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THE GENERALIZED SUNDMAN TRANSFORMATION AND DIFFERENTIAL FORMS FOR LINEARIZING THE VARIABLE FREQUENCY OSCILLATOR EQUATION AND THE MODIFIED IVEY'S EQUATION

[*1,2]Orverem J. M., [2]Haruna Y.

[1]Department of Mathematics, Federal University Dutsin-Ma, Katsina State, Nigeria
[2]Department of Mathematical Sciences, Abubakar Tafawa Balewa University Bauchi, Bauchi State, Nigeria

*Corresponding authors' email: orveremjoel@yahoo.com

ABSTRACT

The linearization approach is used in this contribution to acquire the answers to the variable frequency oscillator equation as well as the modified Ivey's equation. Differential forms (DF) and the generalized Sundman transformation (GST) are two linearization techniques that are considered. It is found that the modified Ivey's equation cannot be linearized using differential forms, while the equation for a variable frequency oscillator can. However, using GST, the modified Ivey's equation can be linearized.

Keywords: Linearization, Generalized Sundman transformation, Differential forms, Ivey's equation, Variable frequency oscillator equation

INTRODUCTION

In the past few decades, nonlinear oscillators have drawn a lot of attention from engineers because they are crucial modeling tools in the biological and physical sciences. The models' capacity to synchronize with other oscillators or with outside driving impulses makes them fascinating. These synchronization skills are however constrained, and it is not always simple to pick the model parameters effectively to achieve proper synchronization with the external driving signals (Righetti et al., 2009). In fact, an oscillator has a finite entrainment area that depends on a variety of factors, including the coupling efficiency and the frequency difference between the oscillator and the driving signal. To solve the Fredholm Integro-Differential Equation, the variational iteration method was used. It was sensible to choose the initial approximation that satisfies the Fredholm Integro-differential Equation's initial condition (Lanlege et al., 2023). Mat-Lab and Maple software were used to assess the numerical results, which were then compared to the exact solution to demonstrate the effectiveness of the Method. When compared to the exact solution, the findings demonstrate how effective, trustworthy, and highly accurate the Variational Iteration Method is for solving the Fredholm Integro-Differential Equation.

In electronics, a variable frequency oscillator (VFO) is an oscillator whose frequency may be tweaked, or varied, throughout a certain range. It controls the frequency to which the device is tuned and is a necessary component in any tunable radio transmitter or receiver that operates on the superheterodyne principle. The variable frequency oscillator equation and the Ivey's equation under consideration here are second-order nonlinear differential equations. For the thorough comprehension and precise prediction of motion and deformation, nonlinear ordinary differential equations (ODEs) have been employed extensively in many fields of physics and engineering (Saravi & Hermann, 2014). They are also crucial in mechanical and structural dynamics. Numerous academics have expressed interest in studying nonlinear ODEs, and numerous solutions have been put forth.

Linearization via differential forms was put in place by authors like (Harrison, 2002), (Orverem et al., 2017) and (Orverem et al., 2022). Also, linearization through the generalized Sundman transformation was first considered by (Duarte et al., 1994) where the Laguerre form was considered. Later, (Nakpim & Meleshko, 2010), treated the non-Laguerre form of linearization via the GST. This procedure was used by (Johnpillai & Mahomed, 2013) to linearized a class of Lienard equations. The Emden differential equation was also solved via the generalized Sundman transformation by (Orverem et al., 2021b). The two approaches were jointly applied to linearize the spheres of gaseous stability equation by (Orverem et al., 2021a).

Linearization of Variable Frequency Oscillator Equation via Differential Forms

The technique of differential forms entails that the general second order nonlinear ordinary differential equation

$$y'' = f(x, y, y') \tag{1}$$

should necessarily be in the form

$$y'' + f_0 + f_1 y' + f_2 y'^2 + f_3 y'^3 = 0, \tag{2}$$

and the coefficients $f_0, f_1, f_2,$ and f_3 must satisfy the conditions

$$f_{0yy} + f_0(f_{2y} - 2f_{3x}) + f_2 f_{0y} - f_3 f_{0x} + \frac{1}{3}(f_{2xx} - 2f_{1xy} + f_1 f_{2x} - 2f_1 f_{1y}) = 0, \tag{3}$$

and

$$f_{3xx} + f_3(2f_{0y} - f_{1x}) + f_0 f_{3y} - f_1 f_{3x} + \frac{1}{3}(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}) = 0. \tag{4}$$

Once the conditions in equations (3) and (4) are satisfied, we proceed to construct a 3×3 matrix

$$M = Pdx + Qdy \tag{5}$$

where

$$P = \left(\frac{1}{3}\right)\begin{bmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{bmatrix},$$

$$Q = \left(\frac{1}{3}\right)\begin{bmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{bmatrix},$$

and solve the equation

$$dr = Mr, \tag{6}$$

where $r = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$, for the three components of r; a special solution is usually enough. We can also construct

$$K = {}^{U}/_{W}, \quad L = {}^{V}/_{W}. \tag{7}$$

Next, we construct the 2×2 matrix

$$Z = \begin{bmatrix} (2K - f_1)dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)dy \end{bmatrix},$$

and solve for R from

$$dR = ZR, \quad (8)$$

where $R = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$. Finally, we solve

$$dF = [dx \quad dy]R; \quad (9)$$

the two independent solutions will be taken as F and G. Note that F and G are the invertible change of independent and dependent variables that will map the nonlinear equation (1) into a linear equation

$$\frac{d^2Y}{dX^2} = 0, \quad (10)$$

where $X = F(x, y)$ and $Y = G(x, y)$.

The variable frequency oscillator equation was considered by (Mustafa et al., 2013) where a new characterization of Sundman linearizable equations in terms of the coefficients of ODE and one auxiliary function was given. The general solutions for the first integral were also obtained explicitly. It is important to note that, the variable frequency oscillator equation can be linearized through the Laguerre form of the generalized Sundman transformation for the case $f_3 = 0, f_4 = 0$. We will use the method of differential forms to linearize this same equation.

This equation is given as

$$y'' + yy'^2 = 0, \quad (11)$$

with the coefficients as in equation (2); $f_0 = f_1 = f_3 = 0, f_2 = y$ satisfying the linearizable conditions (3) and (4). Construction of the matrix M from equation (5) gives

$$M = \begin{bmatrix} -\frac{y}{3}dy & 0 & 0 \\ 0 & \frac{2y}{3}dy & 0 \\ -dx & dy & -\frac{y}{3}dy \end{bmatrix},$$

so that equation (6) is now

$$dr = \begin{bmatrix} -U\frac{y}{3}dy \\ \frac{2}{3}Vydy \\ -Udx + Vdy - \frac{W}{3}ydy \end{bmatrix}.$$

Setting $V = 0$, we have $dU = -\frac{U}{3}ydy,\ dV = 0,$ $dW = -Udx - \frac{W}{3}ydy$, so that

$U_x = 0,\ U_y = -U\frac{y}{3},\ W_x = -U,\ W_y = -\frac{W}{3}y.$

Now, $\frac{\partial W}{W} = -\frac{y}{3}\partial y$ which implies that $W = e^{-\frac{y^2}{6}}a(x)$, for some function $a(x)$. But $W_x = -U = a'(x)e^{-\frac{y^2}{6}}$, using the special solution $a(x) = x$, we have $U = -e^{-\frac{y^2}{6}}$, $V = 0$, $W = xe^{-\frac{y^2}{6}}$. Therefore, equation (7) is now $K = -\frac{1}{x}$ and $L = 0$, and the matrix Z becomes

$$Z = \begin{bmatrix} -\frac{2}{x}dx & -\frac{dy}{x} \\ 0 & -\frac{dx}{x} + ydy \end{bmatrix}.$$

Then, one has from equation (8) that

$$dR = \begin{bmatrix} -\frac{2b}{x}dx - \frac{cdy}{x} \\ -c\frac{dx}{x} + cydy \end{bmatrix},$$

and

$$db = -\frac{2b}{x}dx - \frac{cdy}{x}, \quad (12)$$

$$dc = c\left(-\frac{dx}{x} + ydy\right). \quad (13)$$

Integrating equation (13), we have

$$c = \frac{k}{x}e^{\frac{y^2}{2}},$$

where $k = e^k$ is a constant.

Now, we see from equations (12) and (13) that $b_x = -\frac{2b}{x}$, $b_y = -\frac{c}{x}$, $c_x = -\frac{c}{x}$ and $c_y = cy$. One also notices that, $b_y = c_x$, and therefore,

$$b_y = -kx^{-2}e^{\frac{y^2}{2}}. \quad (14)$$

On integration, we have

$$b = \frac{-k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{2x^2} + g(x), \quad (15)$$

where $erf\left(\frac{y}{\sqrt{2}}\right)$ is the error function.

Differentiating equation (15) with respect to x, one sees that

$$b_x = \frac{k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{x^3} + g'(x). \quad (16)$$

The other expression for b_x is known to be

$$b_x = \frac{k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{x^3} - \frac{2g(x)}{x}. \quad (17)$$

We now compare the expressions for b_x and have the linear differential equation of first order

$$g'(x) + \frac{2}{x}g(x) = 0. \quad (18)$$

The integrating factor of the linear differential equation (18) is x^2, and the solution of the equation is now $g(x) = \frac{m}{x^2}$, where m is another constant.

Substituting this expression of $g(x)$ into equation (15), we have

$$b = \frac{-k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{2x^2} + \frac{m}{x^2}. \quad (19)$$

Therefore $b = F_x = \frac{-k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{2x^2} + \frac{m}{x^2}$ and $c = F_y = \frac{k}{x}e^{\frac{y^2}{2}}$, so that $dF = [dx \quad dy]\begin{bmatrix} b \\ c \end{bmatrix}$ becomes

$$dF = \left(\frac{-k\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{2x^2} + \frac{m}{x^2}\right)dx + \left(\frac{k}{x}e^{\frac{y^2}{2}}\right)dy. \quad (20)$$

Integrating equation (20), we have

$$F = k\left(\frac{\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{x}\right) - m\left(\frac{1}{x}\right). \quad (21)$$

Therefore, we take

$$X = \frac{1}{x},\ Y = \frac{\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right)}{x} \quad (22)$$

to be the linearizing point transformation. With the transformation $Y = c_1X + c_2$, one sees that $\sqrt{\pi}\sqrt{2}\, erf\left(\frac{y}{\sqrt{2}}\right) = c_1 + c_2x$ is the solution of the variable frequency oscillator equation. This solution is in line with the one obtained by (Mustafa et al., 2013), where the two-parameter family solutions of equation was obtained to be $erfi\left(\frac{y}{\sqrt{2}}\right) = c_1x + c_2$, where $erfi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{t^2}dt$ is the imaginary error function. One sees that our method of differential forms eliminated the complexity associated with the error function.

Linearization of the Modified Ivey's Equation via the Generalized Sundman Transformation

A generalized Sundman transformation is a non-point transformation defined by the formulae

$$u(t) = F(x,y),\ dt = G(x,y)dx,\ F_yG \neq 0. \quad (23)$$

The requisite form of a linearizable ordinary differential equation (1) that can be translated into a linear ordinary differential equation

$$u'' + \beta u' + \alpha u = \gamma, \quad (24)$$

through the transformation (23), is given by

$$y'' + f_2 {y'}^2 + f_1 y' + f_0 = 0, \quad (25)$$

where $\alpha(t), \beta(t)$ and $\gamma(t)$ from equation (24) are some functions.

Consider the case $f_3 \neq 0$ and $f_5 \neq 0$, where $f_3 = f_{1y} - 2f_{2x}$, $f_4 = 2f_{0yy} - 2f_{1xy} + 2f_0f_{2y} - f_{1y}f_1 + 2f_{0y}f_2 + 2f_{2xx}$ and $f_5 = f_{2xx} + f_{2x}f_1 + f_{3x} + f_1f_3$. For the equation (25) to be linearizable by (23), the sufficient conditions are that:

$$f_{0x} = 2f_0 \frac{(f_5 - f_1f_3)}{f_3}, \quad (26)$$

$$f_{2xxy} = -f_{2xy}f_1 - f_{3xy} - 2f_{2x}^2 - 2f_{2x}f_3 - f_{3y}f_1 + (f_{3y}f_5)f_3^{-1}, \quad (27)$$

$$f_{2xxx} = -f_{3xx} - f_{1x}f_{2x} - f_{1x}f_3 + f_{2x}f_1^2 + f_1^2f_3 - 2f_1f_5 + f_3^{-1}f_5(f_{3x} + f_5), \quad (28)$$

and

$$f_3f_5(6f_{0y}f_{2x} + 2f_{2xy}f_0 + 4f_{2x}f_0f_2 + 2f_{3y}f_0 + 4f_0f_2f_3 + f_1f_5) - f_3^2(6f_{2x}^2f_0 + 12f_{2x}f_1f_3 - 6f_{0y}f_5 + 6f_0f_3^2) - f_4f_5^2 - 2f_5^3 = 0. \quad (29)$$

One obtains the functions F and G by solving the following equations:

$$F_x = 0, \quad (30)$$

$$F_{yy} = \frac{F_yG_y + f_2F_yG}{G}, \quad (31)$$

$$G_x = \frac{G(f_{2xx} + f_{2x}f_1 + f_{3x})}{f_3}, \quad (32)$$

$$G_y = \frac{Gf_3(f_{2x} + f_3)}{f_5}. \quad (33)$$

The constants from equation (24), α, β and γ are determined from the equations below:

$$\alpha = \frac{G(f_{0y} + f_0f_2) - G_yf_0}{G^3}, \quad (34)$$

$$\beta = \frac{G_x + Gf_1}{G^2}, \quad (35)$$

$$\gamma = \frac{\alpha FG^2 - F_yf_0}{G^2}. \quad (36)$$

The general Ivey's differential equation is given as

$$y'' - \frac{1}{y}{y'}^2 + \frac{2}{x}y' + ky^2 = 0, \quad (37)$$

and the coefficients $f_0 = ky^2$, $f_1 = \frac{2}{x}$, $f_2 = -\frac{1}{y}$ does not satisfy all the linearizability conditions (26), (27), (28) and (29), and hence it is not linearizable using the generalized Sundman transformation.

On a check, one discovers that the modified equation

$$y'' - \frac{1}{y}{y'}^2 + yy' = 0, \quad (38)$$

with $xy = 2,\ k = 0$, is linearizable. The equation (38) is in the form of equation (25) with the coefficients

$$f_0 = 0,\ f_1 = y,\ f_2 = -\frac{1}{y}.$$

We see that $f_3 = 1 \neq 0,\ f_4 = -y,\ f_5 = y \neq 0$.

Since $f_3 = 1 \neq 0$ and $f_5 = y \neq 0$, we consider the case $f_3 \neq 0,\ f_5 \neq 0$. Next, we test the linearizability conditions (26), (27), (28) and (29). On a check, one sees that all the conditions are satisfied. Since all the linearizability conditions are satisfied, we proceed and solve equations (31), (32) and (33). Obviously,

$$F_x = 0, \qquad F_{yy} = \frac{-\frac{GF_y}{y} + \frac{F_yG}{y}}{G} = 0, \qquad G_x = 0,$$

$$G_y = \frac{G}{y}.$$

We take the solution $F = y,\ G = y$ which satisfies $F_{yy},\ G_x$ and G_y. We now obtain the transformation $u = y,\ dt = ydx$.

Next, we find the expressions for $\alpha,\ \beta$ and γ as given from equations (34), (35) and (36) respectively. One sees that

$$\alpha = 0, \qquad \beta = 1, \qquad \gamma = 0,$$

and we have from equation (24) that

$$u'' + u' = 0. \quad (39)$$

The characteristic equation of (39) is $r^2 + r = 0$, which implies that $r = 0,\ r = -1$. One now have the general solution to be $u = c_1 + c_2e^{-t}$, where c_1, c_2 are arbitrary constants.

Applying the generalized Sundman transformation to the modified Ivey's equation, we see that

$$y(x) = c_1 + c_2e^{-t}, \qquad t = \phi(x),$$

so that $y(x) = c_1 + c_2e^{-\phi(x)}$, where $t = \phi(x)$ is a solution of the equation $\frac{dt}{dx} = c_1 + c_2e^{-t}$.

Remark

One can infer that equation (25) is a special case of equation (2) by comparing equations (2) and (25) together. Even though equation (25) is a special case of equation (2), the procedures for the two approaches GST and DF are different. Generalized Sundman transformation is a non-point transformation as opposed to differential forms, which is a point transformation.

CONCLUSION

The variable frequency oscillator equation and the modified Ivey's equation are two nonlinear ordinary differential equations that are taken into consideration. Differential forms (DF) and the generalized Sundman transformation (GST) were used to linearize the aforementioned equations. It was discovered that the non-Laguerre form of GST can linearize the modified Ivey's differential equation, but that DF cannot. Additionally, the differential forms (DF) and the Laguerre-form of the generalized Sundman transformation, in the case when $f_3 = 0, f_4 = 0$, allow for the linearization of the variable frequency oscillator equation. During the linearization procedure, the general solutions to the two nonlinear differential equations were also discovered.

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