\beta\Lambda}{\mu} & -(\mu+\gamma+\zeta) & \zeta & \gamma \\ 0 & 0 & 0 & -(\mu+\theta) & \theta \\ 0 & 0 & 0 & 0 & -\mu \end{pmatrix}$$

The left eigenvector, $V = \ (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, , \mathcal{V}_5)$, associated with this simple zero eigenvalue at $\beta = \beta^*$can be obtained from $VJ(E^0_{cor}) = 0$ which can also be written as $J(E^0_{cor})^T V^T$and is given by

$$\begin{pmatrix} -\mu & 0 & 0 & 0 & 0 \\ 0 & -(\mu+\xi) & \xi & 0 & 0 \\ -\frac{\beta\Lambda}{\mu} & \frac{\beta\Lambda}{\mu} & -(\mu+\gamma+\zeta) & \zeta & \gamma \\ 0 & 0 & 0 & -(\mu+\theta) & \theta \\ 0 & 0 & 0 & 0 & -\mu \end{pmatrix}\begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \mathcal{V}_3 \\ \mathcal{V}_4 \\ \mathcal{V}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the above into a system of equation we have

$-\mu\mathcal{V}_1 = 0$

$-(\mu+\xi)\mathcal{V}_2 + \xi\mathcal{V}_3 = 0$

$-\frac{\beta\Lambda}{\mu}\mathcal{V}_1 + \frac{\beta\Lambda}{\mu}\mathcal{V}_2 - (\mu+\gamma+\zeta)\mathcal{V}_3 + \zeta\mathcal{V}_4 + \gamma\mathcal{V}_5 = 0$

$-(\mu+\theta)\mathcal{V}_4 + \theta\mathcal{V}_5$

$-\mu v_5 = 0$

Now solve for v_1,v_2,v_3,v_4,v_5 we get

$V = \ (0, \frac{\xi\mathcal{V}_3}{\mu+\xi}, \mathcal{V}_3 0,0)^T$

We proceed to compute the value of $\mathcal{A}$ using the formula

$$\mathcal{A} = \sum_{\mathcal{K},i,j=1}^{n} \mathcal{V}_{\mathcal{K}}\, \mathcal{W}_i \mathcal{W}_j \frac{\partial^2 g_{\mathcal{K}}}{\partial x_i\, \partial x_j}(E_0)$$

$$\mathcal{A} = \mathcal{V}_1 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_1}{\partial x_i\, \partial x_j}(E_0) + \mathcal{V}_2 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_2}{\partial x_i\, \partial x_j}(E_0) + \mathcal{V}_3 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_3}{\partial x_i\, \partial x_j}(E_0) + \mathcal{V}_4 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_4}{\partial x_i\, \partial x_j}(E_0) + \mathcal{V}_5 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_4}{\partial x_i\, \partial x_j}(E_0)$$

Since $\mathcal{V}_1 = \mathcal{V}_4 = \mathcal{V}_5 = 0$and the second derivative of $\frac{\partial^2 g_3}{\partial x_i\, \partial x_j}(E_0) = 0$, We are left with

$$\mathcal{A} = \mathcal{V}_2 \sum_{i,j=1}^{5} \mathcal{W}_i\, \mathcal{W}_j \frac{\partial^2 g_2}{\partial x_i\, \partial x_j}(E_0)$$

$$\mathcal{A} = \left(\frac{\beta(2\xi+\mu)\mathcal{V}_3\mathcal{W}_3\mathcal{W}_3}{(\mu+\xi)\mu^2(\mu+\theta)}\right)\mathcal{Q}$$

$$\mathcal{Q} = (1-\delta)\mu(\gamma(\mu+\theta) + \zeta\theta) - \beta\Lambda(\mu+\theta)$$

$$\frac{dx_2}{dt} = \beta x_3 x_1 + (1-\delta)\beta x_3 x_5 - (\alpha+\xi)x_2$$

Now we compute the second partial derivatives of the $f_i{}'s$with respect to the x_i's and got

$\frac{\partial^2 g_2}{\partial x_1\, \partial x_3} = \frac{\partial^2 g_2}{\partial x_3\, \partial x_1} = \beta$, $\frac{\partial^2 g_2}{\partial x_3\, \partial x_5} = \frac{\partial^2 g_2}{\partial x_5\, \partial x_3} = (1-\delta)\beta$

while the rest of the partial derivatives are zero. After some calculation we got the expression for $\mathcal{A}$

Where

Now we compute the value of $\mathcal{B}$ using the formula

$$\mathcal{B} = \sum_{\mathcal{K},i=1}^{n} \mathcal{V}_{\mathcal{K}}\, \mathcal{W}_i \frac{\partial^2 g_{\mathcal{K}}}{\partial x_i\, \partial\beta}(E_0, \beta)$$

Now let compute the values of the second partial derivatives of the $g_i{}'$s with respect to the $x_i{}'$s and β's to get

$\frac{\partial^2 g_2}{\partial x_3\, \partial\beta} = \beta x_1 = \frac{\beta\pi}{\mu}$,

The rest of the partial derivatives are zero.

Substitute the values of $\frac{\partial^2 g_2}{\partial x_3\, \partial\beta}$, $\mathcal{V}_2$ and $\mathcal{W}_3$ into the expression for $\mathcal{B}$, we got

$$\mathcal{B} = \frac{\xi\beta\Lambda}{\mu(\mu+\xi)} v_3 w_3 > 0$$

We now use Castillo-Chavez and Song stated below

Theorem 5: consider the following general system of ordinary differential equation with parameter φ

$$\frac{dx}{dt} = g(x,\varphi), \text{g}: \text{R}^n \text{x R and g} \in C^2(\text{R}^n \text{x R})$$

Where 0 is an equilibrium point of the system is($thatisg(0,\varphi) \equiv 0$ for all φ) and assume:

1. A=$D_x g(0,0) = (\frac{\partial g_i}{\partial x_j}(0,0))$ is the linearization matrix of the system around the equilibrium point 0 with φ evaluated at 0;
2. All other eigenvalues of A have negative real components except for the simple eigenvalue zero.
3. Matrix A has right eigenvector W and a left eigenvector V corresponding to the zero eigenvalue.

   Let $g_{\mathcal{K}}$ be the k^{th} component of G and

$$\mathcal{A} = \sum_{\mathcal{K},i,j=1}^{n} \mathcal{V}_{\mathcal{K}} \mathcal{W}_i \mathcal{W}_j \frac{\partial^2 g_{\mathcal{K}}}{\partial x_i \partial x_j}(E_0), \quad \mathcal{B} = \sum_{\mathcal{K},i=1}^{n} \mathcal{V}_{\mathcal{K}} \mathcal{W}_i \frac{\partial^2 g_{\mathcal{K}}}{\partial x_i \partial \text{a}}(E_0, \beta)$$

Then the local dynamics of the system around the x=0 are totally determined by $\mathcal{A}$ and $\mathcal{B}$. Particularly,

1. $\mathcal{A} > 0, \mathcal{B} > 0, when\ \varphi < 0\ with\ ||\varphi|| \ll 1, (0,0)$ is locally asymptotically stable hence there exists a positive unstable equilibrium. When $0 < \varphi \ll 1$, (0,0) is unstable then there exists a negative and locally asymptotically stable equilibrium.
2. $\mathcal{A} < 0, \mathcal{B} < 0, when\ \varphi < 0\ with\ ||\varphi|| \ll 1, (0,0)$ is unstable, when $0 < \varphi \ll 1$, (0,0) is locally unstable and there exists a positive unstable equilibrium.
3. $\mathcal{A} > 0, \mathcal{B} < 0, when\ \varphi < 0\ with\ ||\varphi|| \ll 1, (0,0)$ is unstable, and there exists locally asymptotically stable equilibrium; when $0 < \varphi \ll 1$, (0,0) is stable and positive unstable equilibrium appears
4. $\mathcal{A} < 0, \mathcal{B} > 0$, when φ changes from negative to positive, x=0 changes its stability from stable to unstable.

Correspondingly, a negative unstable equilibrium becomes locally asymptotically stable.

Now from our values of $\mathcal{A}$ and $\mathcal{B}$, The value of $\mathcal{B}$ is positive so whether the bifurcation will be forward bifurcation or backward at β=β* depends on the values of $\mathcal{A}$ and in turn depend on the values of $\mathfrak{Q}$. If the value of $\mathfrak{Q}$ is negative we will have forward bifurcation while if $\mathfrak{Q}$is positive we will have backward bifurcation.

(a) We have forward bifurcation if

$(1-\delta)\mu(\gamma(\mu+\theta)+\zeta\theta) - \beta\Lambda(\mu+\theta) < 0$ or $(1-\delta)\mu(\gamma(\mu+\theta)+\zeta\theta) < \beta\Lambda(\mu+\theta)$

(b) We have backward bifurcation if

$(1-\delta)\mu(\gamma(\mu+\theta)+\zeta\theta) - \beta\Lambda(\mu+\theta) > 0$ or $(1-\delta)\mu(\gamma(\mu+\theta)+\zeta\theta) > \beta\Lambda(\mu+\theta)$

Global Stability Analysis of the Endemic Equilibrium Point

Theorem 6: If R_{cov}>1, the endemic equilibrium E_{cov}^0 of the model is globally asymptotically stable.

Proof: by Lyapunov's direct method and LaSale's Invariant principle we prove the above theorem by defining a Lyapunov's function

$$\text{L}(S^*, E^*, \boldsymbol{C}^*, \boldsymbol{P}^*, \text{R}^*) = (S - S^* - \ln\frac{\text{S}^*}{S}) + (E - E^* - E^*\ln\frac{E^*}{E}) + (\text{C} - \boldsymbol{C}^* - \boldsymbol{C}^*\ln\frac{\boldsymbol{C}^*}{\text{C}}) + (\text{P} - \boldsymbol{P}^* - \boldsymbol{P}^*\ln\frac{\boldsymbol{C}^*}{\text{C}}) + (\text{R} - \text{R}^* - \text{R}^*\ln\frac{\text{R}^*}{\text{R}})$$

Differentiating L with respect to t produced

$$\frac{dL}{dt} = \frac{(S-S^*)}{S}\frac{dS}{dt} + \frac{(E-E^*)}{E}\frac{dE}{dt} + \frac{(C-C^*)}{C}\frac{dC}{dt} + \frac{(P-P^*)}{P}\frac{dP}{dt} + \frac{(R-R^*)}{R}\frac{dR}{dt}$$

Substitute the values of $\frac{dS}{dt}, \frac{dE}{dt}, \frac{dC}{dt}, \frac{dP}{dt}$ and $\frac{dR}{dt}$ into $\frac{dL}{dt}$ and then simplify to get

$$\frac{dL}{dt} = (\Lambda + S^*\beta C + S^*\mu + E^*(\mu+\xi) + C^*(\mu+\gamma+\zeta) + P^*(\mu+\theta) + R^*(\mu+(1-\delta)\beta C) - \left(\frac{S^*}{S}\Lambda + \mu S + \mu E + \frac{E^*}{E}\beta SC + \frac{E^*}{E}(1-\delta)\beta CR + \mu C + \frac{C^*}{C}\xi E + \mu P + \frac{P^*}{P}\zeta C + \mu R + \frac{R^*}{R}\gamma C + \frac{R^*}{R}\theta P)\right)$$

Which of the form

$$\frac{dL}{dt} = \mathfrak{X}_1 - \mathfrak{X}_2$$

Where

$$\mathfrak{X}_1 = \Lambda + S^*\mu + \beta S^* C + E^*(\mu+\xi) + C^*(\mu+\gamma+\zeta) + P^*(\mu+\theta) + \mu R^* + (1-\delta)\beta C R^*$$

$$\mathfrak{X}_2 = \frac{S^*}{S}\Lambda + \mu S + \frac{E^*}{E}\beta SC + \mu E + \frac{E^*}{E}(1-\delta)\beta CR + \mu C + \frac{C^*}{C}\xi E + \mu P + \frac{P^*}{P}\zeta C + \mu R + \frac{R^*}{R}\gamma C + \frac{R^*}{R}\theta P)$$

$\frac{dL}{dt} \leq 0$ if Q_1 is less then Q_2

$\frac{dL}{dt} = 0$ if and only if $S = S^*, E = E^*, C = C^*, P = P^*, R = R^*$

Therefore, the largest invariant impact invariant set in $\{(S^*, E^*, C^*, P^*, R^*) \in \Omega: \frac{dL}{dt} = 0\}$ is the singleton set E_0^*, where E_0^* is the endemic equilibrium of the system (1). Therefore, by Lasalle's Invariant principle, it implies that E_{cov}^* is globally asymptotically stable in Ω if $\mathfrak{X}_1$ is less than $\mathfrak{X}_2$.

Effect of Recycling Corrupted Individuals into the Society

At $\delta = 1$, there is no recycling of corrupt individual into the society.

We know that $\mathcal{A} = \frac{\beta(2\xi+\mu)\mathcal{V}_3\mathcal{W}_3\mathcal{W}_3}{(\mu+\xi)\mu^2(\mu+\theta)}\big((1-\delta)\mu(\gamma(\mu+\theta)+\zeta\theta) - \beta\Lambda(\mu+\theta)\big)$

Making $\delta = 1$in the expression for the bifurcation coefficient gives the value of $\mathcal{A}$ as

$$\mathcal{A} = -\frac{\Lambda\beta^2(2\xi+\mu)}{(\mu+\xi)\mu^2}\mathcal{V}_3\mathcal{W}_3^2 < 0$$

That is, the value of $\mathcal{A}$ becomes negative while the value of $\mathcal{B}$ is positive. This satisfies the condition of center manifold theorem for forward bifurcation.

$$\mathcal{B} = \frac{\xi\beta\Lambda}{\mu(\mu+\xi)}\mathcal{V}_3\mathcal{W}_3 > 0$$

Also when we differentiate the bifurcation coefficient $\mathcal{A}$ with respect to δ, we get

Let $(1-\delta)\beta = \mathcal{G}$

Then $\frac{\partial\delta}{\partial\mathcal{G}} = -1$

$$\frac{\partial\mathcal{A}}{\partial\delta} = -\frac{\beta(2\xi+\mu)(\gamma(\mu+\theta)+\zeta\theta)}{\mu(\mu+\xi)(\mu+\theta)}\mathcal{V}_3\mathcal{W}_3^2$$

Therefore,

$$\frac{\partial\mathcal{A}}{\partial\mathcal{G}} = \frac{\partial\mathcal{A}}{\partial\delta}\times\frac{\partial\delta}{\partial\mathcal{G}} = -\frac{\beta(2\xi+\mu)(\gamma(\mu+\theta)+\zeta\theta)}{\mu(\mu+\xi)(\mu+\theta)}\mathcal{V}_3\mathcal{W}_3^2 \times -1$$

$$\frac{\partial\mathcal{A}}{\partial\mathcal{G}} = \frac{\beta(2\xi+\mu)(\gamma(\mu+\theta)+\zeta\theta)}{\mu(\mu+\xi)(\mu+\theta)}\mathcal{V}_3\mathcal{W}_3^2$$

This shows that the bifurcation coefficient is an increasing function of the re-cycling coefficient, hence increasing the recycling coefficient will increase the bifurcation coefficient. Hence the recycling coefficient need to be decreased to decrease the bifurcation coefficient to produce a forward bifurcation. This is shown in figure (4b)

Numerical Simulation

Table 4: The Parameters Values and Their Sources

S/No	Parameter	Value	Source
1	ξ	0.12502	(Rwat and Atinah, 2023)
2	β	0.50	(Nathan, 2021)
3	Λ	0.00157	(Rwat and Atinah, 2023)
4	μ	0.0131	(Rwat and Atinah, 2023)
5	γ	0.06	(Alemneh, 2020)
6	d	0.07	(Nathan, *et al., 2021*)
7	δ	(0 1)0.24	Assumed
8	θ	0.56	Assumed
9	ζ	0.07	Assumed

We used the values of the parameters in table (4) to show how the re-infection parameter affect the bifurcation phenomenon of the model. (see Sabastine *et al.*, 2022)

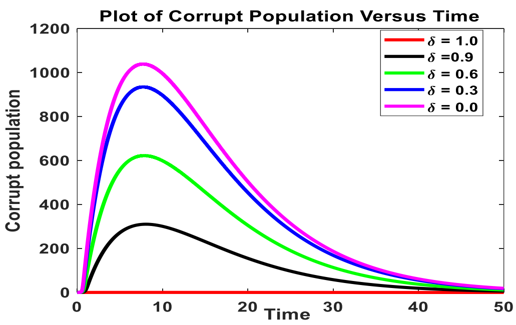

Figure 4a: plot of corrupt population against time with varying values of δ

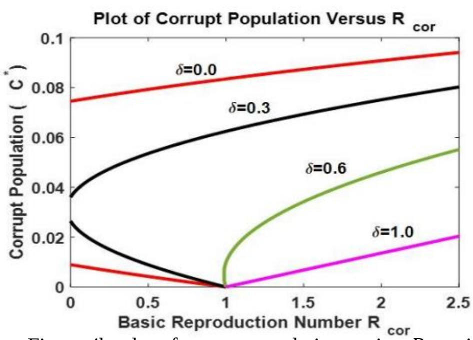

Figure 4b: plot of corrupt population against $R_{cor}$with varying values of δ

DISCUSSION

In this studies we presented a mathematical models for corruption dynamics. The model considered re-cycling of the recovered individuals after some times when they have repented from the act of corruption. The mathematical analysis of the model suggests two different types of bifurcation phenomenon. The model exhibits a backward bifurcation at various values of re-cycling parameter while its exhibits global forward bifurcation when the recycling parameter is removed. Using parameter values in articles associated with the corruption situation, we showed how re-infection can trigger backward bifurcation in the model. A backward bifurcation will make it a lot difficult to control corruption because a critical value of the reproduction number which is much less than 1 is needed to achieve corruption-free state.

The basic reproduction number R_{cor} for the model often defined as the ratio of contact rate to the recovery rate is one of the main indicators used to measure transmissibility of an epidemic in epidemiology. This is due to the believe that the basic reproduction number is necessary and sufficient to determine the fate of an epidemic. However, re-cycling of corruption individuals creates the possibility of a backward bifurcation as shown in figure 3 in this work. Therefore, controlling the basic reproduction number R_{cor} (i.e., controlling contact rate and recovery rate) alone will not be sufficient to control the transmission of corruption. We should therefore control the possibility of re-cycling people that have recovered from corrupt practices into the population in order to eradicate the corrupt individual from the society.

CONCLUSION

In figure 4a we start with the value of the scaling factor (δ) from one and keep on decreasing the values till we reach 0 (1.0, 0.9, 0.6, 0.3, 0.0). It is observed that the population of the corrupt population keep on increasing. Please note that at $\delta = 0$, the re-cycling rate is equal to the exposure rate to corruption, but as the value of δ increases from zero to one, the re-cycling rate decreases. As the value of δ becomes 1, the re-cycling rate becomes zero and therefore no re-cycling occurs.

In figure 4b, as the value of δ increases from $\delta = 0.0$ to 0.6, the model exhibits backward bifurcation. At $\delta = 1$ the bifurcation becomes forward.

REFERENCES

Alemneh, H. T. (2020). Mathematical modelling, analysis, and optimal control of corruption dynamics. *Journal of Applied Mathematics*, 2020, Article ID 5109841, 13 pp.

Binuyo, A. O., & Akinsola, V. O. (2020). Stability analysis of the corruption-free equilibrium of the mathematical model of corruption in Nigeria. *Mathematical Journal of Interdisciplinary Sciences*, 8(2), 61–68.

Bitterhout, S., & Simo-Kengne, B. D. (2020). The effect of corruption on economic growth in the BRICS countries: A panel data analysis, 12(3), 1–23.

Carr, J. (1981). Application of center manifold theory. Springer Verlag, New York, NY, USA.

Castillo-Chavez, C., & Song, B. (2004). Dynamical models of tuberculosis and their applications. *Mathematical Biosciences and Engineering*, 1, Article ID 361404.

Danford, O., Kimathi, M., & Mirau, S. (2020). Mathematical modeling and analysis of corruption dynamics with control measures in Tanzania. *Journal of Mathematics and Informatics*, 1, 58–79.

Egudam, F. Y., Oguntolu, F., & Ashezua, T. (2017). Understanding the dynamics of corruption using a mathematical modeling approach. *International Journal of Innovative Science, Engineering, and Technology*, 4(4), 2348–7968.

Emmanuel, S., Edogbanya, H. O., & Omodara, B. (2020). Numerical Solutions of Elliptic Partial Differential Equations. *International Journal of Science for Global Sustainability*, 6(1), 12. Retrieved from https://fugus-ijsgs.com.ng/index.php/ijsgs/article/view/127

Emmanuel, S., Sathasivam, S., Ali, M. K. M., Kee, T. J., & Ling, Y. S. (2023). Estimating the transmission dynamics of Dengue fever in subtropical Malaysia using SEIR model. *Journal of Quality Measurement and Analysis*, 19(2), 45–56.

Emam, F. (2022). Routh-Hurwitz stability criterion. Retrieved from http://www.aast.edu/cv.php?disp_unit=346&ser=68525.

LaSalle, J. P. (1976). Stability of dynamical systems. In Proceedings of the Regional Conference Series in Applied Mathematics. *SIAM, Philadelphia*, PA, USA, April 1976.

Legesse, L., & Shiferaw, F. (2018). Mathematical modelling and analysis of corruption dynamics. *Ethiopian Journal of Sciences and Sustainable Development*, 5(2), p-ISSN 1998-0531.

Lemecha, L. (2018). Modelling corruption dynamics and its analysis. *Ethiopian Journal of Sciences and Sustainable Development*, 5(2), 13–27.

Li, J., Xiao, Y., Zhang, F., & Yang, Y. (2016). An algebraic approach to proving the global stability of a class of epidemic models. *Nonlinear Analysis*: Real World Applications, 13.

Nathan, O. M., & Jackob, K. O. (2019). Stability analysis in a mathematical model of corruption in Kenya. *Asian Research Journal of Mathematics*, 28, 1–15.

Nathan, O. M., Haileyesus, T. A., Cyrus, G. N., & Grace, G. M. (2021). Mathematical modelling and analysis of corruption of morals among adolescents with control measures in Kenya. April 21, 2021.

Oscar, D., Mark, K., & Silas, M. (2020). Mathematical modelling and analysis of corruption dynamics with control measures in Tanzania. *Journal of Mathematics and Informatics*, 19, 57–79.

Rwat, S. I., & Atinah, A. (2023). Backward bifurcation and hysteresis in a mathematical model of COVID-19 with imperfect vaccine. *Matematika, MJIM*, 39(1), 87–99.

Sabastine, E., Okoye, I., Ezenweke, C. P., Shobanke, D. A., & Adeniyi, I. A. (2022). Estimating nonlinear regression parameters using particle swarm optimization and genetic algorithm. *FUDMA Journal of Sciences (FJS)*, 6(6), 202–213. DOI: https://doi.org/10.33003/fjs-2022-0606-1114.

Transparency International. (2017). Corruption Perception Index. Retrieved September 2018 from *www.transparency.org.*