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REGULARIZED JACOBI-TYPE ADMM-METHOD FOR FINDING SOLUTIONS TO GENERALIZED NASH EQUILIBRIUM PROBLEM

[*1]Emmanuel Akaligwo, [2]Pius Opara, [2]Aharanwa Boniface

[1]Department of Mathematics, Federal University Lokoja, P.M.B 1154 Lokoja, Nigeria
[2]Department of Mathematics/Statistics, Imo State Polytechnic Omuma, Imo State, Nigeria.

*Corresponding authors' email: emmanuel.akaligwo@fulokoja.edu.ng Phone: +2347034929919

ABSTRACT

In this paper, we extended the well-known alternating direction method of multipliers (ADMM) for optimization problems to generalized Nash equilibrium problems (GNEP) with shared constraints. We developed an ADMM-type algorithm with fixed regularization to tackle the problem (GNEP) where an upper estimate for the operator norm is not known and then we apply a multiplier-penalty in order to get rid of the joint constraints. We equipped the Hilbert space with an appropriate weighted scalar product and it turns out to be weakly convergent under a lipschitz and monotonicity assumption. A proximal term is then added to improve the convergence properties. Furthermore, a comparative analysis of quasi-variational inequality method, interior point method, penalty method and the proposed method are discussed.

Keywords: convex optimization, weak convergence, monotonicity, hilbert space

INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $f_i : H \to \mathbb{R}$ be a proper, convex and lower semi-continuous function and $g_i : H \to \mathbb{R}$ be continuously Fréchet-differentiable with $g_i(x_i, x_{-i})$ being convex for any fixed (x_{-i}). Let A_i and B_i be linear operators from H to C, and b $\in$ C is a vector. Thus, C is a non-empty, closed, convex cone and subset of H. A generalized Nash equilibrium problem is of the form

$$\min_{x_n \in \mathbb{R}^n} f_i(u_i, \cdots, u_{i-1}, x_i, u_{i+1}, \cdots, x_N)$$

$$\text{s.t} \quad \sum_{i=1}^{n} A_i X_i = b, \qquad x_i \in X_i \tag{1}$$

For all $\mathrm{i} = 1, \cdots, \mathrm{N}$.

Let

$$\mathrm{L}(\mathrm{x}, \mu) := f_i(x_i, x_{-i}) + (\mu | A_i x_i - b)$$

$$\mathrm{L_A}(\mathrm{x}, \mu) := f_i(x_i, x_{-i}) + (\mu | A_i x_i - b) + \frac{\beta}{2}\|A_i x_i - b\|$$

denote the Lagrangian and the augmented Lagrangian of (1), respectively, where $\beta > 0$is the penalty parameter. Then a standard optimization technique for minimizing the generalized. Nash equilibrium problem of this kind (1) is given by

$$\begin{cases} \alpha^1, \mu^1 \in C \\ x_i^{\,k+1} = arg \min_{x \in X} \mathrm{L_A}(\mathrm{x_i}, \mathrm{x_i}^{\mathrm{k}}) \\ \mu^{k+1} = \mu^k + \beta(A_i, x_i^{\,k+1} - b), \qquad k \geq 1 \end{cases}$$

provided that a minimum of the augmented Lagrangian exists. Another fruitful concept in applied mathematics is the Nash equilibrium problems (NEPs), which were initially developed by John F. Nash in the 1950s, see (Nie, J et.al. 2023). Over the years, Nash's theory has been extended and broadly applied to many fields in biology, economics and engineering, see for instance (Benenati et. al 2023; Chen et. al. 2023; Gahururu et. al. 2023; Nash Jr. et. al. 1950; Shi et. al. 2023; Shehu et. al. 2019) and the references therein. Consequently, the demand for numerical methods tackling these kinds of problems rose. Since Nash and generalized Nash problems are intertwined optimization problems, the optimization theory and certain algorithms were extended to these problem classes. For an introduction to the theory and algorithmic of generalized Nash problems, see for example (Chen et. al. 2023). The structure of these (generalized) Nash problems suggests that a suitable generalization of the splitting methods described above could yield efficient algorithms, which could serve as tools to solve these kinds of problems. Furthermore, it is desirable to develop such splitting methods because they tend to resemble the structure of applications where each player decides on his or her own how to react. There exist many approaches for the numerical solution of GNEPs, and the interested reader is referred to the survey papers (Chen, et. al 2023; Deng, et. al. 2023) for more details. However, these survey papers consider the finite-dimensional case only. Solution methods in an infinite-dimensional Hilbert space (or Banach space) are still in their infancy. The ideas from the finite-dimensional setting can sometimes be generalized to the Hilbert space setting, like the usage of the Nikaido-Isoda function and the application of Moreau-Yosida-type methods (Jordan, et. al. 2023; Laine 2023). The augmented Lagrangian methods from (Lee, 2023; Liu, 2023) may be viewed as extensions of this approach, but they have to solve a (standard) Nash equilibrium problem (NEP) in each iteration. Some other methods operating in an infinite-dimensional context are (Braouezec, et. al. 2023; Cai, et. al. 2023; Catellani, et. al. 2023; Meng, et. al. 2023) but none of them is a splitting-type method and many of them are situated in an optimal control context. Splitting methods for GNEPs that are based on such forward-backward methods can be found in (Borgens, et. al. 2023; Boyd, et. al. 2023; Singh, et. al 2023; Zhu, et. al 2023). These articles consider splitting-type methods that are based on forward-backward methods; in (Boyd, et. al. 2023), the authors focus on standard NEPs and show afterwards how to solve certain GNEPs under a cocoercivity assumption. On the other hand, the closely related algorithms considered in (Borgens, et. al. 2023; Singh, et. al 2023; Zhu, et. al. 2023) (for finite-dimensional problems) are fully distributed, but they use a strong monotonicity and Lipschitz assumption. Taking into account the situation of the standard ADMM-method for optimization problems, one might expect that (a) no regularization is necessary for GNEPs with $\mathrm{N} = 2$ players, and (b) arbitrary (possibly small) regularization parameters $\gamma_i > 0$ are sufficient for the global convergence of GNEPs with $\mathrm{N} \geq 3$ players. The subsequent discussion shows that none of these statements hold. Hence, regularization is also necessary for two players, and the corresponding regularization parameters have to be sufficiently large.

Preliminaries

This section contains some definitions and basic results that will be used in our subsequent analysis. The letter H always denotes a real Hilbert space. We first state the formal definition of some classes of functions that play an essential role in our analysis.

Definition 1. Let $X \subseteq H$ be a nonempty subset. Then a mapping $A: X \to H$ is called
(a) monotone on X if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in X$
(b) Lipschitz continuous on X if there exists a constant $L > 0$ such that
$\|Ax - Ay\| \leq L\|x - y\|$,
for all $x, y \in X$.

We next recall some properties of the projection, let $C \subseteq H$ be a nonempty, closed, and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that
$\|u - P_C u\| \leq \|x - y\|, \ \forall\, y \in C$.

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall\, x, y \in H. \qquad (2)$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \text{ and } \langle x - P_C x, y - P_C y\rangle \qquad (3)$$

This characterization implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C y\|^2, \ \forall\, x \in H, y \in C. \qquad (4)$$

The following elementary lemma will be used in our convergence analysis.

Lemma 1. In H, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in H.$$

Lemma 2. The following identities hold in H:

(a) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$, for all $x, y \in H$.
(b) $\|\lambda x + (1 - \lambda y)\|^2 = \lambda\|x\|^2 + (1 - \lambda y)\|y\|^2 + \lambda(1 - \lambda y)\|x - y\|^2$ for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 3. Let C be a nonempty set of H and $\{x_n\}$ be a sequence in Hsuch that the following two conditions hold:
(a) for any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$exists;
(b) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

MATERIALS AND METHODS

This section is devoted to developing an algorithm for finding the solution of a convex generalized Nash equilibrium problem (GNEP). The method presented here uses a fixed regularization parameter. The precise statement of the algorithm is given below.

Algorithm 1 (Proposed Method).
Initialization: Choose a starting point $(x_0, \mu_0) \in C$, parameters $\beta > 0$, $\gamma_i > 0$ for all $i = 1, \cdots, N$, and set $k := 0$
Step 1: If a suitable termination criterion is satisfied: STOP
Step 2: For $i = 1, \cdots, N$, compute
Step 3:

$$x_i^{k+1} := arg \min_{x_i \in X_i} \{f_i(x_i) + g_i(x_1^{k+1}, \cdots, x_{i-1}^{k+1}, x_i, x_{i+1}^{k+1}, \cdots, x_N{}^k) + \langle \mu^k | A_i x_I\rangle_C + \tfrac{\gamma_i}{2}\|x_i - x_i^k\|_H^2 + \tfrac{\beta}{2}\| A_i x_i + \textstyle\sum_{i=1}^{k-1} A_i x_i^{k+1} + \sum_{i=k+1}^{N} A_i x_i^k - b \|_H^2\} \qquad (5)$$

Step 4: Define

$$\mu^{k+1} := \mu^k + \beta\left(\textstyle\sum_{i=k+1}^{N} A_i x_i^k - b\right) \qquad (6)$$

Step 5: Set $k \leftarrow k + 1$, and go to **Step 2.**

We note that every subproblem (5) is strongly convex for all i and all iterations k . Hence, all iterates $x^{k+1} := (x^{k+1}, \cdots, x_N^{k+1})$ are well defined and uniquely determined.

RESULT AND DISCUSSION

Numerical Experiments

In this section, we provide some concrete example including numerical results of the problem considered in Section 3 of this paper. All codes were written in Matlab 2016b, and installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 GB (7.78 GB usable) RAM running on Windows 10 operating system.

Example 1. Consider the 2-player GNEP
1st player

$$\min_{x_1 \in \mathbb{R}^2} (x_{1,1})^4 + 2(x_{1,2})^2 + \sum_{j=1}^{2} x_{1,j}(x_{2,j})^2$$

$$\text{s.t.} \quad x_{1,1} + 2x_{1,2} - x_{2,1} \leq 1$$
$$3x_{1,1} + 2x_{1,2} - x_{2,1} \leq 1.5,$$
$$(x_{1,1})^2 + (x_{2,1})^2 \leq 3$$
$$(x_{1,1}) \geq 0.$$

2nd player

$$\min_{x_2 \in \mathbb{R}^2} \tfrac{1}{2}\|x\|^2 \cdot \|x\|^2 + x_{2,1} - x_{1,1} + 2x_{1,1}x_{1,2}x_{2,1}x_{2,2}$$

$$\text{s.t.} \quad (x_{2,1})^2 + x_{1,2}x_{2,2} \leq 2,$$
$$(x_{1,1})^2 + (x_{2,2})^2 \leq 3,$$
$$(x_{2,2}) \geq 0.$$

By Algorithm 1, we got the GNE, $u = (u_1, u_2)$ with

$$u_1 \approx (0.0000, -0.7500), u_2 \approx (-1.0881, 1.7321), q_1(u) \approx 0.2591, q_2(u) \approx 2.4028.$$

It took around 0.34 second in solving for both players, and 8.40 seconds to find the GNE.

Table 1: Numerical results of example 1 and comparing with different methods

Algorithms	u	time	$\|y^k - y^{exact}\|$
QVI	(0.4897, 1.0259, 0.7077)	1.83	2.5563×10^{-7}
Penalty	(0.4897, 1.0259, 0.7077)	6.49	3.8589×10^{-6}
IPM	(0.4897, 1.0259, 0.7077)	0.02	7.9280×10^{-8}
Alg. 1	(0.0000, -0.7500, -1.0881, 1.7321)	0.34	9.4049×10^{-9}

Example 2. Consider the 3-player GNEP

1st player

$$\begin{cases} \min_{x_1 \in \mathbb{R}^2} x_{2,2}(x_{1,1})^2 - x_{1,2}x_{3,1} \\ s.t. \quad x_1^T x_1 + x_2^T x_2 + (x_{2,1} + x_{2,2})x_3^T x_3 \le 1 \end{cases}$$

2nd player

$$\begin{cases} \min_{x_1 \in \mathbb{R}^2} (x_{2,1})^2 + (x_{1,1} - 1)x_{2,1} + (x_{3,2}x_{2,2})^2 - x_{3,1}x_{2,2} \\ s.t. \; x_{1,1}x_{2,1}x_{3,1} + x_{1,1}x_{2,1}x_{3,1} + 0.1 \ge 0, \quad 1 - \sum_{j=1}^{2} x_{2,j} \ge 0 \end{cases}$$

3rd player

$$\begin{cases} \min_{x_1 \in \mathbb{R}^2} (x_3 - x_1 + x_2)^T x_3 \\ s.t. \quad (x_{3,1} - x_{3,2})^2 \le x_{2,1}, \quad (x_{3,1} + x_{3,2})^2 \le 3 \end{cases}$$

The first player's Lagrange multipliers have a rational expression, that

$$\lambda_1 = \frac{-x_1^T \nabla_{x_1} f_1}{2q_1(x)}, \quad q_1(x) = 1 - x_2^T x_2 - (x_{2,1} + x_{2,2})x_3^T x_3$$

For the second player, with $s_2 = 1$. For λ_3, if we let $q_3 = 2x_{2,1} - 2(x_{3,1})^2 + 2(x_{3,2})^2$, then

$$\lambda_{3,1} = \frac{1}{q}\left(x_3^T \nabla_{x_3} f_3\right)$$

Note that $q_1(x) \ngeq 0$ on X. So we change the constraint, to make it work. By Algorithm 1, we got the GNE $\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)$ with

$u_1 \approx (0.0000, -0.7993)$, $u_2 \approx (0.5000, 0.0000)$, $u_3 \approx (-0.2500, -0.3997)$,

$u_1(u) \approx 0.6389$, $u_2(u) = 1$, $u_3 \approx 1.1944$

It took around 10.44 seconds.

Table 2: Numerical results of example 2 and comparing with different methods.

Algorithms	u	time	$\|y^k - y^{exact}\|$
QVI	(0.0000, −1.1339, 1.5875)	Not convergent	Not convergent
Penalty	(0.3776, 0.0700, 1.5318)	Not convergent	Not convergent
IPM	(0.9407, 1.2194, −0.0360)	Not convergent	Not convergent
Alg. 1	(0.6389, 1.0000, 1.1944)	10.44	5.4972×10^{-7}

Remark 1. We could observe that Algorithm 1 is robust, converges faster and easy to implement.

CONCLUSION

In this paper, we proposed a method for finding common solution to convex GNEP and give numerical illustrations of our results and show the numerical improvements of Algorithm 1 over quasi-variational inequality method, interior point method and penalty Method.

CONFLICT OF INTEREST

The authors have no potential conflict of interest to disclose.

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