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NUMERICAL SOLUTIONS OF THIRD ORDER FREDHOLM INTEGRO DIFFERENTIAL EQUATION VIA LINEAR MULTISTEP-QUADRATURE FORMULAE

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ABSTRACT

In this study, we present a 6-step linear multistep mehod in union with some newton-cote quadrature family for solving third order Linear Fredholm Integro-Differential Equation(LIDE). The schemes were derived using Vieta-Pell-Lucas Polynomial as the approximating function. The linear multi-step component is for the non integral part while the quadrature family is for the Integral part. The quaudrature methods Boole , Simpson 3/8 and Trapezoidal rule were separately combined with the linear multistep method. The qualititaive analysis of the scheme revealed that the method is consistence, stable and convergent. In order to further attest to the behavioural attribute of the methods, numerical experiments were carried out on some selected Initial Value Problems.The results from the tested problems and their absolute errors of deviation revealed that the new method is very suitbale for solution to the tested problems. The scheme when combined with Boole and Simpson 3/8 merhod, performed better with Tracendental function than when combined with Trapezoidal rule and vice –versa. The results further showed that the proposed method perform creditably well with lesser computional steps when compared with some existing methods when applied to the selected examples.

Keywords: Booles, Fredholm, Integro-differential equation, Simpson 3/8, Trapezoidal, Vieta-Pell-Lucas, Newton Cote Quadrature method

INTRODUCTION

Fredholm Integro differential equation has a far reaching applications in the area of sciences, engineering and in all facet of human endeavors such as kinetic theory of gases, geophysics, communication theory, mathematical economics, queuing theory, hereditary phenomena in physics and biology.It is however difficult if not impossible to solve some of the mathematical models representing the Fredholm, Volterra, Fredholm-Volterra analytically, hence the need to solve them numerically using any suitable method. Many reseachers has proposed different methods for solving the differential and integral parts of integro diferrential equations .Such works include Ogunrinde (2010), Ogunrinde et al. (2020), Obayomi (2012),Obayomi and Ogunrinde(2015) in which both Finite Difference and Non Standard Finite Difference methods were proposed for Ordinary Differential Equations. In the same vein,Salawu et al. (2022) worked on linear multistep method for solving ordinary differential equations. Many scholars have devoted their time in studying Integral equations, Fredholm Integro differential equations,Volterra integro differential equations since its discovery in 1900.Such works include the celebrated works of Feldstein and Sopka.(1974), Behrouz (2010) in which numerical methods for nonlinear volterra equations was introduced. Bruner (1984) presented Implicit Runge- Kutta methods for optimal order for Volterra integro differential equations ,Yalcinbas and Sezer (2000) considered the approximate solution of higher linear Volterrra-Fredholm integro differential equations in terms of Taylor Polynomials, Al-Timeme and Atifa (2003) presented the quadrature rules for finding the numerical solutions of Initial Value Problems to Volterra integro differential equation. Kamoh, et al (2019) and Kamoh, et al (2017) proposed Continuous Linear Multistep Method for First and Second order integro differential equation using different orthogonal polynomial as approximating function. This research work proposed a 6-step continuous linear multistep method for third order Fredholm integro differential equations with Vieta-Pell-Lucas as approximating polynomial.

METHODOLOGY

We derived a 6 –step Linear Multistep Method (LMM) which combined with newton-cotes quadrature to solve third order linear integro-differential equation.

Derivation of the 6-step Linear Multistep Methods(LMM)

We consider a reliable method for solving linear Fredholm Integro-Differential Equation of the form:

$$y'''(x) = f(x) + \int_{\alpha}^{\beta} \varphi(x,t)y(t)dt, y'(a) = b_{1_1}, y''(a) = b_2 \quad (1)$$

where $K(x,t)$ is the kernel, $\alpha(x)$ and $\beta(x)$ are the limits of the integral and $f(x)$ are given in advance.The solution to ordinary derivative part of (1) is approximated by using Horadam family of Orthogonal polynomials called Vieta-Pell-Lucas which is of the of the form:

$$y(x) = \sum_{i=0}^{m} a_i\, s_i(x) \quad (2)$$

Where $a_i(x)$ are real undetermined coefficient and $s_i(x)$ are the terms of the Vieta-Pell-Lucas Polynomials given by recurrence formula$S_i(x) = 2xS_{i-1}(x) - S_{i-2}(x)$,where $i \geq 2$,$S_0(x) = 2$and $S_1(x) = 2x, m = 9$.
The third derivative of (2.1) is obtained as a system of equation of the form:

$$y'''(x) = \sum_{i=0}^{m} a_i\, s_i(x) \quad (3)$$

The method is derived by interpolating (2) and collocating (3) at x_{n+j} where $j = 0,2,4,5$ and $j = 0,1,3,4,5,6$ respectively which yield a system of interpolation and collocation equations. The resulting system of equations are solved using linear algebra suitable method via MAPLE 18 software in order to obtain the values of unknown$a_i's$.
The values of $a_i's$are simplified and substituted into (2) to obtain a continuous scheme of the form

$y(x) = \sum_{i=0}^{k} \alpha_i(x) y_{n+i} +$
$h^3 \sum_{i=0}^{k} \beta_i\,(x) f(x_{n+i}, y(x_{n+i}, z(x_{n+i})))$ (4)

where h is the step size and
$z(x_{n+i}) = h \sum \alpha_i\,(x) w_{ni} \tau(x_n, x_i, y_i)$ (5)

are the embedded weight functions of Boole,Simpson 3/8 and Trapezoidal quadrature formulae obtained.
Consequently, the continuous scheme of the form (4) is evaluated at points $x_{n+1}, x_{n+3}, x_{n+5}$, x_{n+7} . Further simplification with collection of terms in power of $h$gives the first four discrete schemes:

$$\left.\begin{aligned}
y_{n+1} &= \left(\frac{4553}{2523600} f_n + \frac{90529}{841200} f_{n+1} - \frac{90433}{252360} f_{n+3} - \frac{27977}{168240} f_{n+4} - \frac{1217}{280400} f_{n+5} \right.\\
&\quad \left. + \frac{199}{1261800} f_{n+6}\right) h^3 + \frac{848}{701} y_{n+2} - \frac{571}{701} y_{n+4} + \frac{1289}{3505} y_{n+5} + \frac{831}{3505} y_n \\
y_{n+3} &= \left(-\frac{26887}{85802400} f_n + \frac{129589}{28600800} f_{n+1} + \frac{1592087}{8580240} f_{n+3} + \frac{958543}{5720160} f_{n+4} \right.\\
&\quad \left. + \frac{243}{9533600} f_{n+5} + \frac{15259}{42901200} f_{n+6}\right) h^3 + \frac{3827}{11917} y_{n+2} + \frac{12135}{11917} y_{n+4} \\
&\quad - \frac{20443}{59585} y_{n+5} + \frac{218}{59585} y_n \\
y_{n+6} &= \left(\frac{1676}{2681325} f_n - \frac{10232}{893775} f_{n+1} + \frac{43888}{536265} f_{n+3} + \frac{126856}{178755} f_{n+4} + \frac{138936}{297925} f_{n+5} \right.\\
&\quad \left. + \frac{20516}{2681325} f_{n+6}\right) h^3 + \frac{4441}{11917} y_{n+2} - \frac{24537}{11917} y_{n+4} + \frac{160768}{59585} y_{n+5} - \frac{703}{59585} y_n
\end{aligned}\right\} \quad (6)$$

Evaluating the first and second derivative of (4) at all the grid points leads to a system of discrete schemes:

$$\left.\begin{aligned}
b[n] &= \left(-\frac{32356}{11261565}f_{n+6} - \frac{459373}{1501542}f_{n+4} + \frac{974}{59585}f_{n+5} + \frac{2289382}{3753855}f_{n+1}\right.\\
&\quad \left. + \frac{1299043}{22523130}f_n - \frac{773000}{2252313}f_{n+3}\right)h^2\\
&\quad + \frac{-\frac{225283}{238340}y_n + \frac{59015}{35751}y_{n+2} - \frac{58445}{47668}y_{n+4} + \frac{93056}{178755}y_{n+5}}{h}\\[2ex]
b[n+1] &= \left(\frac{777061}{1801850400}f_{n+6} + \frac{1332829}{240246720}f_{n+4} - \frac{1260563}{400411200}f_{n+5}\right.\\
&\quad \left. - \frac{190492349}{1201233600}f_{n+1} - \frac{3568459}{514814400}f_n - \frac{8885323}{360370080}f_{n+3}\right)h^2\\
&\quad + \frac{-\frac{119909}{238340}y_n + \frac{18245}{35751}y_{n+2} - \frac{739}{47668}y_{n+4} + \frac{1478}{178755}y_{n+5}}{h}\\[2ex]
b[n+2] &= \left(\frac{1823}{18769275}f_{n+6} + \frac{131027}{500514}f_{n+4} + \frac{7838}{2085475}f_{n+5} - \frac{155326}{6256425}f_{n+1}\right.\\
&\quad \left. - \frac{1619}{37538550}f_n + \frac{1808332}{3753855}f_{n+3}\right)h^2\\
&\quad + \frac{-\frac{9921}{238340}y_n - \frac{24832}{35751}y_{n+2} + \frac{61581}{47668}y_{n+4} - \frac{99328}{178755}y_{n+5}}{h}\\[2ex]
b[n+3] &= \left(-\frac{103939}{1801850400}f_{n+6} - \frac{137461}{34320960}f_{n+4} + \frac{128279}{133470400}f_{n+5}\right.\\
&\quad \left. + \frac{4393691}{1201233600}f_{n+1} - \frac{64973}{3603700800}f_n - \frac{9307327}{72074016}f_{n+3}\right)h^2\\
&\quad + \frac{\frac{1361}{238340}y_n - \frac{18556}{35751}y_{n+2} + \frac{25195}{47668}y_{n+4} - \frac{2722}{178755}y_{n+5}}{h}\\[2ex]
b[n+4] &= \left(\frac{412}{1608795}f_{n+6} - \frac{1737751}{7507710}f_{n+4} - \frac{12454}{1251285}f_{n+5} - \frac{6694}{3753855}f_{n+1}\right.\\
&\quad \left. + \frac{433}{4504626}f_n - \frac{1139672}{11261565}f_{n+3}\right)h^2\\
&\quad + \frac{-\frac{383}{238340}y_n - \frac{5767}{35751}y_{n+2} - \frac{24217}{47668}y_{n+4} + \frac{119936}{178755}y_{n+5}}{h}\\[2ex]
b[n+5] &= \left(-\frac{370661}{120123360}f_{n+6} + \frac{5683187}{16016448}f_{n+4} + \frac{2029523}{26694080}f_{n+5} - \frac{127813}{11440320}f_{n+1}\right.\\
&\quad \left. + \frac{170333}{240246720}f_n + \frac{270875}{24024672}f_{n+3}\right)h^2\\
&\quad + \frac{-\frac{2553}{238340}y_n + \frac{7235}{35751}y_{n+2} - \frac{74055}{47668}y_{n+4} + \frac{243446}{178755}y_{n+5}}{h}\\[2ex]
b[n+6] &= \left(\frac{3657253}{56307825}f_{n+6} + \frac{7165267}{7507710}f_{n+4} + \frac{2201046}{2085475}f_{n+5} + \frac{474974}{18769275}f_{n+1}\right.\\
&\quad \left. - \frac{227299}{112615650}f_n + \frac{600428}{1608795}f_{n+3}\right)h^2\\
&\quad + \frac{\frac{4931}{238340}y_n + \frac{15410}{35751}y_{n+2} - \frac{114239}{47668}y_{n+4} + \frac{347648}{178755}y_{n+5}}{h}
\end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned}
c[n] &= \left(\frac{213154}{18769275}f_{n+6} - \frac{518783}{6256425}f_{n+5} + \frac{1144039}{5005140}f_{n+4} - \frac{2818071}{2085475}f_{n+1} - \frac{2521761}{8341900}f_n - \frac{496900}{750771}f_{n+3}\right)h \\
&+ \frac{\frac{53921}{238340}y_n - \frac{10029}{23834}y_{n+2} + \frac{6253}{47668}y_{n+4} + \frac{3776}{59585}y_{n+5}}{h^2} \\
c[n+1] &= \left(-\frac{390239}{150154200}f_{n+6} + \frac{2771351}{100102800}f_{n+5} + \frac{11686471}{30030840}f_{n+4} - \frac{25331497}{300308400}f_{n+1} + \frac{612349}{37538550}f_n + \frac{16434661}{15015420}f_{n+3}\right)h \\
&+ \frac{\frac{146843}{238340}y_n - \frac{41003}{23834}y_{n+2} + \frac{99175}{47668}y_{n+4} - \frac{58172}{59585}y_{n+5}}{h^2} \\
c[n+2] &= \left(\frac{3524}{2252313}f_{n+6} - \frac{16454}{1251285}f_{n+5} + \frac{142019}{15015420}f_{n+4} + \frac{520342}{3753855}f_{n+1} - \frac{45593}{45046260}f_n - \frac{3642476}{11261565}f_{n+3}\right)h \\
&+ \frac{\frac{10573}{47668}y_n - \frac{9677}{23834}y_{n+2} + \frac{5197}{47668}y_{n+4} + \frac{896}{11917}y_{n+5}}{h^2} \\
c[n+3] &= \left(-\frac{270511}{150154200}f_{n+6} + \frac{930799}{100102800}f_{n+5} - \frac{862247}{2002056}f_{n+4} - \frac{1110537}{33367600}f_{n+1} + \frac{6109}{4170950}f_n - \frac{6862907}{15015420}f_{n+3}\right)h \\
&+ \frac{-\frac{8773}{238340}y_n + \frac{10869}{23834}y_{n+2} - \frac{56441}{47668}y_{n+4} + \frac{45572}{59585}y_{n+5}}{h^2} \\
c[n+4] &= \left(\frac{51146}{18769275}f_{n+6} - \frac{221647}{6256425}f_{n+5} + \frac{2267261}{15015420}f_{n+4} + \frac{395009}{18769275}f_{n+1} - \frac{94081}{75077100}f_n + \frac{1233676}{3753855}f_{n+3}\right)h \\
&+ \frac{\frac{4769}{238340}y_n + \frac{6355}{23834}y_{n+2} - \frac{42899}{47668}y_{n+4} + \frac{36544}{59585}y_{n+5}}{h^2} \\
c[n+5] &= \left(-\frac{5222429}{450462600}f_{n+6} + \frac{37945847}{100102800}f_{n+5} + \frac{25532359}{30030840}f_{n+4} - \frac{13027849}{300308400}f_{n+1} + \frac{310039}{112615650}f_n - \frac{1068925}{9009252}f_{n+3}\right)h \\
&+ \frac{-\frac{9829}{238340}y_n + \frac{11221}{23834}y_{n+2} - \frac{57497}{47668}y_{n+4} + \frac{46276}{59585}y_{n+5}}{h^2} \\
c[n+6] &= \left(\frac{5708392}{18769275}f_{n+6} + \frac{9124186}{6256425}f_{n+5} + \frac{1487641}{5005140}f_{n+4} + \frac{299862}{2085475}f_{n+1} - \frac{85103}{8341900}f_n + \frac{3737884}{3753855}f_{n+3}\right)h \\
&+ \frac{\frac{30593}{238340}y_n - \frac{2253}{23834}y_{n+2} - \frac{17075}{47668}y_{n+4} + \frac{19328}{59585}y_{n+5}}{h^2}
\end{aligned}\right\} \quad (8)$$

Where $b[n+k] = y'_{n+k}$, $c[n+k] = y''_{n+k}$, $0 \le k \le 6$.

Evaluating the third derivative of the continuous scheme at x_{n+2} and further simplification gives the remaining part of the scheme:

$$\left.\begin{aligned} f_{n+2} = -\frac{703}{178755}f_n - \frac{73}{178755}f_{n+6} - \frac{484}{59585}f_{n+5} - \frac{7675}{11917}f_{n+4} - \frac{14764}{59585}f_{n+1} \\ - \frac{52880}{35751}f_{n+3} + \frac{-\frac{6048}{11917}y_n + \frac{20160}{11917}y_{n+2} - \frac{30240}{11917}y_{n+4} + \frac{16128}{11917}y_{n+5}}{h^3} \end{aligned}\right\} \quad (9)$$

The derived block of eighteen discrete schemes of (6),(7), (8) and (9) simultaneously solves third order linear integro-differential equation with the choices of Boole, Simpson 3/8 and Trapezoidal quadrature rules.

Quadrature formulae

In order to solve the third order linear integro –differential equations,quadrature formulae are employed to approximate the integral parts of the LIDEs.Thus,the choice of quadrature rules: Boole, Simpson 3/8 and Trapezoidal quadrature rules defined respectively as follows:

$$\int_a^b y(x)dx = \frac{2h}{45}\left[7(y_0+y_n) + 32(y_1+y_3+y_5+\cdots) + 12(y_2+y_6+y_{10}+\cdots) + 14(y_4+y_8+y_{12}+\cdots) - \frac{8(h)^7y^{vi}(\varepsilon)}{945}\right] \quad (10)$$

$$\int_a^b y(x)dx = \frac{3h}{8}\left[(y_1+3y_2+3y_3+2y_4+3y_5+3y_6+\cdots+2y_{n-2}+3y_{n-1}+3y_n+y_{n+1}) - \frac{(b-a)h^4y^{iv}(\varepsilon)}{180}\right] \quad (11)$$

$$\int_a^b y(x)dx = \frac{h}{2}\left[(y_1+2y_2+2y_3+2y_4+2y_5+2y_6+\cdots+2y_{n-2}+2y_{n-1}+2y_n+y_{n+1}) - \frac{(b-a)h^2y''(\varepsilon)}{12}\right] \quad (12)$$

Where $x_i i = a + ij, (i = 9 = 0,1,2,\dots,n)$ are the abscissas of the partition points of the integration in interval $[a,b]$,h is the step size which is given by

$$h = \frac{b-a}{n} \quad (13)$$

where n is the number of subinterval in the interval $[a,b]$ and the terms $-\frac{8(h)^7y^{vi}(\varepsilon)}{945}$, $\frac{(b-a)h^4y^{iv}(\varepsilon)}{180}$ and $\frac{(b-a)h^2y''(\varepsilon)}{12}$ are the error terms for Boole,Simpson 3/8 and Trapezoidal rules respectively.

Analysis of the 6-Step LMM

The basic behavioural properties of the methods are analyzed to establish the efficiency, reliability and validity of the method. These properties include order, error constant, consistence, stability and convergence.

Order and Error constant of the main schemes and its derivatives

Employing the Taylor series of the main scheme $y_{n+6}(x)$ of (4), its first and second derivatives $y'_{n+6}(x)$ and $y''_{n+6}(x)$ of (7) and (8) respectively by collecting the like terms in term of h ,we obtained the error constants as

$$\left.\begin{aligned} c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = \frac{1256}{3312225} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = -\frac{20417}{9936675} \\ c_0 = c_1 = c_2 = c_3 = c_4 = c_{.5} = c_6 = 0, c_7 = -\frac{184859}{19873350} \end{aligned}\right\} \quad (14)$$

By extension, the block of eighteen discrete schemes in (6), (7), (8) and (9) is found to be of uniform order $\rho = 7$with a varying error constant.

Consistency of the method

Definition 1: A linear multistep method is consistent if the following conditions hold:

(a) The order is greater than one, i.e. $\rho \geq 1$
(b) $\sum_{i=0}^{k} \alpha_i = 0$
(c) $\rho(1) = p'(1) = \rho''(1) = 0$
(d) $\rho'''(1) = 3!\,\delta(1)$

The main scheme $y_{n+6}(x)$ of (4) satisfies all the conditions in definition 1, hence the main scheme is consistent and by extension, other schemes are found to be consistent.

Zero stability

Definition 2: A linear multistep method is said to be zero stable if no roots of the first characteristic polynomial $\rho(r)$ has modulus greater one, i.e $|r| \leq 1$and every root with modulus is simple. Following thisdefinition,the roots of the first characteristic polynomial

$$r^6 - \frac{160768}{59585}r^5 + \frac{24537}{11917}r^4 - \frac{4441}{11917}r^2 + \frac{703}{59585} = 0 \quad (15)$$

associated with main scheme are:

$$\left.\begin{aligned} r_1 &= 0.1940115796 - 4.\,10^{-11}\,\mathrm{I} \\ r_2 &= -0.2736853527 - 3.244101616\,10^{-10}\,\mathrm{I} \\ r_3 &= -0.2221975033 + 3.684101616\,10^{-10}\,\mathrm{I} \\ r_4 &= 1 \\ r_5 &= 1 \\ r_6 &= 1 \end{aligned}\right\} \quad (16)$$

Hence, the derived method is zero stable.

Region of Absolute Stability of the main scheme

Using the function $\prod(r,H) = \rho(r) - H\delta(r)$.the region of absolute stability is sketched for $H(r) = \frac{\rho(r)}{\delta(r)}$,where $r = e^{i\theta}$, $0 \le \theta \le 2\pi$ and presented

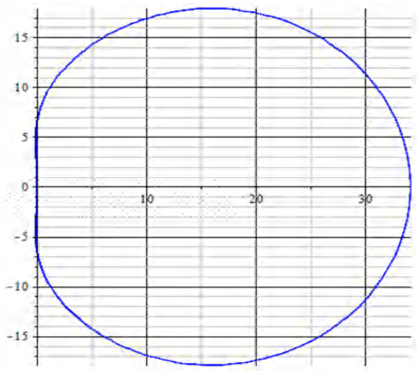

Figure 1: Region of absolute stability of the main scheme

Convergence

According to Lambert (1973) and Fatunla (1991), the necessary and sufficient conditions for a Linear Muiltistep Method (LMM) to be convergent are that it must be consistent and zero- stable.Hence,the proposed method is convergent.

Numerical Experiment

The selected third order initial value problems of FIDE are solved to support our theoretical discussion on the derived method; the method is implemented with the choice of Boole,Simpson 3/8 and Trapezoidal rules. The experiments are performed with the aid of MAPLE 18 software package while the graphs are plotted with MATLAB R2016a software. The error is defined as follows:

$error = |y(exact) - y(numerical)|$

Problem 1

Consider the linear FIDE $y'''(x) = 6 + x - \int_0^1 xy''(t)dt\, , y(0) = -1, y'(0) = 1, y''(0) = -2$ with exact solution $y(x) = x^3 - x^2 + x - 1$

Source: Wazwaz (1997)

Table 1: Solution to problem 1 using the derived scheme with Boole (6LMBL), Simpson 3/8 (6LMSP) and Trapezoidal (6MTR) with N=12 partitions

x	y(exact)	6LMBL	6LMSP	6LMTR	E6LMBL	E6LMSP	E6LMTR
0.08333	-0.9230324074	-0.9230324067	-0.9230324066	-0.9230324073	7.00E-10	8.00E-10	1.00E-10
0.16667	-0.8564814815	-0.8564814800	-0.8564814799	-0.8564814810	1.50E-09	1.60E-09	5.00E-10
0.25	-0.7968750000	-0.7968749978	-0.7968749975	-0.7968749992	2.20E-09	2.50E-09	8.00E-10
0.33333	-0.7407407408	-0.7407407377	-0.7407407373	-0.7407407395	3.10E-09	3.50E-09	1.30E-09
0.41667	-0.6846064815	-0.6846064775	-0.6846064770	-0.6846064798	4.00E-09	4.50E-09	1.70E-09
0.5	-0.6250000001	-0.6249999951	-0.6249999945	-0.6249999978	5.00E-09	5.60E-09	2.30E-09
0.58333	-0.5584490742	-0.5584490679	-0.5584490672	-0.5584490710	6.30E-09	7.00E-09	3.20E-09
0.66667	-0.4814814817	-0.4814814739	-0.4814814731	-0.4814814774	7.80E-09	8.60E-09	4.30E-09
0.75	-0.3906250002	-0.3906249908	-0.3906249899	-0.3906249947	9.40E-09	1.03E-08	5.50E-09
0.83333	-0.2824074077	-0.2824073964	-0.2824073955	-0.2824074009	1.13E-08	1.22E-08	6.80E-09
0.91667	-0.1533564819	-0.1533564686	-0.1533564677	-0.1533564735	1.33E-08	1.42E-08	8.40E-09
1	-0.0000000006	0.0000000149	0.0000000158	0.0000000095	1.55E-08	1.64E-08	1.01E-08

Table 2: Maximum Absolute Error (MaxE) of the Derived method with different quadrature rule for Problem 1 with varying partition N

Quadrature Formula	MaxE N=24	MaxE N=36	MaxE N=48
E6LMBL	8.8698E-09	1.2121E-08	1.7083E-07
E6LMSP	5.3000E-09	7.900E-09	6.6986E-08
E6LMTR	7.3493E-09	1.2206E-08	5.9391E-08

Problem 2

Consider the LFIDE $y'''(x) = sin\, x - x - \int_0^{\frac{\pi}{2}} xty''(t)dt\, , y(0) = 1, y'(0) = 0, y''(0) = -1$ with exact solution $y(x) = cos\, x$

Source: Darania and Ali (2007)

Table 3: Solution to problem 2 using the derived scheme with Boole (6LMBL), Simpson 3/8 (6LMSP) and Trapezoidal (6LMTR) with N=12 partitions

x	y(exact)	6LMBL	6LMSP	6LMTR	E6LMBL	E6LMSP	E6LMTR
0.13093	0.9914412786	0.9914412839	0.9914412838	0.9914412971	5.30E-09	5.20E-09	1.85E-09
0.26185	0.9659116177	0.9659116954	0.9659116938	0.9659119068	7.77E-08	7.61E-08	2.89E-07
0.39278	0.9238480199	0.9238484103	0.9238484019	0.9238494805	3.90E-07	3.82E-07	1.46E-06
0.52371	0.8659705064	0.8659717380	0.8659717114	0.8659751203	1.23E-06	1.21E-06	4.61E-06
0.65464	0.7932697922	0.7932727969	0.7932727322	0.7932810547	3.00E-06	2.94E-06	1.13E-05
0.78556	0.7069903277	0.7069965561	0.7069964219	0.7070136795	6.23E-06	6.10E-06	2.34E-05
0.91649	0.6086089966	0.6086205340	0.6086202854	0.6086522572	1.15E-06	1.13E-05	4.33E-05

1.04742	0.4998098358	0.4998295159	0.4998290918	0.4998836341	1.97E-05	1.93E-05	7.37E-05
1.17834	0.3824552087	0.3824867296	0.3824860503	0.3825734166	3.15E-05	3.08E-04	1.18E-04
1.30927	0.2585539264	0.2586019661	0.2586009307	0.2587340908	4.80E-05	4.70E-04	1.80E-04
1.44020	0.1302268620	0.1302971935	0.1302956776	0.1304906373	7.03E-05	6.88E-04	2.64E-04
1.57113	-0.0003293532	-0.0002297466	-0.0002318937	0.0000442271	9.96E-05	9.75E-04	3.74E-04

Table 4: Maximum Absolute Error (MaxE) of the Derived method with different quadrature rule for Problem 2 with varying partition N

Quadrature Formula	MaxE N=24	MaxE N=36	MaxE N=48
E6LMBL	1.4362E-05	9.9587E-05	2.1403E-04
E6LMSP	1.4495E-05	9.9561E-05	2.1402E-04
E6LMTR	5.4275E-05	1.3008E-04	2.3118E-04

Table 5: Comparison of Maximum Absolute Error (MaxE) of the Derived method with Darania and Ali (2007) for Problem 2 with varying partition N

Method	No of partition	MaxE
6LMSP	6	1.6280E-04
6LMTR	6	1.1327E-03
Darania and Ali (2007)	10	5.2938E-01

Problem 3

Consider the LFIDE $y'''(x) = 1 - e + e^x + \cos x + \int_0^1 y(t)dt$, $y(0) = y'(0) = y''(0) = 1$ with exact solution $y(x) = e^x$
Source: Gegele et al. (2014)

Table 7: Solution to problem 3 using the derived scheme with Boole (6LMMBL), Simpson 3/8 (6LMMSP) and Trapezoidal (6LMMTR) with N=12 partitions

x	y(exact)	6LMBL	6LMSP	6LMTR	E6LMBL	E6LMSP	E6LMTR
0.08333	1.086904050	1.086904050	1.086904049	1.086904150	0.00E+00	1.00E-10	1.00E-07
0.16667	1.181360413	1.181360414	1.181360413	1.181361214	1.00E-09	0.00E+00	8.01E-07
0.25	1.284025417	1.284025418	1.284025420	1.284028120	1.00E-09	3.00E-09	2.70E-06
0.33333	1.395612425	1.395612427	1.395612431	1.395618831	2.00E-09	6.00E-09	6.41E-06
0.41667	1.516896796	1.516896798	1.516896809	1.516909308	2.00E-09	1.30E-08	1.25E-05
0.5	1.648721271	1.648721273	1.648721294	1.648742891	2.00E-09	2.30E-08	2.16E-05
0.58333	1.792001825	1.792001829	1.792001860	1.792036160	4.00E-09	3.50E-08	3.43E-05
0.66667	1.947734041	1.947734045	1.947734092	1.947785292	4.00E-09	5.10E-08	5.13E-05
0.75	2.117000016	2.117000023	2.117000090	2.117072989	7.00E-09	7.40E-08	7.30E-05
0.83333	2.300975890	2.300975896	2.300975989	2.301075990	6.00E-09	9.90E-08	1.00E-04
0.91667	2.500940013	2.500940018	2.500940144	2.501073244	5.00E-09	1.31E-07	1.33E-04
1	2.718281828	2.718281835	2.718282000	2.718454797	7.00E-09	1.72E-07	1.73E-04

Table 8: Maximum Absolute Error (MaxE) of the Derived method with different quadrature rule for Problem 3 with varying partition N

Quadrature Formula	MaxE N=24	MaxE N=36	MaxE N=48
E6LMBL	2.0400E-07	3.5700E-07	7.9000E-08
E6LMSP	1.800E-08	3.6700E-07	9.1900E-09
E6LMTR	4.3300E-05	1.9930E-05	1.1341E-05

Table 9: Comparison of Maximum Absolute Error (MaxE) of the Derived method with Gegele et al. (2014) for Problem 3 with varying partition N

Method	No of Partition(N)	MaxE
E6LMSP	6	2.8610E-06
Gegele et al. (2014) (CSSCM)	10	2.2747E-05
Gegele et al. (2014) (CSCGLCM)	10	3.1782E-04

Problem 4

Consider the LFIDE $y'''(x) = 5\text{lin}2 - 3 - x + 4\cosh x + \int_0^{\text{lin}2}(x-t)y(t)dt$, $y(0) = y''(0) = 0, y'(0) = 4$ with exact solution $y(x) = 4\sinh x$
Source: Wazwaz (2011).

Table 10: Solution to problem 4 using the derived scheme with Boole (6LMMBL), Simpson 3/8 (6LMMSP) and Trapezoidal (6LMMTR) with N=12 partitions

x	y(exact)	6LMBL	6LMSP	6LMTR	E6LMBL	E6LMSP	E6LMTR
0.05776	0.2311775634	0.2311775638	0.2311775636	0.2311775066	4.00E-10	2.00E-10	5.68E-08
0.11552	0.4631266604	0.4631266612	0.4631266607	0.4631262044	8.00E-10	3.00E-10	4.56E-07
0.17329	0.6966213996	0.6966214013	0.6966213994	0.6966198613	1.70E-09	2.00E-10	1.54E-06
0.23105	0.9324410480	0.9324410494	0.9324410458	0.9324374061	1.40E-09	2.20E-09	3.64E-06
0.28881	1.171372631	1.171372633	1.171372626	1.171365531	2.00E-09	5.00E-09	7.10E-06
0.34657	1.414213562	1.414213563	1.414213551	1.414201317	1.00E-09	1.10E-08	1.22E-05
0.40434	1.661774299	1.661774299	1.661774280	1.661754895	0.00E-09	1.90E-08	1.94E-05
0.46210	1.914881053	1.914881054	1.914881024	1.914852149	1.00E-09	2.90E-08	2.89E-05
0.51986	2.174378545	2.174378547	2.174378503	2.174337478	2.00E-09	4.20E-08	4.11E-05
0.57762	2.441132823	2.441132824	2.441132764	2.441076607	1.00E-09	5.90E-08	5.62E-05
0.63538	2.716034155	2.716034153	2.716034074	2.715959489	2.00E-09	8.10E-08	7.47E-05
0.69315	2.999999998	2.999999992	2.999999891	2.999903268	6.00E-09	1.07E-07	9.67E-05

Table 11: Maximum Absolute Error (MaxE) of the Derived method with different quadrature rule for Problem 4 with varying partition N

Quadrature Formula	MaxE N=24	MaxE N=36	MaxE N=48
E6LMBL	6.6800E-08	1.4000E-07	4.3600E-07
E6LMSP	9.0000E-09	1.3700E-07	2.8000E-08
E6LMTR	2.4248E-05	1.0757E-05	6.8090E-06

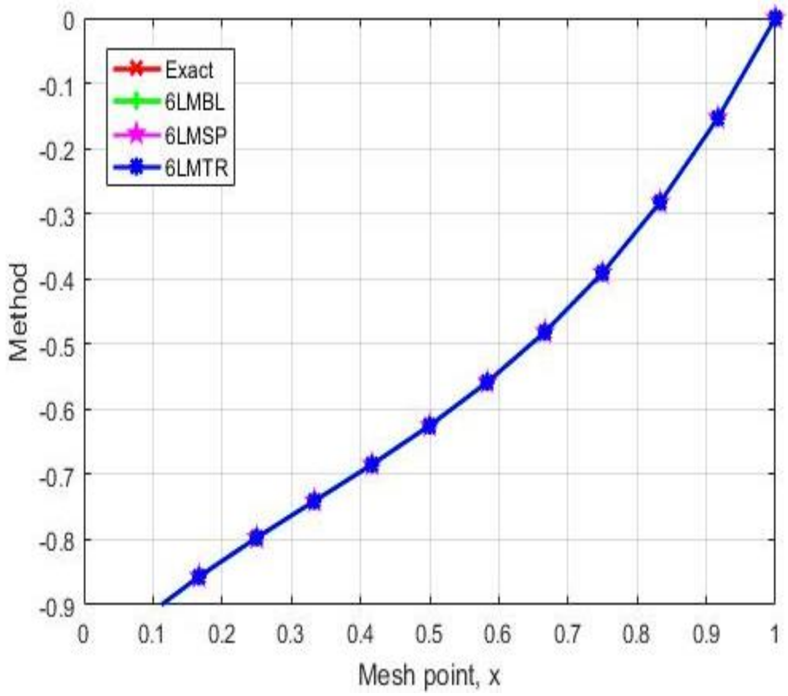

Figure 2: A plot of Exact, 6LMBL, 6LMSP and 6LMTR Solutions for Problem 1

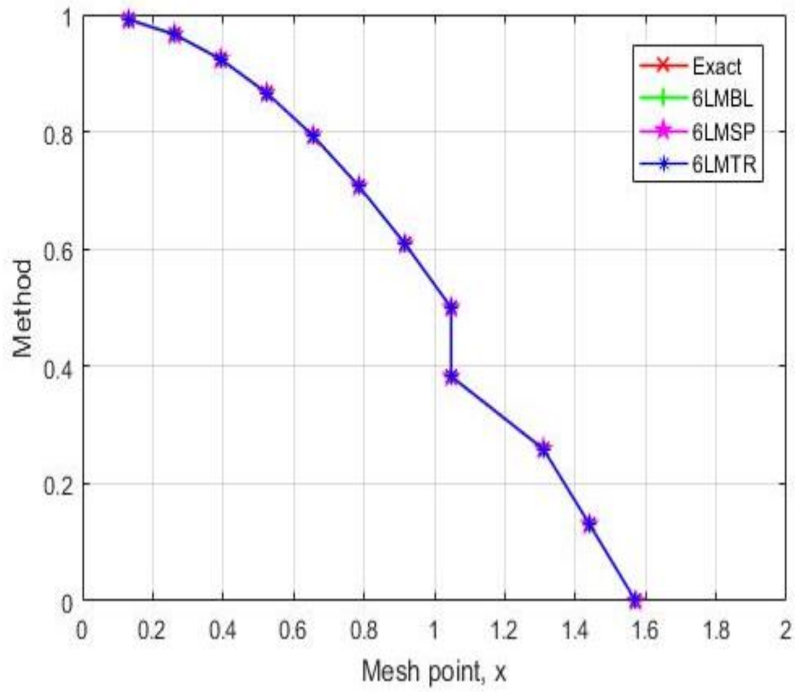

Figure 3: A plot of Exact, 6LMBL, 6LMSP and 6LMTR Solutions for Problem 2

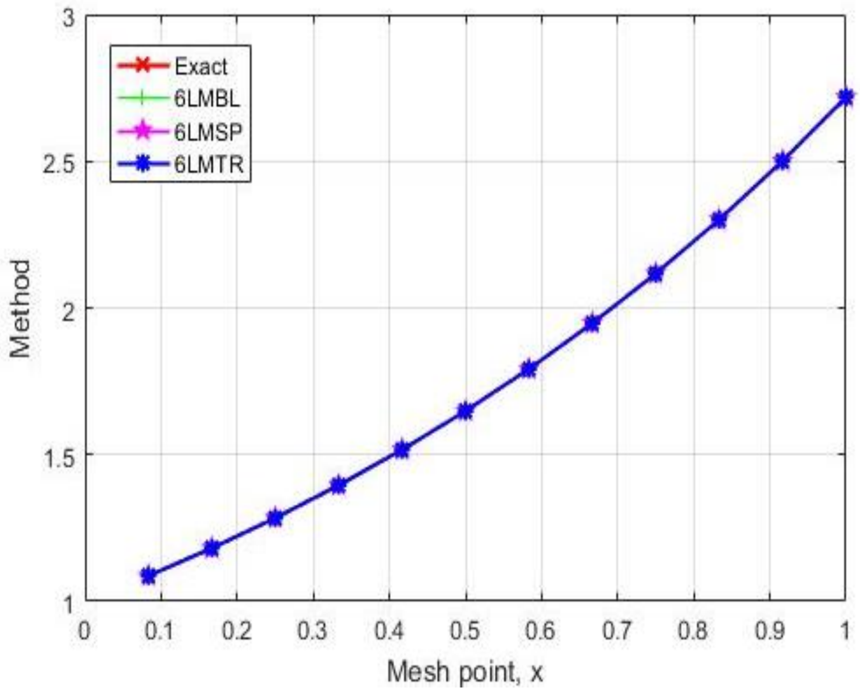

Figure 4: A plot of Exact, 6LMBL, 6LMSP and 6LMTR Solutions for Problem 3

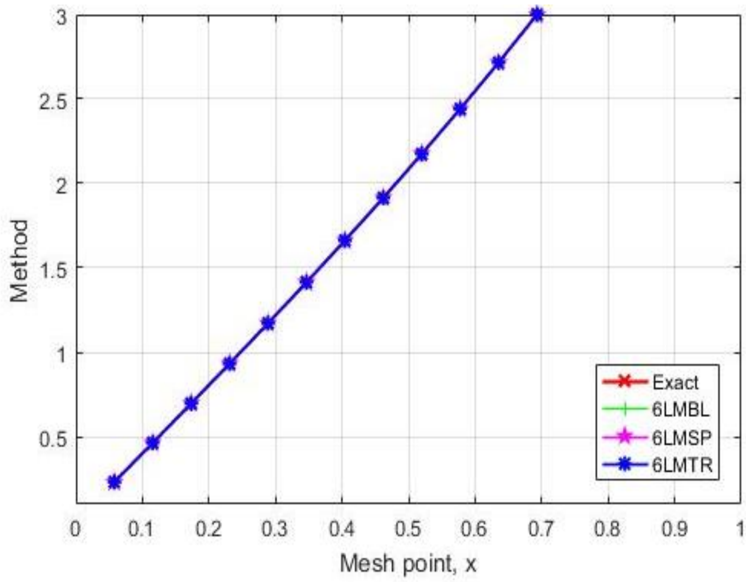

Figure 5: A plot of Exact, 6LMBL, 6LMSP and 6LMTR Solutions for Problem 4

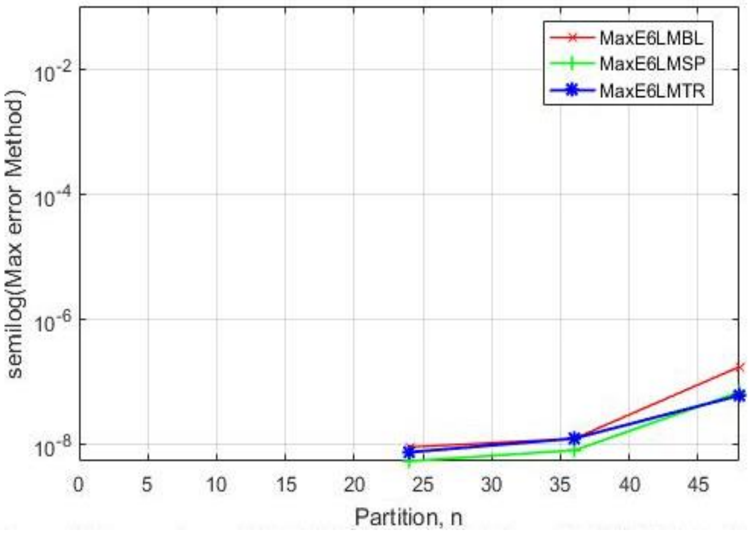

Figure 6: Comparison of Mx6LMBL, Mx6LMSP and Mx6LMTR Solutions for Problem 1

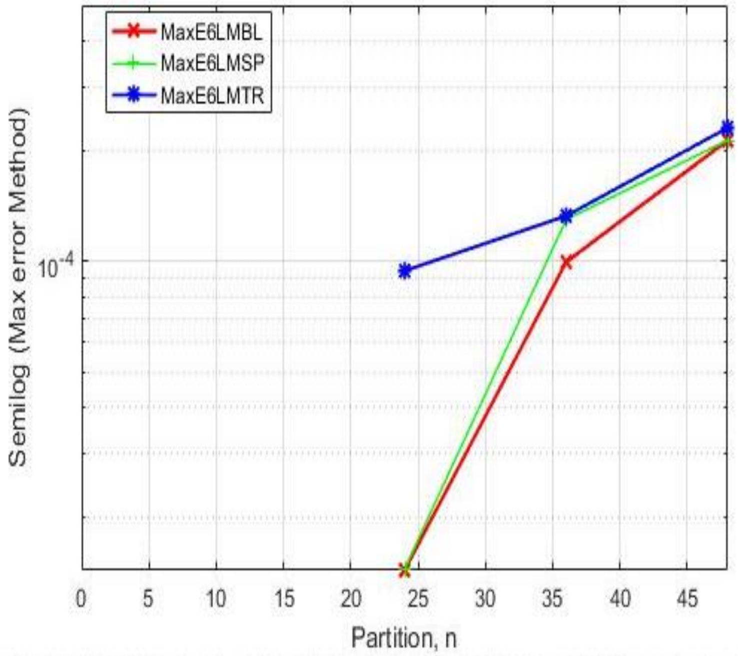

Figure 7: Comparison of Mx6LMBL, Mx6LMSP and Mx6LMTR Solutions for Problem 2

RESULT AND DISCUSSION

The tables and figures above represent the numerical experiment of the 6-step linear multistep method in combination with some Newton cote quadrature family that can solve third order Fredholm Integro-differential equation.The method is employed together with Booles (6LMBM).Simpson 3/8 (6LMSP) and Tarpezoidal(6LMTR) to solve some Initial Value Problems of third order LFIDEs. The method was derived using Vieta-Pell-lucas polynomial as an approximate solution of the ordinary derivative part of the third order LFIDEs.The behavioural properties of the method have been analysed and depicted in diagrams as well .The analysis revealed that the method is of uniform order (error of order seven) .It was also found that the method is consistent,stable and convergent. Four numerical examples were solved using the methods in order to confirm the theorical claims of consistence,stability and accuracy.

The result of Problem 1 to 4 is presented in tabular form.The results have been compared with Exact solutions of the Integro-differential equations. Table 1,3,7 and 10 show the numerical solutions alongside the exact solutions and absolute errors and the various step sizes h.

From table 1,it can be observed from the absolute error that 6LMTR performs slightly better when compared with 6LMBL and 6LMSP. Furthemore,Tables 2,4,8 and 11 displayed the performance of each of the schemes with different various number of partitions (N=24,N=36 and N=48) and the resulting maximum errors.

The results of the problems are also displayed graphically in Figure 2,Figure 3, Figure 4 and Figure 5 respectively. It's obvious fron the figures that the methods perform favourably with the exact solutions.

Figure 6 and Figure7 represent the graphical presentation of the maximum errors in each of the problems 1 and 2 with varying number of partition N=24,N=36 and N=48.

Table 5 and 9 displayed the comparison between the derived methods and some earlier methods such as : Darania and Ali (2007) ,Gegele et al. (2014). The comaparative analysis of the derived method with some earlier methods also shows that the method perform better than that of Darania and Ali (2007) and Gegele et al. (2014) even when lesser number of partition (N=6) has being used. The use of lesser number of partition also confirm that lesser number of computions are used to get a better result than these earlier methods.

The following notations have been used as short form of :

6LMBL= 6-step linear multistep method with Boole quadrature formula

6LMSP = 6-step linear multistep method with Simpson 3/8 quadrature formula

6LMTR = 6-step linear multistep method with Trapezoidal quadrature formula

MaxE = Maximum Error

CONCLUSION

It's a known fact that some of Integro –diiferential Equation are difficult to solve analytically if not impossible to solve, hence the need to approximate their solutions numerically. In this research,we present a 6-step linear multistep mehod which are implemented with the family of newton-cote quadrature formulae. The results show that the new method perform slightly better when implemented with Trapezoidal rule on problems that have linear and Non tracendental form in Problem1.The results further revealed superiority of the method when implemented with both Boole and Simpson 3/8 on Problems involving tracendental functions of the form in Problems 2 to 4.The comaparative analysis of the derived method with some earlier methods also shows that the method perform better than that of Darania and Ali (2007) and Gegele et al. (2014) even when lesser number of partition (N=6) has being used. The use of lesser number of partition also confirm that lesser number of computions are used to get a better result than these earlier methods.

Conflict of Interests

The authors declare no conflict of interests regarding the publication of this paper.

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The authors wish to state that all the cited materials are duly referenced.

REFERENCES

Al-Timeme and Atifa (2003). Approximated methods for first order Volterra Integro-differential equations, M.Sc, Thesis, University of Technology.

Behrouz, R (2010) "Numerical Solution of the Linear Volterra Integro- Diiferential Equation by Homotopy Perturbation Method and Finite Difference Method".World Applied Science Journal 9 (Special Issue of Applied Maths) :07-12

Bruner, H (1984) "Implicit Runge –Kutta Methods of Optimal Order for Volterra Integro-Differenetial Equations",Mathematics of Computation,42 :(165),95-109

Darania, P. and Ali Ebadian (2007). A method for the numerical solution of the integro-differential equations,Applied Mathematics and Computation 188,657-668.

Fatunla, S.O. (1991). A Block method for second order IVPs, International Journal of Computer Mathematics, Vol. 41, pp. 55-63. Available at http:dx.doi.org/10.1080/00207169108804026

Feldstein, A and Sopka J.R. (1974)"Numerical Methods for Non-lnear Volterra Integro-Differential Equations".SIAM Journal Numerical Analysis,11:826-846.

Gegele, O.A,Evans, O.P.and Akoh, D. Numerical Solution of Higher Order Linear Fredholmn Integro-Differential Equations.Americal Journal of Engineering Research (AJER),e-ISSN:2320-0847,p-ISSN:2320-0936,volume-03,Issue-08,pp243-247.

Kamoh, N. M,Aboiyah,T. and Onah, E.S. (2017). On one investigating some quadrature rules for the solution of second order Volterra Integro-differential equations. IOSR Journal of Mathematics (IOSR-JM), Vol.13, e-ISSN:2278-5728.p-ISSN:2319-765X., Issue 5 Ver.iii.pp.45-50.www.iosrjournals.org, DOI:10.9790/5728-1305034550.

Kamoh, N.M., Gyemang, D.G. and Soomiyol, M.C. (2019). Comparing the efficiency of Simpson's 1/3 and Simpson's 3/8 rules for the numerical solution of first order Volterra Integro-differential equations, *World* Academy of Science. Engineering and Technology International Journal of Mathematical and Computational Sciences, Vol.,13. No5, ISNI:0000000091950263. pp.136-139.

Lambert, J.D. (1973). Computational Methods in Ordinary Differential Equations: John Wiley & Sons Incorporation New York.

Obayomi, A.A. (2012). A set of non-standard finite difference schemes for the solution of an equation of the type $y' = y(1 - y^n)$, International Journal of Pure and Applied Sciences and Technology, 12(20), pp. 34-42.

Obayomi, .A.A.,Ogunrinde R.B,(2015) Nonstandard discrete models for some initial value problems arising from nonlinear oscillator equations.Algorithmsresearch,4(1):8-13 DOI:10.5923/j.algorithms.20150401.02

Ogunrinde, R.B.(2010) A new numerical scheme for the solution of initial value problem (IVP) in ordinary differential equations.Ph.D Thesis,Ekiti State University,Ado-Ekiti.

Ogunrinde, R.B., Olayemi,K.S, Isah I.O. and Salawu A.S. (2020). A numerical solver for first order initial value problems of ordinary differential equation via the combination of Chebyshev polynomial and exponential function. *Journal of Physical Sciences*, ISSN2520-084X (online), Vol. 2, Issue No.1, pp17-32.

Salawu, A.S,Isah,I.O.,Olayemi, K.S**.,**Paul, R.V,. A comparative Study of Orthogonal polynomials for Numerical Solution of Ordinary Differential Equations. FUDMA Journal of Sciences(FJS). ISSN online:2616-1370,ISSN Print:2645-2944.Vol.,6 No.1,March,2022,pp 282-290.DOI:https://doi.org/1033003/fjs-2022-0601-898

Wazwaz, A.M. (1997). A First Course in Integral Equations. Singapore: World Scientific Publishing Company

Wazwaz, A.M. (2011). Linear and Non Linear Integral Equations Method and Applications. Higher Education Press, Beijing and Springer-Verlag Berlin Heieberg.

Yalcinbas and Sezer (2000) "The ApproximatenSolution of High Order Linear Volterra- Fredholm integro-Differential Equations in terms of Taylor Polynomials". Applied Mathematics and Computation,112:291-308