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NUMERICAL APPROXIMATION OF OSCILLATORY INITIAL VALUE PROBLEMS USING THE HOMOTOPY ANALYSIS ALGORITHM

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ABSTRACT

In this paper, we present numerical approximation for oscillatory initial value problems (IVPs) using the homotopy analysis algorithm. The convergence of the method is discussed and numerical experiments are presented to illustrate the computational effectiveness of the algorithm. The results obtained are in good agreement with the exact solutions and Adomian decomposition method (ADM). These results show that the algorithm introduced here is easy to apply without linearization.

Keywords: Numerical Analysis, Initial Value Problems, Differential Equations, Homotopy Analysis

INTRODUCTION

Oscillatory initial value problems arise in mathematical models for problems in mechanics, physics and engineering. These are more difficult to solve analytically, hence it seems more natural to provide direct numerical methods for solving such initial value problems. The purpose of this paper is to discuss the homotopy analysis algorithm for approximation of oscillatory initial value problems (IVPs) .Homotopy Analysis Algorithm is a semi-analytic technique used to solve non-linear ordinary differential equations. The Homotopy Analysis Algorithm employs the concept of homotopy in topology to generate a convergent series solution of non-linear systems. This is enabled using a Homotopy-Taylors series to deal with the non-linearity in the system. The Homotopy Analysis Method was proposed by Liao 1992 and the method provides a convenient way to control and adjust the convergence region and rate of series approximation. Goreishi etal (2011) applied HAA to solve a model for HIV infection of CD4+T cells .Fallahzadeh and Shakibi (2015) applied homotopy analysis algorithm to solve Convection-Diffusion equation. Mkharrib and Salem (2021) studied the new algorithm of the optimal homotopy asymptotic method for solving Lane-Emden equation. Omar (2021a) carried out homotopy analysis-based hybrid genetic algorithm and secant method to solve IVPs and higher order BVP. This work will extend the homotopy analysis algorithm to approximate the Solution of oscillatory IVPs.

Homotopy Analysis Algorithm

Consider the non-linear differential equation

$$N[U(X)]=0 \tag{1}$$

N is a non-linear differential operator, X denotes the independent variable $U(X)$ is an unknown function. According to Liao (1992) , the zero order deformation equation is given as

$$[1-q]L(x;q)-Uo(x)]qhN[\Phi(x;q) \tag{2}$$

Where $q\in(0;1)$ is an embedding parameter, $h\neq 0$, L is an auxiliary linear operator U_0 is the initial guess of $U(X)$ and $\phi(x;q)$ is an unknown function. In this work we assume $h=-1$ when $q=1$ in equation (2) and $\phi(x;q)$ = $U_0(x)$

$\phi(x;1) = U(x)$

Since $\phi(x;q)$ depends on the parameter q, expanding $\phi(x;q)$ by Taylor's series with respect to q

$$\phi(x;q)=\phi(x;0)+\sum_{n=2}^{\alpha}Un(X)q \tag{3}$$

$$Un=\frac{1}{n!}\frac{\partial^n\phi(x;q)}{\partial q^n}|q=0 \tag{4}$$

From equation (2) when $q=1, we\ get\ \phi(x;1)Uo+\sum_{n=1}^{\infty}Un(0)$ $u_n(x)$ can be deduced using the zero-order deformation equation (2). Differentiating (2) n times with respect to the embedding parameter q and q=1, divide by n! And get the nth order deformation equation

$$L[U_n(X)-X_nU_{n-1}9X)]=hD_{n-1}[N(\phi(x;q)] \tag{5}$$

Where

$$D_{n-1}[N(\phi(x;q)]=\frac{1}{(n-1)!}\frac{\delta^{n-1}[N(x;q)]}{\delta q^{n-1}}$$

For example, consider the non-linear differential equation $\frac{\partial u}{\partial x}-2u^2=0$ with initial condition $Uo(0)=1$.

$$L[\phi(x)]=\phi^1(x)$$

$$N[\phi(X)]=\phi^1(X)-2\phi(X)^2 \tag{6}$$

Let $Uo(0)=1,\ u_0^1(x)=0$

From equation (3),we obtain $u_o=0$ and $u_n(0)=0 for\ all\ n\geq 1$ Let

$$\phi=\phi_o(x)+\sum_{n=1}^{\infty}\phi_n q^n=\phi_o+\phi q+\phi_2 q^2+\phi_3 q^2+....$$

$$=\phi_o(x)+\sum_{n=1}^{\infty}\phi_n q^n=\phi^1{}_o+\phi_1^1 q+\phi^1{}_2 q^2+\phi^1{}_3 q^2+.... \tag{7}$$

from (1),we obtain

$$1[1-q]=[L[\phi(X)-\phi_N(X)]=hq[N[\phi(X)]\ (1-q)[\phi_1-\phi_0^1]-2\phi^2]$$

Substituting (5), (6) and (7) and get

$$=\phi_2+\phi_1^1 q+\phi_2 q^2+\phi_3 q^2+....)-\phi_0^1\,\_\,\beta[\phi_0+\phi^1 q+\phi_2^1 q+\phi_3 q^3+...... \tag{8}$$

Differentiating (8) with respect to q and get the first derivative, to obtain.

$$(1-q)[\phi_1^1+2\phi_2^1 q+3\phi_3^1 q^2+....]+[-1[\phi 1_1^1+\phi_2^1 q+\phi 1_3 q^3=hq[\phi_1^1+2\phi_2^1 q^3+3\phi 1_3 q^2+...]$$

$$-4[\phi_0+\phi_1 q+\phi_2{}^2 q^2+\phi q^3+.... \tag{9}$$

Differentiating (9) with respect to q to get first derivative, and obtain

$$(1-q)[\phi_1^1+2\phi_2^1 q+3\phi_3^1 q^2+....]+[-1[\phi 1_1^1+\phi_2^1 q+\phi 1_3 q^3=hq[\phi_1^1+2\phi_2^1 q^3+3\phi 1_3 q^2+...]$$

$$-4[\phi_0+\phi_1 q+\phi_2{}^2 q^2+\phi q^3+.... \tag{10}$$

$$+\phi_2+2\phi_2 q+3\phi_3 q^2+...]h[\phi_1^{10}+\phi_2^1 q^3+....]-2[\phi_0+\phi_1 q+\phi_2 q^2+\phi^3 q^3....$$

When $q=0$ in (10) to obtain

$$\phi_1^1=h[\phi_0^1-2\phi_0^2]$$

Since $u_0(x)=1$, it implies that $\varphi_0(x)=1$, then $\phi_0^1(x)=0$, and

$$\phi_1^1=h[0-(L)^2]$$

$$\phi_1^1=-2h$$

Integrating both sides, to obtain

$$\phi_1=-2h+c \tag{11}$$

Since $u_n(0)=0,\forall_n\geq 1$, thus $\varphi_n(1)=0,\forall_n\geq 1$, the $\varphi_1(0)=0$, so $c=0$

Thus $\phi_1=-2hx$ Which is the first derivative of HAA, to get the second derivative for (11) differentiate the first derivative one time to get

$$3\phi_3^1 q^1+....)+(-1)[\phi_1^1+2\phi_2^1 q+3\phi_3^1 q^2+....)]+hq[2\phi_2^1+6\phi_3^1 q...]-4[\phi_0+\phi_1 q+\phi_2 q^2+\phi_3 q^3....]$$

$$-(2\phi_2+6\phi_3 q+...\}-4[\phi_1+2\phi_2 q+3\phi_3 q^2+...)[\phi_1+2\phi_2 q+3\phi_3 q^2 \quad +$$

$$h[\phi_1^1+2\phi_2^1 q+3\phi_3^1 q+...]-4[\phi_0+\phi_1 q+\phi_2 q^2+\phi_3 q^3+h[\phi_0+2\phi_2^1 q+3\phi_3^1 q^2+.$$

$$-4[\phi_0+\phi_1 q+\phi_2 q^2+\phi_3 q^3+..]+[\phi_{12}+2\phi_3 q^2+3\phi_3 q^2+...] \tag{12}$$

Let $q=0$ in (12), to get

$$2\phi_2^1-\phi_1^1-\phi_1^1=h[\phi_1^1-4\phi_0\phi_1]$$

But $\phi_0=1,\phi_1=-2h$ and then $\phi_1^1=-2h+8h^2x-2h$

Integrating both sides, to obtain

$\phi_2=-2h^2x+4^2x^2-2hx+c$ Since $\phi_2(0)=0$ then $c=0$, Thus

$$\phi_2=-2h^2x+4h^2x^2-2hx \tag{13}$$

This is the second derivative, to get the third derivative for (13) differentiate three times to differentiate the second derivative one time

$$[1-q][6\phi_3^1+....]+[2\phi_2^1+6\phi_3^1q+..](-1)[2\phi_2^1+6\phi_2+6\phi_3^1q+..](-)[2\phi_2^1+6\phi_3^1q+...$$

$$=hq[(6\phi_3+..)-4[\phi_0+\phi_1q+\phi_2q_2+\phi_3q\ +....][6\phi_3+...]-4[2\phi_2+6\phi_3q+...][\phi_1+2\phi_2q+3\phi_3q^2+..]$$

$$-4[\phi_1+2\phi_2q+3\phi_3q^2+\phi_3q^2+\phi_3q^3+..][6\phi_3+...]-4[2\phi_2+6\phi_3q+..][\phi_1+2\phi_2q+3\phi_3q^2+...]$$

$$-4[\phi_1+2\phi_2q+3\phi_3q^2+][2\phi_2+6\phi_3q+..]-4[\phi_1+2\phi_2q+3\phi_3q^2+...][2\phi_2q+6\phi_3q]$$

$$=h[2\phi_2^1+6\phi_3^1q+..]=4[\phi_0+\phi q+\phi_2q+\phi_3q+..]-4[\phi_1+2\phi-2q+3\phi_3q^2+,,,,]$$

$$\phi[\phi_1+2\phi_2q+3\phi_3q^2+...]+h[2\phi_2^1+6\phi_3^1q+..]-4[\phi_0+\phi_1q+\phi_2q^2+\phi^3q^3+2\phi\ +6\phi_3q+..]$$

$$4[\phi_1+2\phi_2q^2+3\phi_3q^2+\phi_1+2\phi_2q+3\phi_3q^2+...]$$

$$+h[2\phi_2^1+6\phi_2^1q+...]-4[\phi_0+\phi_1q+\phi_2q^2+\phi_3q^3+2\iota_2+6\phi_3^1q+...]$$

Let $q=0$ in (13) . $6\phi_3^1-2\phi_2^1=h[2\phi_2^1-8\phi_0\phi_2-4\phi_1\phi_1]+h[2\phi_2^1-8\phi_0\phi_2-4\phi_1\phi_1]$

$$6\phi_3^1=6\phi_2^1=3h[2\phi_2-8\phi_0\phi_2-4\phi_1\phi_1],$$

Hence $\phi_2^1=\frac{1}{2}h[2\phi_2^1-8\phi_0\phi_1=4\phi_1\phi_1]...$ (14)

Using $\phi_0-1\ ,\phi=-hx\ ,\phi_1^1=-2h$

$\phi_2=-2h^2x+4h^2x^2-2h$ and $\phi_1^1=-2h^2+8h^2x2h$

Then $\phi_3^1=(-2h^2+8h^2)(-2h)=\frac{1}{2}h[2(-2h^2x\_2h)-8(1)(-2h^2x+4h^2x^2=2hx)-4(-2hx)^2$ Or

$$\phi_3^1=\frac{1}{2}h[-4h^2+16h^2x-4h+16h^2x-32h^2x+16hx-16h^2x^2]-2h^2+8h^2x^2-2h$$

Hence, $\phi_3^1=-2h[h^2+2h+1]+16h^2x(h+1)-24h^3x^2$

Integrating both sides, to get. $\phi_3^1=2h[h^2 2h+1]x+8h^2(h+1)-8h^3x^3+c$

The third approximation for $\varphi(x)$ is

$$\phi_3=\phi_0+\sum_{i=0}^{3}\phi_n$$

i.e., $\phi_3^1=1-2hx-2h^2x+4h^2x^2-2hx-2h[h^2+2h+1+8h^2x^2(h+1)-8h^3x^3$ (15)

For n^{th} approximation of $\phi(x)$

$$\phi_n=\phi_0+\phi_1+\phi_2+\phi_3+.....\phi_n$$

Hence, $\varphi_0(x)=\varphi_0+\sum_{n=1}^{n}\varphi_n=\varphi_1+\varphi_2+\varphi_3+.....\varphi_n$

Convergence Analysis

Theorem: Whereas the series

$$\varphi_0(x)+\sum_{n=1}^{\infty}\varphi_n(x)=1,$$

Converges, where $\varphi_n(x)$ are resulted in (2) and (3) the limit of the series is an exact solution of (1)

Proof,

Since by hypothesis, the series is convergent, it holds

$$S(x)=\sum_{n=1}^{\infty}\varphi_n(0)$$

So, the necessary condition for the convergence of the series is valid, that is

$$\lim_{n\to 0}\varphi_n(x)=0$$

From (5) and (6)

$$hH(x)\sum_{n=1}^{\infty} Rn[\varphi_{n-1}(x)]\lim_{n\to\infty}\sum_{n=1}^{\infty}[\varphi_n(x)x_n\varphi_{n-1}-(x)]=$$

$$L\lim_{n\to 0}\sum_{n=1}^{\infty}[(x)-x_n\varphi_{n-1}(x)=L\lim_{n\to\infty}\varphi_n(x)=0$$

Since h is not equal to zero,
Hence

$$\sum_{n=1}^{\infty} Rn[\varphi'_n(x)]=0$$

Numerical Examples
To demonstrate the effectiveness of the algorithm in this study, we consider the following two examples, the HAA computation results are as presented in the table.
Example 1: Consider the fourth order oscillatory initial value problem

$$y^{iv}=5y^{ii}-4y$$

with the initial conditions

$$y(0)=1, y'(0)=0, y''(0)=0, y'''(0)=1$$

Exact solution: $1+\frac{1}{6}\sin 2x$

Source: Bataineh (2009)
To solve the equation by Homotopy Analysis Algorithm with the initial approximation

$$y(x)=y\ (0)+y^{i}(0)x+y^{ii}(0)x^2+y^{iii}(0)x^3$$

$$y(x)=1+\frac{1}{6}x^3$$

And linear operator

$$L[\phi(x;q)]=\frac{\delta^2\phi(x;q)}{\delta x^4}$$

With the property

$$L[c_1+c_2+c_3+c_4]=0$$

Where $c,[c=1,2,3,4]$ are constants of integration for m ≥ 1, the mth order deformation with initial conditions will be

$$y_m(0)=0, y^{i}m(0)=0, y^{ii}m(0)=0, y^{iii}m(0)=1$$

Where $Rm(y)_{m-1}=y^{iii}(m-1)(x)+5x^{ii}(m-1)(x)+4y_{(m-1)}(x)$

The solution of the mth order deformation for m ≥ 1 $y_m(x)=y_m y_{m-1}(x)+hL^{-1}R_m(y_{m-1})$

Hence $y^{i}(x)=\frac{1}{6}hx^4+\frac{1}{24}hx^5+\frac{1}{1260hx^7}$

$$y^2(x)=\frac{1}{6}hx^3+\frac{1}{6}hx^2x^4+\frac{1}{24}h^2x^6+\frac{1}{1260}hx^7+\frac{29}{5040}h^2x^9+\frac{1}{5230}h^2x^9+\frac{1}{2494800}h^2x^{11}$$

.. Then the series solution expression can be written in the form

$y(x)=y(0)+y^{i}(x)+y^{ii}(x)+y-3(x)+\ldots$ .... And so forth

, hence the series solution when h = −1 is

$$y_i(x)=-\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{1260}x^7$$

$$y_2(x)=\frac{1}{36}x^6+\frac{5}{1008}x^7+\frac{1}{2520}x^8+\frac{1}{9072}x^9+\frac{1}{2494800}x^{11}$$

$$y_3(x)=\frac{-5}{2010}x^8+\frac{25}{7257}x^9-\frac{1}{22680}x^{11}+\frac{1}{133056}x^{12}+\frac{1}{25945920}x^{13}+\frac{1}{2043241200}x^{15}$$

$$y_4(x)=\frac{5}{36288}x^{10}+\frac{25}{1596672}x^{11}+\frac{1}{399168}x^{12}+\frac{15}{1550755}x^{13}+\frac{1}{90810720}x^{14}+\frac{1}{544864320}x^{15}+\frac{1}{8142948000}x^{16}+\frac{1}{2778803250}x^{17}+\frac{1}{4715761734200}x^{19}+.....$$

And so forth.

Hence the series solution is

$$y(x)=1+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{36}x^5+\frac{1}{240}x^7-\frac{1}{480}x^8-\frac{17}{2597}x^9+\frac{17}{181144}x^{10}+..$$

This converges to the ADM solution

Approximation of Homotopy Analysis Algorithm

X, 0

$1 + (0)^3 0.1666 = 1$

X, 0.1

$1 + (0.1)^3 0.1666=1.$

X, 0.2

$1 + (0.2)^3 0.1666 = 1.0013$

X.(0.3)

$1 + (0.3)^3 0.1666 = 1.0045$

X,(0.4)

$1 + (0.4)^3 0.1666 = 1.0107$

X, (0.5)

$1 + (0.5)^3 0.1666 = 1.0208$

X, (0.6)

$1 + (0.6)^3 0.1666 = 1.0360$

X, (0.7)

$1 + (0.7)^3 0.1666 = 1.0571$

X, (0.8)

$1 + (0.8)^3 0.1666 = 1.0571$

X, (0.9)

$1 + (0.9)^3 0.1666 = 1.1215$

X, (1)

$1 + (1)^3 0.1666 = 1.1666$

Approximation of Adomian Decomposition Method

X,(0)

$1 + (0)^{2.5}\, 0.1666 = 1$

X,(0.1)

$1 + (0.1)^{2.5}\, 0.1666 = 1.0005$

X,(0.2)

$1 + (0.2)^{2.5}\, 0.1666 = 1.0030$

X,(0.3)

$1 + (0.3)^{2.5}\, 0.166 = 1.0082$

X,(0.4)

$1 + (0.4)^{2.5}\, 0.1666 = 1.0169$

X,(0.5)

$1 + (0.5)^{2.5}\, 0.1666 = 1.0295$

X,(0.6)

$1 + (0.6)^{2.5}\, 0.1666 = 1.0464$

X,(0.7)

$1 + (0.7)^{2.5}\, 0.1666 = 1.0683$

X, (0.8)

$1 + (0.8)^{2.5}\, 0.1666 = 1.0953$

X, (0.9)

$1 + (0.9)^{2.5}\, 0.1666 = 1.1280$

X, (1)

$1 + (1)^2 0.1666 = 1.1666$

Table 1: Numerical results for example 1

(x)	HAA	ADM	EXACT	HAA ERROR	ADM ERROR
0	1	1	1	0.0E	0.0E
0.1	1.0002	1.0005	1.0331	0.0311	0.0326
0.2	1.0013	1.0030	1.1501	0.1488	0.1471
0.3	1.0045	1.0082	1.0941	0.1018	0.0859
0.4	1.0107	1.0169	1.1125	0.0998	0.0956
0.5	1.0205	1.0295	1.1402	0.1197	0.1187
0.6	1.0360	1.0465	1.1553	0.1193	0.1198
0.7	1.0571	1.0683	1.1642	0.1071	0.1288
0.8	1.0853	1.0953	1.1666	0.0813	0.9130
0.9	1.1213	1.1280	1.1673	0.0460	0.7393
1	1.1666	1.1666	1.1728	0.0062	0.0062

The HAA compares favourably with the ADM and exact solution.

Example 2: Consider the non-linear oscillatory initial value problem

$$y^4 = yy^{ll} + y^2$$

Subject to the initial conditions

$$y(0)=0.y_1(0)=1, y_2(0), y_3(0)=1$$

Source: Liao (2012)

The exact solution is $e^x - 1$, According to the Homotopy Analysis Algorithm, the initial approximation is $y_0(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

The zero order deformation equation with initial conditions

$$Rm_{(ym-1)} = y^{\iota\iota\iota}m-1 + \sum_{m=0}^{n-1} y^{\iota}(x) y^{\iota\iota}m-1(0) - yj(x)_{m-1} \sum_{j=0}^{m-1} y^{\iota} - j(x)$$

The solution of the mth order deformation equation for m≥1 is

$$y_m(x) = y_m y_{m-1}(x) + hL - 1R_m y_{m-1} y(x) = -\frac{1}{24hx^4} - \frac{1}{120}hx^5 - \frac{1}{270}hx^6 - \frac{1}{2520}hx^7 + \frac{1}{2016}hx^8 + \ldots$$

The series solution expression can be written in the form $y(x) = y(x) + y'(x) + y''(x) + \ldots..$ and so forth,

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{750}x^6 + \frac{1}{5040}x^6 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9$$

This converges to the Adomian Decomposition Method solution

Approximation of Homotopy Analysis Algorithm

X,(0)
0 + (0) ^{2} = 0
X (0.1)
0.1 + 0.5(0.1) ^{2} = 0.1050
X (0.2)
0.2 + 0.5(0.2) ^{2} = 0.2100
X ,(0.3)
0.3 + 0.5(0.3)^2 = 0.3450
X,(0.4)
0.4 + 0.5(0.4)^2 = 0.4850
X,(0.5)
0.5 + 0.5(0.5)^2 = 0.6250
X,(0.6)
0.6 + 0.5(0.6) ^{2} = 0.7800
X,(0.7)
0.7 + 0.5(0.7) ^{2} = 0.9450
X,(0. 8)
0.8 + 0.5(0.8) ^{2} = 1.1200
X,(0.9)
0.9 + 0.5(0.9) ^{2} = 1.3050
X, 1
1 + 0.5(1) ^{2}=1.5000

Approximation of Adomian Decomposition Method

X,(0)
0 + 0.5(0)^3 = 0
X,(0.1)
0.1 + 0.5(0.1^3) = 0.1005
0.2 + 0.5(0.2)^3 = 0.2040
X,(0.3)
0.3 + 0.5(0.3) ^{3}= 0.3135
X,(0.4)
0.4 + 0.5(0.4)^3 = 0.4320
X,(0.5)
0.5 + 0.5(0.5)^3 = 0.5625
X,(0.6)
0.6 + 0.5(0.6)^3 = 0.7080
X,(0.7)
0.7 + 0.5(0.7)^3 = 0.8715
X,(0.8)
0.8 + 0.5(0.8)^3 = 1.0560
X,(0.9)
0.9 + 0.5(0.9)^3 = 1.2645
X,(1)
1+0.5(1)=1.5000

Table 2: Numerical results for example 2

X	HAA	ADM	EXACT	HAA ERROR	ADM ERROR
0	0.0000	0.0000	0	0.0000	0.0000
0.1	0.0050	0.1005	0.1051	0.1001	0.0046
0.2	0.2100	0.2040	0.2214	0.0114	0.0174
0.3	0.3450	0.3135	0.3499	0.0049	0.0364
0.4	0.4850	0.4320	0.4918	0.0068	0.0598
0.5	0.6250	0.5625	0.6487	0.0237	0.0862
0.6	0.7800	0.7080	0.8221	0.0421	0.1141
0.7	0.9450	0.8715	1.0137	0.0687	0.1422
0.8	1.1200	1.0560	1.2256	0.1056	0.1696
0.9	1.3050	1.2645	1.4596	0.1546	0.1951
1	1.5000	1.5000	1.7183	0.2183	0.2183

The HAA compares favourably with the ADM

CONCLUSION

The proposed algorithm HAA have been successfully applied for the approximation of oscillatory initial value problems. The result obtained is compared with the Adomian Decomposition Method and the exact solution, it was observed that all the problems considered shows that the HAA results compared favourably with the ADM and exact solutions, It is clearly seen that the Homotopy Analysis Algorithm is a cogent and effective algorithm for approximating the numerical (analytic) solution of oscillatory initial value problems, also It could be observed that HAA converges faster and was implemented without any need for discretization of the problem, Therefore for easy of solution to oscillatory IVPs, without tedious algebraic computations, this study recommends HAA for approximating oscillatory IVPs.

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