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STABILITY ANALYSIS OF A SHIGELLA INFECTION EPIDEMIC MODEL AT ENDEMIC EQUILIBRIUM

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ABSTRACT

In this study, we modified continuous mathematical model for the dynamics of shigella outbreak at constant recruitment rate πformulated by (Ojaswita et al., 2014). In their model, they partitioned the population into Susceptible (S), Infected (I) and recovered (R) individuals. We incorporated a vaccinated class (V), educated class (G), exposed class (E), asymptomatic (A) hospitalized class (H) and Bacteria class (B) with their corresponding parameters. We analyzed a SVGEAIHRB compartmental nonlinear deterministic mathematical model of shigella epidemic in a community with constant population. Analytical studies were carried out on the model using the method of linearized stability. The basic reproductive number R_0 that governs the disease transmission is obtained from the largest eigenvalue of the next-generation matrix. The endemic equilibrium is computed and proved to be locally and globally asymptotically stable if $R_0 \leq 1$ and unstable if $R_0 > 1$. Finally, we simulate the model system in MATLAB and obtained the graphical behavior of the infected compartments. From the simulation, we observed that the shigella infection was eradicated when $R_0 \leq 1$while it persist in the environment when $R_0 > 1$.

Keywords: SVGEAIHRB Model, Basic reproduction number, endemic equilibrium, Local stability, global stability, numerical simulation, transmission

INTRODUCTION

Shigellosis is an infection of the intestines caused by *Shigella* bacteria. (CDC, 2017) Symptoms generally start one to two days after exposure and include diarrhea, fever, abdominal pain, and feeling the need to pass stools even when the bowels are empty. (CDC, 2017) The diarrhea may be bloody. (CDC, 2017) Symptoms typically last five to seven days and it may take several months before bowel habits return entirely to normal. (CDC, 2017) Complications can include reactive arthritis, sepsis, seizures, and hemolytic uremic syndrome. (CDC, 2017)

Shigellosis is caused by four specific types of *Shigella*. (WHO, 2005).These are typically spread by exposure to infected feces. (CDC, 2017) This can occur via contaminated food, water, or hands or sexual contact. (CDC, 2017) (CDC, 2019) Contamination may be spread by flies or when changing diapers (nappies). (CDC, 2017) Diagnosis is by stool culture. (CDC, 2017)

The risk of infection can be reduced by properly washing the hands. (CDC, 2017) Currently, no licensed vaccine targeting *Shigella* exists. Several vaccine candidates for *Shigella* are in various stages of development including live attenuated, conjugate, ribosomal, and proteosome vaccines (Mani *et. al.*, 2016; WHO, 2016; VRD, 1997).In clinical trials, these O-specific polysaccharide conjugate vaccines appeared safe and immunogenic in adults (Taylor *et al.*, 1993; Cohen *et al.*, 1996; Passwell *et al.*, 2001) and in children 4 to 7 years of age (Ashkenazi *et al.*, 1999), but the antibody responses were lower for children 3 years of age (Passwell *et al.*, 2003 & Passwell *et al.*, 2010). (CDC, 2017) Shigellosis usually resolves without specific treatment. (CDC, 2017) Rest and sufficient fluids by mouth are recommended. (CDC, 2017) Bismuth subsalicylate may help with the symptoms; however, medications that slow the bowels such as loperamide are not recommended. (CDC, 2017) In severe cases, antibiotics may be used but resistance is common. (CDC, 2017) (CDC, 2018).Commonly used antibiotics include ciprofloxacin and azithromycin. (CDC, 2017)

Mathematical Model Literatures

(Ojaswita et al., 2014) developed a continuous mathematical model for shigella diarrhea outbreak. According to the pathogenesis of shigella, they partitioned the population into Susceptible (S), Infected (I) and recovered (R) individuals. They computed the disease-free equilibrium state and the basic reproduction number R_0such that $R_0 < 1$ indicates the possibility of shigella diarrhea eradication in the community while $R_0 > 1$ represents uniform persistence of the disease.

(Ebenezer et al., 2019) developed a compartmental mathematical model of (SITR) to investigate the effect of saturation treatment in the dynamical spread of diarrhea in the community. Their mathematical analysis showed that the disease free and the endemic equilibrium points of the model exist. They also showed that the disease-free equilibrium is locally and globally asymptotically stable if $R_0 < 1$ and unstable if$R_0 > 1$. From simulation, the efficacy of the treatment also showed a great impact in the total eradication of diarrhea epidemic.

(Hailay et al., 2019a) developed and investigated dysentery dynamics model with incorporating controls. The system is considered as SIRSB deterministic compartmental model with treatment and sanitation. They obtained the threshold number R_0 such that $R_0 \leq 1$ indicates the possibility of dysentery eradication in the community while $R_0 > 1$ represents uniform persistence of the disease. They used Lyapunov–LaSalle method to prove the global stability of the disease-free equilibrium. Moreover, they used geometric approach method to obtain the sufficient condition for the global stability of the unique endemic equilibrium for$R_0 > 1$.

Mathematical formulation

In this section, we formulate and analyze a mathematical model of Shigella disease. The modeled populations include humans and pathogens. The human population is subdivided into eight classes. These classes of individual are: Susceptible(S), Vaccinated (V), Education campaign (G), Exposed (E), Asymptomatic (A), Infected (I), Hospitalized (H) and Recovered (R). The pathogen population (concentration of shigella dysenteriae) is represented by B. The formulation of the model is based on the following assumptions:

Assumptions of the model

i. the recruitment is through birth only and it is constant.
ii. all individuals are born susceptible.
iii. an individual can be infected through contact with the infectious individuals' faeces and contaminated water or food.
iv. infected individuals die either naturally or due to the disease.
v. vaccination is strictly on susceptible adult and susceptible children between the ages of 4 to 7years.
vi. Vaccinated individuals move back to the susceptible class when they lose immunity due to the vaccine.
vii. there is no permanent recovery.
viii. there is homogenous mixture in the population.
ix. the interaction of individuals in the human population is panmictic.
x. the recruitment of bacteria in the environment is constant.
xi. humans and primate animals are the only source of pathogens.
xii. in the environment, pathogens interact to replicate and hence are recruited through birth only.
xiii. pathogen population in the environment diminishes through natural death and environmental contamination.
xiv. environmental sanitation will be enforced so that shigella pathogen death can be approximated to be constant at a rate σ_3.

Flow diagram of the model with constant control

We demonstrate the dynamical transfer of the population with the flow diagram in Figure 1 below

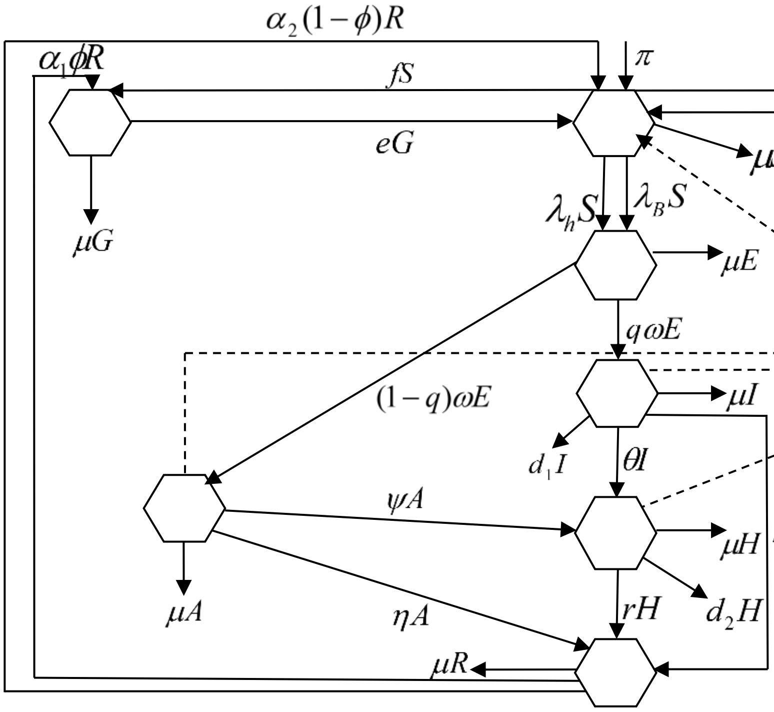

Figure 1: A schematic representation of flow of individuals (solid lines) among states and flow of pathogen in the environment (dotted lines) for the environmental infect transmission system (EITS) of the modified model.

Table 1: Description of the variables of the models

Variables	Description
$S(t)$	Number of susceptible individuals at time (t).
$V(t)$	Number of vaccinated individuals at time (t).
$G(t)$	Number of educated individuals at time (t).
$E(t)$	Number of exposed individuals at time (t).
$A(t)$	Number of asymptomatic individuals at time (t).
$I(t)$	Number of infected individuals at time (t).
$H(t)$	Number of hospitalized individuals at time (t).
$R(t)$	Number of Recovered individuals at time (t).
$B(t)$	Number of bacteria in the environment at time (t).
$N_h(t)$	The total human population size at time (t).

Table 2: Description of the parameters of the models

Parameters	Description
π	The recruitment rate.
m	The vaccination rate at which the susceptible individuals move to the vaccinated class.
n	The vaccine immunity loss rate at which the vaccinated individuals move to the susceptible class.
f	The education rate at which the susceptible individuals move to the educated class.
e	The recovering rate at which the educated individuals (who failed to adhere to the education they received) moved back to the susceptible class.
r	The rate at which the hospitalized individuals moved to the recovered class.
θ	The rate at which the infected individuals moved to the hospitalized class.
η	The rate at which the asymptomatic individuals moved to the recovered class.
μ	The natural death rate.
d_1	The death rate due to the disease in the infected class.
d_2	The death rate due to the disease in the hospitalized class.
φ	The proportion of the recovered individuals who moved to the educated class at a rateα_1.
$(1-\varphi)$	The proportion of the recovered individuals who moved to the susceptible class at a rateα_2.
q	The proportion of the exposed individuals who moved to the infected class at a rateω.
$(1-q)$	The proportion of the exposed individuals who moved to the asymptomatic class at a rate.
ω	The incubation rate (rate at which exposed individuals, $E(t)$, progress to either asymptomatic class $A(t)$ or infected $I(t)$).
ψ	The rate at which the asymptomatic individuals moved to the hospitalized class.
ρ	The recovering rate at which the infected individuals moved to the recovered class.
K	The concentration of *Shigella* in the environment that yields 50% chance of catching dysentery diarrhea (Berhe et al., 2019).
λ_h	The force of infection in the human to human interaction.
λ_B	The force of infection in the environment to human interaction.
β_1	The transmission rate of shigella for the infected individuals due to human to human interaction.
β_2	The transmission rate of shigella for the asymptomatic individuals due to human to human interaction.
β_3	The transmission rate of shigella for the hospitalized individuals due to human to human interaction.
β_B	The ingestion rate of shigella by human from the environment.
ε	Shigella pathogen shedding rate for the infected individuals.
δ	Shigella pathogen shedding rate for the asymptomatic individuals.
γ	Shigella pathogen shedding rate for the hospitalized individuals.
σ_1	Shigella pathogen growth rate.
σ_2	Shigella pathogen natural death rate.
σ_3	Death rate of shigella pathogen due to environmental decontamination.

Equations of the model

$$\frac{dS}{dt} = \pi + nV + eG + \alpha_2(1-\varphi)R - (\lambda_h + \lambda_B)S - (m + f + \mu)S \quad (1)$$
$$\frac{dV}{dt} = mS - (n + \mu)V \quad (2)$$
$$\frac{dG}{dt} = fS + \alpha_1\varphi R - (e + \mu)G \quad (3)$$
$$\frac{dE}{dt} = (\lambda_h + \lambda_B)S - (\omega + \mu)E \quad (4)$$
$$\frac{dA}{dt} = (1-q)\omega E - (\eta + \psi + \mu)A \quad (5)$$
$$\frac{dI}{dt} = q\omega E - (\theta + \rho + d_1 + \mu)I \quad (6)$$
$$\frac{dH}{dt} = \theta I + \psi A - (r + d_2 + \mu)H \quad (7)$$
$$\frac{dR}{dt} = rH + \eta A + \rho I - \alpha_1\varphi R - \alpha_2(1-\varphi)R - \mu R \quad (8)$$
$$\frac{dB}{dt} = \varepsilon I + \delta A + \gamma H + (\sigma_1 - \sigma_2 - \sigma_3)B \quad (9)$$
$$N = S + V + G + E + I + A + H + R \quad (10)$$

$S(0) = S_0 > 0, V(0) = V_0 \geq 0, E(0) = E_0 \geq 0, G(0) = G_0 \geq 0, I(0) = I_0 \geq 0, A(0) = A_0 \geq 0,$
$H(0) = H_0 \geq 0, R(0) = R_0 \geq 0,\ B(0) = B_0 > 0.$

The force of infection for human to human interaction (λ_h) and the force of infection for environment to human interaction (λ_β) are (11) and (12) respectively:

$$\lambda_h = \beta_1 I + \beta_2 A + \beta_3 H \quad (11)$$

$$\lambda_B = \frac{\beta_B B}{K+B} \tag{12}$$

$$\lambda_0 = \beta_1 I + \beta_2 A + \beta_3 H + \frac{\beta_B B}{K+B} \tag{13}$$

Where K is the shigella concentration that yields $25-50\%$ chance of catching dysentery diarrhea (Cabral & Joao, 2010). β_1,β_1 and β_1 are human to human interaction while $\beta_B$is the ingesting rate of shigella from the contaminated environment. Infected humans contribute to the concentration of shigella at a rate of ε , asymptomatic humans contribute to the concentration of shigella at a rate of$\delta$and hospitalized humans contribute to the concentration of shigella at a rate ofγ.The pathogen population is growing at a rateσ_1, natural death rate σ_2 and death rate of shigella pathogen due to environmental decontamination is σ_3. We assume that $\sigma_1-\sigma_2-\sigma_3>0$ $\sigma_1>\sigma_2+\sigma_3$represents the net death rate of the pathogen population in the environment (Bani-Yaghoub et al., 2012).

Model analysis

Endemic equilibrium point of the model equations

The endemic equilibrium state is the state where the disease cannot be totally eradicated but remains in the population. An endemic equilibrium exists if and only if the value of (E°) is less than$(E^{*})$and this is equivalent to(E°). For the disease to persist in the population, the susceptible, vaccinated, educated, exposed, asymptomatic, infected, hospitalized, recovered and bacteria class must not be zero at equilibrium state. In other words, if $E^*=(S^*,V^*,G^*,E^*,A^*,I^*,H^*,H^*,R^*)$ is the endemic equilibrium state, then $E^*=(S^*,V^*,G^*,E^*,A^*,I^*,H^*,H^*,R^*)\neq(0,0,0,0,0,0,0,0,0)$. In order to obtain the endemic equilibrium points of the system of non-linear ordinary differential equation, we solve equation (4.18 – 4.26) simultaneously by setting the total derivatives of the model equations to zero$\left(i.e. \frac{dS}{dt}=\frac{dV}{dt}=\frac{dG}{dt}=\frac{dE}{dt}=\frac{dA}{dt}=\frac{dI}{dt}=\frac{dH}{dt}=\frac{dR}{dt}=\frac{dB}{dt}=0\right)$. The system of equations (1) to (9) at endemic equilibrium point can be simplified to obtain:

$$\Theta^* = \left(S^*,V^*,G^*,E^*,A^*,I^*,H^*,R^*,B^*\right)$$

$$\Theta^* = \begin{Bmatrix} \frac{\pi}{P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*}, \frac{m\pi}{P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*}, \frac{\pi\left(f+\frac{\alpha_1\varphi P_{17}\lambda^*}{P_{13}}\right)}{P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*}, \\ \frac{\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \frac{P_{14}\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \frac{P_{15}\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \\ \frac{P_{16}\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \frac{P_{17}\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \frac{P_{18}\pi\lambda^*}{P_{13}(P_{10}-P_{19}-P_{20}+(1-P_{22})\lambda^*)}, \end{Bmatrix} \tag{14}$$

where

$P_1=(1-q)\omega,\ P_2=\eta+\psi+\mu,\ P_3=q\omega,\ P_4=\theta+\rho+d_1+\mu,\ P_5=r+d_2+\mu,\ P_6=\alpha_1\phi,\ P_7=\alpha_2(1-\phi),\ P_8=\sigma_2+\sigma_3-\sigma_1,\ P_9=\beta_1 I^*+\beta_2 A^*+\beta_3 H^*+\frac{\beta_B B^*}{K+B^*},\ P_{10}=m+f+\mu,\ P_{11}=n+\mu,\ P_{12}=e+\mu,\ P_{13}=\omega+\mu.$

Computation of the Basic Reproduction Number R_0

The basic reproduction number R_0 is the average number of new infections, that one infected case will generate during their entire infectious lifetime (Nelson & Williams, 2013; Addo, 2009; Heffernan et al., 2005).

It is very important in determining whether the disease persists in the population or die out. We use the next generation matrix to compute the basic reproduction number R_0 which is formulated in (Van den Driessche & Watmough, 2002). Let us assume that there are n compartments of which the first m compartments correspond to infected individuals. Let

- $F_i(y)$ be the rate of appearance of new infections in compartment i,
- ${V_i}^+(y)$ be the rate of transfer of individuals into compartment i by all other means, and
- ${V_i}^-(y)$ be the rate of transfer of individuals out of compartments$i$.

It is assumed that each function is continuously differentiable at least twice in each variable. The disease transmission model consists of nonnegative initial conditions together with the following system of equations:

$$\frac{dy_i}{dt}=f_i(y)=F_i(y)-V_i(y),\ i=1,2,3,\cdot\ \ \cdot\ \ \cdot,n \tag{15}$$

where$V_i(y)={V_i}^-(y)-{V_i}^+(y)$. (16)

$$\frac{d}{dt}=F-V=\begin{pmatrix}(\lambda_h+\lambda_B)S\\0\\0\\0\\0\end{pmatrix}-\begin{pmatrix}(\omega+\mu)E\\(\eta+\psi+\mu)A-(1-q)\omega E\\(\theta+\rho+d_1+\mu)I-q\omega E\\(r+d_2+\mu)H-\theta I-\psi A\\(\sigma_2+\sigma_3-\sigma_1)B-\varepsilon I-\delta A-\gamma H\end{pmatrix}$$

$$R_0=\rho(FV^{-1})=\rho\left(\left(\left.\frac{\partial F_i}{\partial y_j}\right|_{E^0}\right)\left(\left.\frac{\partial V_i}{\partial y_j}\right|_{E^0}\right)^{-1}\right), \tag{17}$$

where $F$are the new infection transfer terms and $V$is the non-singular matrix of the remaining transfer terms. The basic reproduction number $R_0$of the model (1) – (9) is calculated using the next generation matrix (Van den Driessche & Watmough, 2002). In using their approach (Van den Driessche & Watmough, 2002), we have:

$$F=\left(\frac{\partial F_i}{\partial y_j}\bigg|_{E^\circ}\right)=\begin{pmatrix}0 & \beta_2 S^\circ & \beta_1 S^\circ & \beta_3 S^\circ & \frac{\beta_B S^\circ}{K}\\ 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\Rightarrow F=\begin{pmatrix}0&y_1&y_2&y_3&y_4\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix} \tag{18}$$

Let $y_1=\beta_2 S^\circ, y_2=\beta_1 S^\circ,\ y_3=\beta_3 S^\circ$ and $y_1=P_{30}=\frac{\beta_B S^\circ}{K}$

Similarly,

$$V=\left(\frac{\partial V_i}{\partial y_j}\bigg|_{E^\circ}\right)=\begin{pmatrix}\omega+\mu & 0 & 0 & 0 & 0\\ -(1-q)\omega & \eta+\psi+\mu & 0 & 0 & 0\\ -q\omega & 0 & \theta+\rho+d_1+\mu & 0 & 0\\ 0 & -\psi & -\theta & r+d_2+\mu & 0\\ 0 & -\delta & -\varepsilon & -\gamma & \sigma_2+\sigma_3-\sigma_1\end{pmatrix}$$

$$|V|=T_{15}=P_2P_4P_5P_8P_{13}$$

$$C^T=\begin{pmatrix}T_1&0&0&0&0\\T_2&T_6&0&0&0\\T_3&0&T_9&0&0\\T_4&T_7&T_{10}&T_{12}&0\\T_5&T_8&T_{11}&T_{13}&T_{14}\end{pmatrix}=AdjV$$

$$V^{-1}=\frac{1}{|V|}AdjV\Rightarrow V^{-1}=\frac{1}{T_{15}}.\begin{pmatrix}T_1&0&0&0&0\\T_2&T_6&0&0&0\\T_3&0&T_9&0&0\\T_4&T_7&T_{10}&T_{12}&0\\T_5&T_8&T_{11}&T_{13}&T_{14}\end{pmatrix}\Rightarrow V^{-1}=\begin{pmatrix}\frac{T_1}{T_{15}}&0&0&0&0\\\frac{T_2}{T_{15}}&\frac{T_6}{T_{15}}&0&0&0\\\frac{T_3}{T_{15}}&0&\frac{T_9}{T_{15}}&0&0\\\frac{T_4}{T_{15}}&\frac{T_7}{T_{15}}&\frac{T_{10}}{T_{15}}&\frac{T_{12}}{T_{15}}&0\\\frac{T_5}{T_{15}}&\frac{T_8}{T_{15}}&\frac{T_{11}}{T_{15}}&\frac{T_{13}}{T_{15}}&\frac{T_{14}}{T_{15}}\end{pmatrix}. \tag{19}$$

Substitute (18) and (19) in (17), we have

$$FV^{-1}=.\begin{pmatrix}\frac{T_2y_1}{T_{15}}+\frac{T_3y_2}{T_{15}}+\frac{T_4y_3}{T_{15}}+\frac{T_5y_4}{T_{15}} & \frac{T_6y_1}{T_{15}}+\frac{T_7y_3}{T_{15}}+\frac{T_8y_4}{T_{15}} & \frac{T_9y_2}{T_{15}}+\frac{T_{10}y_3}{T_{15}}+\frac{T_{11}y_4}{T_{15}} & \frac{T_{12}y_3}{T_{15}}+\frac{T_{13}y_4}{T_{15}} & \frac{T_{14}y_4}{T_{15}}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$$T_{16}=\frac{T_2y_1}{T_{15}}+\frac{T_3y_2}{T_{15}}+\frac{T_4y_3}{T_{15}}+\frac{T_5y_4}{T_{15}},\ T_{17}=\frac{T_6y_1}{T_{15}}+\frac{T_7y_3}{T_{15}}+\frac{T_8y_4}{T_{15}},\ T_{18}=\frac{T_9y_2}{T_{15}}+\frac{T_{10}y_3}{T_{15}}+\frac{T_{11}y_4}{T_{15}},$$

$$T_{19}=\frac{T_{12}y_3}{T_{15}}+\frac{T_{13}y_4}{T_{15}}\text{ and }T_{20}=\frac{T_{14}y_4}{T_{15}}$$

$$FV^{-1}=\begin{pmatrix}T_{16}&T_{17}&T_{18}&T_{19}&T_{20}\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\ \left|FV^{-1}-\lambda I\right|=\begin{vmatrix}T_{16}-\lambda&T_{17}&T_{18}&T_{19}&T_{20}\\0&-\lambda&0&0&0\\0&0&-\lambda&0&0\\0&0&0&-\lambda&0\\0&0&0&0&-\lambda\end{vmatrix}=0$$

$$\lambda^4(T_{16}-\lambda)=0\Rightarrow T_{16}-\lambda=0\Rightarrow\lambda=T_{16}$$

$$\lambda=\rho(FV^{-1})=R_0=T_{16}=+\frac{T_2y_1}{T_{15}}+\frac{T_3y_2}{T_{15}}+\frac{T_4y_3}{T_{15}}+\frac{T_5y_4}{T_{15}}$$

$$R_0=\frac{T_2y_1}{T_{15}}+\frac{T_3y_2}{T_{15}}+\frac{T_4y_3}{T_{15}}+\frac{T_5y_4}{T_{15}}$$

$$R_0=\frac{(1-q)\omega\beta_2S^\circ}{(\eta+\psi+\mu)(\omega+\mu)}+\frac{q\omega\beta_1S^\circ}{(\theta+\rho+d_1+\mu)(\omega+\mu)}+\frac{\big((1-q)\omega(\theta+\rho+d_1+\mu)\psi+(\eta+\psi+\mu)q\omega\theta\big)\beta_3S^\circ}{(\eta+\psi+\mu)(\theta+\rho+d_1+\mu)(r+d_2+\mu)(\omega+\mu)}$$
$$+\frac{(1-q)\omega\big((\theta+\rho+d_1+\mu)(\psi\gamma+(r+d_2+\mu)\delta)+(\eta+\psi+\mu)q\omega(\theta\gamma+(r+d_2+\mu)\varepsilon)\big)\beta_BS^\circ}{K(\eta+\psi+\mu)(\theta+\rho+d_1+\mu)(r+d_2+\mu)(\sigma_2+\sigma_3-\sigma_1)(\omega+\mu)}$$

The local stability analysis of the endemic equilibrium of the model

To examine the local stability of the endemic (E^*) equilibrium, we obtain the Jacobian matrix by differentiating the functions ($f_i; = 1,2,3,\dots,9$) partially with respect to the variables in the system of the modified equations.

Theorem

The endemic equilibrium point Θ^* is locally asymptotically stable when $R_0 > 1$.

Proof

The Jacobian matrix from the partial derivatives of (1) to (9) at endemic equilibrium (J_{Θ^*}) is given by:

$$J_{\Theta^*} = \begin{pmatrix} -(\lambda_1 + P_{10}) & n & e & 0 & -P_{26} & -P_{27} & -P_{28} & P_7 & -P_{30} \\ m & -P_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & -P_{12} & 0 & 0 & 0 & 0 & P_6 & 0 \\ \lambda_1 & 0 & 0 & -P_{13} & P_{26} & P_{27} & P_{28} & 0 & P_{30} \\ 0 & 0 & 0 & P_1 & -P_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_3 & 0 & -P_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi & \theta & -P_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & \rho & r & -P_{31} & 0 \\ 0 & 0 & 0 & 0 & \delta & \varepsilon & \gamma & 0 & -P_8 \end{pmatrix}$$

where $P_{26} = \beta_2 S^\circ$, $P_{27} = \beta_1 S^\circ$, $P_{28} = \beta_3 S^\circ$, $P_{30} = \frac{\beta_B S^\circ}{K} = y_4$, $P_{31} = \alpha_1\phi + \alpha_2(1-\phi) + \mu$

$$\left|J_{E^*} - \lambda I\right| = 0$$

$$\left|J_{E^*} - \lambda I\right| = \begin{vmatrix} -(\lambda_1 + P_{10}) - \lambda & n & e & 0 & -P_{26} & -P_{27} & -P_{28} & P_7 & -P_{30} \\ m & -P_{11} - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & -P_{12} - \lambda & 0 & 0 & 0 & 0 & P_6 & 0 \\ \lambda_1 & 0 & 0 & -P_{13} - \lambda & P_{26} & P_{27} & P_{28} & 0 & P_{30} \\ 0 & 0 & 0 & P_1 & -P_2 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_3 & 0 & -P_4 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi & \theta & -P_5 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & \rho & r & -P_{31} - \lambda & 0 \\ 0 & 0 & 0 & 0 & \delta & \varepsilon & \gamma & 0 & -P_8 - \lambda \end{vmatrix} = 0 \quad (20)$$

From equation (20), we obtain

$$\left|J_{E^*} - \lambda I\right| = \Phi_1 + \Phi_2 = 0 \quad (21)$$

where

$$\Phi_1 = -n\begin{vmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f & -P_{12} - \lambda & 0 & 0 & 0 & 0 & P_6 & 0 \\ \lambda_1 & 0 & -P_{13} - \lambda & P_{26} & P_{27} & P_{28} & 0 & P_{30} \\ 0 & 0 & P_1 & -P_2 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & P_3 & 0 & -P_4 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi & \theta & -P_5 - \lambda & 0 & 0 \\ 0 & 0 & 0 & \eta & \rho & r & -P_{31} - \lambda & 0 \\ 0 & 0 & 0 & \delta & \varepsilon & \gamma & 0 & -P_8 - \lambda \end{vmatrix}$$

$$\Phi_2 = -(P_{11} + \lambda)\begin{vmatrix} -(\lambda_1 + P_{10}) - \lambda & e & 0 & -P_{26} & -P_{27} & -P_{28} & P_7 & -P_{30} \\ f & -P_{12} - \lambda & 0 & 0 & 0 & 0 & P_6 & 0 \\ \lambda_1 & 0 & -P_{13} - \lambda & P_{26} & P_{27} & P_{28} & 0 & P_{30} \\ 0 & 0 & P_1 & -P_2 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & P_3 & 0 & -P_4 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi & \theta & -P_5 - \lambda & 0 & 0 \\ 0 & 0 & 0 & \eta & \rho & r & -P_{31} - \lambda & 0 \\ 0 & 0 & 0 & \delta & \varepsilon & \gamma & 0 & -P_8 - \lambda \end{vmatrix}$$

$$\Phi_1 = mn\left(P_{12}P_{31} + \left(P_{12}+P_{31}\right)\lambda + \lambda^2\right)\left(\begin{array}{l} P_2P_4P_5P_8P_{13} - \begin{bmatrix} \psi P_1P_{28}P_4P_8 + \psi\gamma P_1P_{30}P_4 + P_1P_{26}P_4P_5P_8 + \delta P_1P_{30}P_4P_5 + \theta P_3P_{28}P_2P_8 \\ + \theta\gamma P_3P_{30}P_2 + P_3P_{27}P_2P_5P_8 + \varepsilon P_3P_{30}P_2P_5 \end{bmatrix} \\ \begin{bmatrix} \left[P_2\left(P_4P_5\left(P_8+P_{13}\right) + P_8P_{13}\left(P_4+P_5\right)\right) + P_4P_5P_8P_{13}\right] - \\ \begin{bmatrix} \psi P_1P_{28}\left(P_4+P_8\right) + \psi\gamma P_1P_{30} + P_1P_{26}\left(P_4P_5 + P_8\left(P_4+P_5\right)\right) + \delta P_1P_{30}\left(P_4+P_5\right) \\ + \theta P_3P_{28}\left(P_2+P_8\right) + \theta\gamma P_3P_{20} + P_3P_{27}\left(P_2P_5 + P_8\left(P_2+P_5\right)\right) + \varepsilon P_3P_{30}\left(P_2+P_5\right) \end{bmatrix} \end{bmatrix}\lambda \\ \begin{bmatrix} \left[P_4P_5\left(P_8+P_{13}\right) + P_8P_{13}\left(P_4+P_5\right) + P_2\left(P_4P_5 + \left(P_4+P_5\right)\left(P_8+P_{13}\right) + P_8P_{13}\right)\right] - \\ \begin{bmatrix} \psi P_1P_{28} + P_1P_{26}\left(P_8+P_4+P_5\right) + \delta P_1P_{30} + \theta P_3P_{28} + P_3P_{27}\left(P_8+P_2+P_5\right) + \varepsilon P_3P_{30} \\ + P_4P_5\left(P_8+P_{13}\right) + \psi P_1P_{28} + P_1P_{26}\left(P_8+P_4+P_5\right) + \delta P_1P_{30} + \theta P_3P_{28} \\ + P_3P_{27}\left(P_8+P_2+P_5\right) + \varepsilon P_3P_{30} \end{bmatrix} \end{bmatrix}\lambda^2 \\ + \left[\left[P_2\left(P_4+P_5+P_8+P_{13}\right) + P_4P_5 + \left(P_4+P_5\right)\left(P_8+P_{13}\right) + P_8P_{13}\right] - \left[P_1P_{26} + P_3P_{27}\right]\right]\lambda^3 \\ + \left[P_2+P_4+P_5+P_8+P_{13}\right]\lambda^4 \\ + \lambda^5 \end{array}\right)$$

Let $X_1 = P_2P_4P_5P_8P_{13} - \begin{bmatrix} \psi P_1P_{28}P_4P_8 + \psi\gamma P_1P_{30}P_4 + P_1P_{26}P_4P_5P_8 + \delta P_1P_{30}P_4P_5 + \theta P_3P_{28}P_2P_8 \\ + \theta\gamma P_3P_{30}P_2 + P_3P_{27}P_2P_5P_8 + \varepsilon P_3P_{30}P_2P_5 \end{bmatrix}$

$$X_2 = \begin{bmatrix} \left[P_2\left(P_4P_5\left(P_8+P_{13}\right) + P_8P_{13}\left(P_4+P_5\right)\right) + P_4P_5P_8P_{13}\right] - \\ \begin{bmatrix} \psi P_1P_{28}\left(P_4+P_8\right) + \psi\gamma P_1P_{30} + P_1P_{26}\left(P_4P_5 + P_8\left(P_4+P_5\right)\right) + \delta P_1P_{30}\left(P_4+P_5\right) \\ + \theta P_3P_{28}\left(P_2+P_8\right) + \theta\gamma P_3P_{20} + P_3P_{27}\left(P_2P_5 + P_8\left(P_2+P_5\right)\right) + \varepsilon P_3P_{30}\left(P_2+P_5\right) \end{bmatrix} \end{bmatrix}$$

$$X_3 = \begin{bmatrix} \left[P_4P_5\left(P_8+P_{13}\right) + P_8P_{13}\left(P_4+P_5\right) + P_2\left(P_4P_5 + \left(P_4+P_5\right)\left(P_8+P_{13}\right) + P_8P_{13}\right)\right] - \\ \begin{bmatrix} \psi P_1P_{28} + P_1P_{26}\left(P_8+P_4+P_5\right) + \delta P_1P_{30} + \theta P_3P_{28} + P_3P_{27}\left(P_8+P_2+P_5\right) + \varepsilon P_3P_{30} \\ + P_4P_5\left(P_8+P_{13}\right) + \psi P_1P_{28} + P_1P_{26}\left(P_8+P_4+P_5\right) + \delta P_1P_{30} + \theta P_3P_{28} \\ + P_3P_{27}\left(P_8+P_2+P_5\right) + \varepsilon P_3P_{30} \end{bmatrix} \end{bmatrix}$$

$$X_4 = \left[P_2\left(P_4+P_5+P_8+P_{13}\right) + P_4P_5 + \left(P_4+P_5\right)\left(P_8+P_{13}\right) + P_8P_{13}\right] - \left[P_1P_{26} + P_3P_{27}\right]$$

$$X_5 = P_2 + P_4 + P_5 + P_8 + P_{13}$$

$$X_6 = 1$$

$$= mn\left(P_{12}P_{31} + \left(P_{12}+P_{31}\right)\lambda + \lambda^2\right)\left(X_1 + X_2\lambda + X_3\lambda^2 + X_4\lambda^3 + X_5\lambda^4 + X_6\lambda^5\right)$$

$$\Phi_1 = \left(\begin{array}{l} mnP_{12}P_{31}X_1 + mn\left(P_{12}P_{31}X_2 + \left(P_{12}+P_{31}\right)X_1\right)\lambda + mn\left(P_{12}P_{31}X_3 + \left(P_{12}+P_{13}\right)X_2 + X_1\right)\lambda^2 \\ + mn\left(P_{12}P_{31}X_4 + \left(P_{12}+P_{31}\right)X_3 + X_2\right)\lambda^3 + mn\left(P_{12}P_{31}X_5 + \left(P_{12}+P_{31}\right)X_4 + X_3\right)\lambda^4 \\ + mn\left(P_{12}P_{31}X_4 + \left(P_{12}+P_{31}\right)X_5 + X_4\right)\lambda^5 + mn\left(P_{12}+P_{31}+X_5\right)\lambda^6 + mn\lambda^7 \end{array}\right) \tag{22}$$

Similarly,

$$\Phi_3 = \left(P_{11}+\lambda\right)eP_6\lambda_1 \begin{vmatrix} P_1 & -P_2-\lambda & 0 & 0 & 0 \\ P_3 & 0 & -P_4-\lambda & 0 & 0 \\ 0 & \psi & \theta & -P_5-\lambda & 0 \\ 0 & \eta & \rho & r & 0 \\ 0 & \delta & \varepsilon & \gamma & -P_8-\lambda \end{vmatrix}$$

$$\Phi_3 = \left(P_{11}+\lambda\right)\left(\begin{array}{l} \left(P_1P_4\psi r + P_1P_4P_5\eta + \theta rP_2P_3 + \rho P_2P_3P_5\right)eP_6P_8\lambda_1 \\ + \left[\begin{bmatrix} P_8\left(P_1\left(\psi r + \eta\left(P_4+P_5\right)\right) + \theta rP_3 + \rho P_3\left(P_2+P_5\right)\right) \\ + \left(P_1P_4\psi r + P_1P_4P_5\eta + \theta rP_2P_3 + \rho P_2P_3P_5\right) \end{bmatrix} eP_6\lambda_1\right]\lambda \\ + \left[\left[P_8\left(P_1\eta + \rho P_3\right) + P_1\left(\psi r + \eta\left(P_4+P_5\right)\right) + \theta rP_3 + \rho P_3\left(P_2+P_5\right)\right]eP_6\lambda_1\right]\lambda^2 \\ + \left[\left(P_1\eta + \rho P_3\right)eP_6\lambda_1\right]\lambda^3 \end{array}\right)$$

Let $Y_1 = \left(P_1P_4\psi r + P_1P_4P_5\eta + \theta rP_2P_3 + \rho P_2P_3P_5\right)eP_6P_8\lambda_1$

$$Y_2 = \left[P_8\left(P_1\left(\psi r + \eta\left(P_4+P_5\right)\right) + \theta rP_3 + \rho P_3\left(P_2+P_5\right)\right) + \left(P_1P_4\psi r + P_1P_4P_5\eta + \theta rP_2P_3 + \rho P_2P_3P_5\right)\right]eP_6\lambda_1$$

$$Y_3 = \left[P_8\left(P_1\eta + \rho P_3\right) + P_1\left(\psi r + \eta\left(P_4+P_5\right)\right) + \theta rP_3 + \rho P_3\left(P_2+P_5\right)\right]eP_6\lambda_1$$

$$Y_4 = \left(P_1\eta + \rho P_3\right)eP_6\lambda_1$$

$$\Phi_3 = (P_{11}+\lambda)(Y_1 + Y_2\lambda + Y_3\lambda^2 + Y_4\lambda^3)eP_6\lambda_1$$

$$\Phi_3 = P_{11}Y_1eP_6\lambda_1 + (P_{11}Y_2 + Y_1)eP_6\lambda_1\lambda + (P_{11}Y_3 + Y_2)eP_6\lambda_1\lambda^2 + (P_{11}Y_4 + Y_3)eP_6\lambda_1\lambda^3 + eP_6\lambda_1Y_4\lambda^4 \tag{23}$$

Moreso,

$$\Phi_4 = -(P_{11}+\lambda)ef(P_{31}+\lambda)\begin{vmatrix} -P_{13}-\lambda & P_{26} & P_{27} & P_{28} & P_{30} \\ P_1 & -P_2-\lambda & 0 & 0 & 0 \\ P_3 & 0 & -P_4-\lambda & 0 & 0 \\ 0 & \psi & \theta & -P_5-\lambda & 0 \\ 0 & \delta & \varepsilon & \gamma & -P_8-\lambda \end{vmatrix}$$

$$\Phi_4 = (P_{11}+\lambda)ef(P_{31}+\lambda)\begin{pmatrix} P_{13}P_2P_4P_5P_8 - \begin{bmatrix} P_1P_{30}P_4(\psi\gamma+\delta P_5) + P_{30}P_3P_2(\theta\gamma+\varepsilon P_5) + P_{13}P_2P_4P_5P_8 + P_1P_4P_8(P_{26}P_5+\psi P_{28}) \\ + P_8P_2P_3(P_{27}P_5+\theta P_{28}) \end{bmatrix} \\ + \begin{bmatrix} [P_{13}P_2P_4P_5 + P_8P_{13}P_2(P_4+P_5) + P_8P_4P_5(P_2+P_{13})] - \\ \begin{bmatrix} P_1P_{30}P_4\delta + P_1P_{30}(\psi\gamma+\delta P_5) + \varepsilon P_{30}P_3P_2 + P_{30}P_3(\theta\gamma+\varepsilon P_5) + P_1P_4(P_{26}P_5+\psi P_{28}) \\ + P_8P_1P_4P_{26} + P_1P_8(P_{26}P_5+\psi P_{28}) + P_2P_3(P_{27}P_5+\theta P_{28}) + P_8P_2P_3P_{27} + P_8P_3(P_{27}P_5+\theta P_{28}) \end{bmatrix} \end{bmatrix}\lambda \\ + \begin{bmatrix} [P_8(P_{13}P_2 + (P_2+P_{13})(P_4+P_5) + P_4P_5) + P_{13}P_2(P_4+P_5) + P_4P_5(P_2+P_{13})] - \\ [P_1P_{30}\delta + \varepsilon P_{30}P_3 + P_8P_1P_{26} + P_1(P_{26}P_5+\psi P_{28}) + P_8P_3P_{27} + P_3(P_{27}P_5+\theta P_{28})] \end{bmatrix}\lambda^2 \\ + [P_8(P_2+P_{13}+P_4+P_5) + P_{13}P_2 + (P_2+P_{13})(P_4+P_5) + P_4P_5 - P_1P_{26} - P_3P_{27}]\lambda^3 \\ + [P_8+P_2+P_{13}+P_4+P_5]\lambda^4 \\ + \lambda^5 \end{pmatrix}$$

Let $W_1 = P_{13}P_2P_4P_5P_8 - \begin{bmatrix} P_1P_{30}P_4(\psi\gamma+\delta P_5) + P_{30}P_3P_2(\theta\gamma+\varepsilon P_5) + P_{13}P_2P_4P_5P_8 + P_1P_4P_8(P_{26}P_5+\psi P_{28}) \\ + P_8P_2P_3(P_{27}P_5+\theta P_{28}) \end{bmatrix}$

$$W_2 = \begin{bmatrix} [P_{13}P_2P_4P_5 + P_8P_{13}P_2(P_4+P_5) + P_8P_4P_5(P_2+P_{13})] - \\ \begin{bmatrix} P_1P_{30}P_4\delta + P_1P_{30}(\psi\gamma+\delta P_5) + \varepsilon P_{30}P_3P_2 + P_{30}P_3(\theta\gamma+\varepsilon P_5) + P_1P_4(P_{26}P_5+\psi P_{28}) \\ + P_8P_1P_4P_{26} + P_1P_8(P_{26}P_5+\psi P_{28}) + P_2P_3(P_{27}P_5+\theta P_{28}) + P_8P_2P_3P_{27} + P_8P_3(P_{27}P_5+\theta P_{28}) \end{bmatrix} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} [P_8(P_{13}P_2 + (P_2+P_{13})(P_4+P_5) + P_4P_5) + P_{13}P_2(P_4+P_5) + P_4P_5(P_2+P_{13})] - \\ [P_1P_{30}\delta + \varepsilon P_{30}P_3 + P_8P_1P_{26} + P_1(P_{26}P_5+\psi P_{28}) + P_8P_3P_{27} + P_3(P_{27}P_5+\theta P_{28})] \end{bmatrix}$$

$$W_4 = [P_8(P_2+P_{13}+P_4+P_5) + P_{13}P_2 + (P_2+P_{13})(P_4+P_5) + P_4P_5 - P_1P_{26} - P_3P_{27}]$$

$$W_5 = P_8 + P_2 + P_{13} + P_4 + P_5$$

$$W_6 = 1$$

$$= (P_{11}+\lambda)ef(P_{31}+\lambda)(W_1 + W_2\lambda + W_3\lambda^2 + W_4\lambda^3 + W_5\lambda^4 + W_6\lambda^5)$$

$$\Phi_4 = (P_{11}+\lambda)\begin{pmatrix} efP_{31}W_1 + ef(P_{31}W_2+W_1)\lambda + ef(P_{31}W_3+W_2)\lambda^2 + ef(P_{31}W_4+W_3)\lambda^3 \\ + ef(P_{31}W_5+W_4)\lambda^4 + ef(P_{31}+W_5)\lambda^5 + ef\lambda^6 \end{pmatrix}$$

$$\Phi_4 = \begin{pmatrix} efP_{11}P_{31}W_1 + efP_{11}(P_{31}W_2+W_1)\lambda + efP_{11}(P_{31}W_3+W_2)\lambda^2 + efP_{11}(P_{31}W_4+W_3)\lambda^3 \\ + efP_{11}(P_{31}W_5+W_4)\lambda^4 + efP_{11}(P_{31}W_6+W_5)\lambda^5 + efP_{11}\lambda^6 + efP_{31}W_1\lambda + ef(P_{31}W_2+W_1)\lambda^2 \\ + ef(P_{31}W_3+W_2)\lambda^3 + ef(P_{31}W_4+W_3)\lambda^4 + ef(P_{31}W_5+W_4)\lambda^5 + ef(P_{31}W_6+W_5)\lambda^6 + ef\lambda^7 \end{pmatrix}$$

$$\begin{aligned} \Phi_4 = {} & ef\lambda^7 + ef((P_{31}W_6+W_5) + P_{11})\lambda^6 + ef((P_{31}W_5+W_4) + P_{11}(P_{31}W_6+W_5))\lambda^5 \\ & + ef((P_{31}W_4+W_3) + P_{11}(P_{31}W_5+W_4))\lambda^4 + ef((P_{31}W_3+W_2) + P_{11}(P_{31}W_4+W_3))\lambda^3 \\ & + ef((P_{31}W_2+W_1) + P_{11}(P_{31}W_3+W_2))\lambda^2 + ef(P_{31}W_1 + P_{11}(P_{31}W_2+W_1))\lambda + efP_{11}P_{31}W_1 \end{aligned} \tag{24}$$

More so,

$$\Phi_5 = (P_{11}+\lambda)(P_{12}+\lambda)P_7\lambda_1 \begin{vmatrix} P_1 & -P_2-\lambda & 0 & 0 & 0 \\ P_3 & 0 & -P_4-\lambda & 0 & 0 \\ 0 & \psi & \theta & -P_5-\lambda & 0 \\ 0 & \eta & \rho & r & 0 \\ 0 & \delta & \varepsilon & \gamma & -P_8-\lambda \end{vmatrix}$$

$$\Phi_5 = (P_{11}+\lambda)(P_{12}+\lambda)\begin{pmatrix} P_7P_8\lambda_1\left[P_1P_4(\psi r+\eta P_5)+P_2P_3(\theta r+\rho P_5)\right] \\ +P_7\lambda_1\begin{bmatrix} P_1P_4(\psi r+\eta P_5)+P_2P_3(\theta r+\rho P_5) \\ +P_8\left[\eta P_1P_4+\rho P_2P_3+P_1(\psi r+\eta P_5)+P_3(\theta r+\rho P_5)\right]\end{bmatrix}\lambda \\ +P_7\lambda_1\left[\eta P_8(\rho+P_1)+\left[\eta P_1P_4+\rho P_2P_3+P_1(\psi r+\eta P_5)+P_3(\theta r+\rho P_5)\right]\right]\lambda^2 \\ +P_7\lambda_1(\eta P_1+\rho P_3)\lambda^3 \end{pmatrix}$$

Let $U_1 = P_7P_8\lambda_1\left[P_1P_4(\psi r+\eta P_5)+P_2P_3(\theta r+\rho P_5)\right]$,

$$U_2 = P_7\lambda_1\begin{bmatrix} P_8\left[\eta P_5P_5+\rho P_2P_3+P_1(\psi r+\eta P_5)+P_3(\theta r+\rho P_5)\right]+P_1P_4(\psi r+\eta P_5) \\ +P_2P_3(\theta r+\rho P_5)\end{bmatrix},$$

$$U_3 = P_7\lambda_1\left[\eta P_8(\rho+P_1)+\left[\eta P_1P_4+\rho P_2P_3+P_1(\psi r+\eta P_5)+P_3(\theta r+\rho P_5)\right]\right],$$

$$U_4 = P_7\lambda_1(\eta P_1+\rho P_3)$$

$$\Phi_5 = (P_{11}+\lambda)(P_{12}+\lambda)(U_1+U_2\lambda+U_3\lambda^2+U_4\lambda^3)$$

$$\Phi_5 = \left(P_{11}P_{12}+(P_{11}+P_{12})\lambda+\lambda^2\right)(U_1+U_2\lambda+U_3\lambda^2+U_4\lambda^3)$$

$$\Phi_5 = P_{11}P_{12}U_1+\left[P_{11}P_{12}U_2+(P_{11}+P_{12})U_1\right]\lambda+\left[P_{11}P_{12}U_3+(P_{11}+P_{12})U_2+U_1\right]\lambda^2$$

$$+\left[P_{11}P_{12}U_4+(P_{11}+P_{12})U_3+U_2\right]\lambda^3+\left[(P_{11}+P_{12})U_4+U_3\right]\lambda^4+U_4\lambda^5 \qquad (25)$$

Furthermore,

$$\Phi_6 = (P_{11}+\lambda)(P_{12}+\lambda)(P_{31}+\lambda)\begin{pmatrix} (\lambda_1+P_{10}+\lambda)\begin{vmatrix} -P_{13}-\lambda & P_{26} & P_{27} & P_{28} & P_{30} \\ P_1 & -P_2-\lambda & 0 & 0 & 0 \\ P_3 & 0 & -P_4-\lambda & 0 & 0 \\ 0 & \psi & \theta & -P_5-\lambda & 0 \\ 0 & \delta & \varepsilon & \gamma & -P_8-\lambda \end{vmatrix} \\ +\lambda_1\begin{vmatrix} -P_{13}-\lambda & P_{26} & P_{27} & P_{28} & P_{30} \\ P_1 & -P_2-\lambda & 0 & 0 & 0 \\ P_3 & 0 & -P_4-\lambda & 0 & 0 \\ 0 & \psi & \theta & -P_5-\lambda & 0 \\ 0 & \delta & \varepsilon & \gamma & -P_8-\lambda \end{vmatrix} \end{pmatrix}$$

Let $J_{11} = \lambda_1 + P_{10}$ and $J_{12} = (P_{11}+\lambda)(P_{12}+\lambda)(P_{31}+\lambda)$

$$\Phi_6 = J_{12}\left(\begin{array}{l} J_{11}+\lambda\left(\begin{array}{l} \psi P_1P_4\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_4P_5\left(P_{26}P_8+\delta P_{30}\right)+P_2P_4P_5P_8P_{13}-P_2P_3\theta\left(P_{28}P_8+\gamma P_{30}\right) \\ -P_2P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right) \\ +\left[\begin{array}{l} \psi P_1P_4P_{28}+P_1P_4\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+P_1P_5\left(P_{26}P_8+\delta P_{30}\right)+P_4P_5P_8P_{13} \\ \psi P_1\left(P_8P_{28}+\gamma P_{30}\right)+P_2\left[P_4P_5\left(P_8+P_{13}\right)+P_8P_{13}\left(P_4+P_5\right)\right]-P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right) \\ -P_2P_3\left(\theta P_{28}+P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right)-\theta P_3\left(P_{28}P_8+\gamma P_{30}\right) \end{array}\right]\lambda \\ +\left[\begin{array}{l} P_1P_4P_{26}+\psi P_1P_{28}+P_1\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+P_4P_5\left(P_8+P_{13}\right) \\ +P_2\left(P_4P_5+\left(P_4+P_5\right)\left(P_8+P_{13}\right)+P_8P_{13}\right)+P_4P_5\left(P_8+P_{13}\right)+P_8P_{13}\left(P_4+P_5\right) \\ -P_3\left(\theta P_{28}+P_5P_{27}+P_{27}P_8+P_2P_{27}+\varepsilon P_{30}\right) \end{array}\right]\lambda^2 \\ +\left[P_1P_{26}+P_2\left(P_4+P_5+P_8+P_{13}\right)+P_4P_5+\left(P_4+P_5\right)\left(P_8+P_{13}\right)+P_8P_{13}-P_3P_{27}\right]\lambda^3 \\ +\left[P_2+P_4+P_5+P_8+P_{13}\right]\lambda^4+\lambda^5 \end{array}\right) \\ +\lambda_1\left(\begin{array}{l} \psi P_1P_4\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_4P_5\left(P_{26}P_8+\delta P_{30}\right)+P_2P_3\theta\left(P_{28}P_8+\gamma P_{30}\right)+P_2P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right) \\ +\left[\begin{array}{l} P_1P_4\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+\psi P_1\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_5\left(P_{26}P_8+\delta P_{30}\right) \\ +P_2P_3\left(P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right)+\psi P_1P_4P_{28}+\theta P_3\left(P_{28}P_8+\gamma P_{30}\right)+P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right) \\ +\theta P_2P_3P_{28} \end{array}\right]\lambda \\ +\left[\begin{array}{l} P_1P_4P_{26}+\psi P_1P_{28}+P_2P_3P_{27}+\theta P_3P_{28}+P_1\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right) \\ +P_3\left(P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right) \end{array}\right]\lambda^2+\left[P_1P_{26}+P_3P_{27}\right]\lambda^3 \end{array}\right) \end{array}\right)$$

Let $J_1=\psi P_1P_4\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_4P_5\left(P_{26}P_8+\delta P_{30}\right)+P_2P_4P_5P_8P_{13}-P_2P_3\theta\left(P_{28}P_8+\gamma P_{30}\right)-P_2P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right)$

$$\begin{aligned} J_2 &= \psi P_1P_4P_{28}+P_1P_4\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+P_1P_5\left(P_{26}P_8+\delta P_{30}\right)+P_4P_5P_8P_{13}+\psi P_1\left(P_8P_{28}+\gamma P_{30}\right) \\ &\quad +P_2\left[P_4P_5\left(P_8+P_{13}\right)+P_8P_{13}\left(P_4+P_5\right)\right]-P_3P_5\left(P_{27}P_8+\varepsilon P_{30}\right)-\theta P_3\left(P_{28}P_8+\gamma P_{30}\right) \\ &\quad -P_2P_3\left(\theta P_{28}+P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right) \end{aligned}$$

$$\begin{aligned} J_3 &= P_1P_4P_{26}+\psi P_1P_{28}+P_1\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+P_4P_5\left(P_8+P_{13}\right)+P_4P_5\left(P_8+P_{13}\right)+P_8P_{13}\left(P_4+P_5\right) \\ &\quad +P_2\left(P_4P_5+\left(P_4+P_5\right)\left(P_8+P_{13}\right)+P_8P_{13}\right)-P_3\left(\theta P_{28}+P_5P_{27}+P_{27}P_8+P_2P_{27}+\varepsilon P_{30}\right) \end{aligned}$$

$$J_4=\left[P_1P_{26}+P_2\left(P_4+P_5+P_8+P_{13}\right)+P_4P_5+\left(P_4+P_5\right)\left(P_8+P_{13}\right)+P_8P_{13}-P_3P_{27}\right]$$

$$J_5=P_2+P_4+P_5+P_8+P_{13}$$

$$J_6=1$$

$$J_7=\psi P_1P_4\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_4P_5\left(P_{26}P_8+\delta P_{30}\right)+P_2P_3\theta\left(P_{28}P_8+\gamma P_{30}\right)+P_2P_3P_5\left(P_{27}P_8+\varepsilon P_{20}\right)$$

$$\begin{aligned} J_8 &= P_1P_4\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+\psi P_1\left(P_{28}P_8+\gamma P_{30}\right)+P_1P_5\left(P_{26}P_8+\delta P_{30}\right)+P_2P_3\left(P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right) \\ &\quad +\psi P_1P_4P_{28}+\theta P_3\left(P_{28}P_8+\gamma P_{30}\right)+P_3P_5\left(P_{27}P_8+\varepsilon P_{20}\right)+\theta P_2P_3P_{28} \end{aligned}$$

$$J_9=P_1P_4P_{26}+\psi P_1P_{28}+P_2P_3P_{27}+\theta P_3P_{28}+P_1\left(P_5P_{26}+P_{26}P_8+\delta P_{30}\right)+P_3\left(P_5P_{27}+P_{27}P_8+\varepsilon P_{30}\right)$$

$$J_{10}=P_1P_{26}+P_3P_{27}$$

$$\Phi_6=\left(P_{11}+\lambda\right)\left(P_{12}+\lambda\right)\left(P_{31}+\lambda\right)\left(\begin{array}{l} \left(J_{11}+\lambda\right)\left(J_1+J_2\lambda+J_3\lambda^2+J_4\lambda^3+J_5\lambda^4+J_6\lambda^5\right) \\ +\lambda_1\left(J_7+J_8\lambda+J_9\lambda^2+J_{10}\lambda^3\right) \end{array}\right)$$

$$=\left(P_{11}P_{12}+\left(P_{11}+P_{12}\right)\lambda+\lambda^2\right)\left(P_{31}+\lambda\right)\left(\begin{array}{l} J_{11}J_1+\left(J_{11}J_2+J_1\right)\lambda+\left(J_{11}J_3+J_2\right)\lambda^2+\left(J_{11}J_4+J_3\right)\lambda^3 \\ +\left(J_{11}J_5+J_4\right)\lambda^4+\left(J_{11}J_6+J_5\right)\lambda^5+J_6\lambda^6 \\ +J_7\lambda_1+J_8\lambda_1\lambda+J_9\lambda_1\lambda^2+J_{10}\lambda_1\lambda^3 \end{array}\right)$$

$$= \left(P_{11}P_{12} + \left(P_{11} + P_{12}\right)\lambda + \lambda^2\right)\left(P_{31} + \lambda\right)\begin{pmatrix} J_{11}J_1 + \left(J_{11}J_2 + J_1\right)\lambda + \left(J_{11}J_3 + J_2\right)\lambda^2 + \left(J_{11}J_4 + J_3\right)\lambda^3 \\ + \left(J_{11}J_5 + J_4\right)\lambda^4 + \left(J_{11}J_6 + J_5\right)\lambda^5 + J_6\lambda^6 \\ + J_7\lambda_1 + J_8\lambda_1\lambda + J_9\lambda_1\lambda^2 + J_{10}\lambda_1\lambda^3 \end{pmatrix}$$

$$\Phi_6 = \left(P_{11}P_{12} + \left(P_{11} + P_{12}\right)\lambda + \lambda^2\right)\left(P_{31} + \lambda\right)\begin{pmatrix} J_{11}J_1 + J_7\lambda_1 + \left(J_{11}J_2 + J_1 + J_8\lambda_1\right)\lambda \\ + \left(J_{11}J_3 + J_2 + J_9\lambda_1\right)\lambda^2 + \left(J_{11}J_4 + J_3 + J_{10}\lambda_1\right)\lambda^3 \\ + \left(J_{11}J_5 + J_4\right)\lambda^4 + \left(J_{11}J_6 + J_5\right)\lambda^5 + J_6\lambda^6 \end{pmatrix}$$

$$\Phi_6 = P_{11}P_{12}P_{31}\left(J_{11}J_1 + J_7\lambda_1\right) + \left[P_{11}P_{12}P_{31}\left(J_{11}J_1 + J_7\lambda_1\right) + \left(J_{11}J_1 + J_7\lambda_1\right)\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right)\right]\lambda$$

$$+ \begin{bmatrix} P_{11}P_{12}P_{31}\left(J_{11}J_3 + J_2 + J_9\lambda_1\right) + \left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\left(P_{11}P_{11} + P_{11} + P_{11}P_{11}\right)\right) \\ + \left(P_{11} + P_{12} + P_{31}\right)\left(J_{11}J_1 + J_7\lambda_1\right) \end{bmatrix}\lambda^2$$

$$+ \left[P_{11}P_{12}P_{13}\left(J_{11}J_4 + J_3 + J_{10}\lambda_1\right) + \left(J_{11}J_2 + J_1 + J_8\lambda_1\right)\left(P_{11} + P_{12} + P_{31}\right) + \left(J_{11}J_1 + J_7\lambda_1\right)\right]\lambda^3$$

$$+ \begin{bmatrix} P_{11}P_{12}P_{31}\left(J_{11}J_5 + J_4\right) + \left(J_{11}J_4 + J_3 + J_{10}\lambda_1\right)\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right) \\ + \left(J_{11}J_3 + J_2 + J_9\lambda_1\right)\left(P_{11} + P_{12} + P_{31}\right) + \left(J_{11}J_2 + J_1 + J_8\lambda_1\right) \end{bmatrix}\lambda^4$$

$$+ \begin{bmatrix} \left(J_{11}J_5 + J_4\right)\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right) + \left(J_{11}J_4 + J_3 + J_{10}\lambda_1\right)\left(P_{11} + P_{12} + P_{31}\right) \\ + \left(J_{11}J_3 + J_2 + J_9\lambda_1\right) \end{bmatrix}\lambda^5$$

$$+ \begin{bmatrix} P_{11}P_{12}P_{31}J_6 + \left(J_{11}J_6 + J_5\right)\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right) + \left(J_{11}J_5 + J_4\right)\left(P_{11} + P_{12} + P_{31}\right) \\ + \left(J_{11}J_4 + J_3 + J_{10}\lambda_1\right) \end{bmatrix}\lambda^6$$

$$+ \left[J_6\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right) + \left(J_{11}J_6 + J_5\right)\left(P_{11} + P_{12} + P_{31}\right) + \left(J_{11}J_5 + J_4\right)\right]\lambda^7$$

$$+ \left[J_6\left(P_{11} + P_{12} + P_{31}\right) + \left(J_{11}J_6 + J_5\right)\right]\lambda^8 + J_6\lambda^9 \qquad (26)$$

Adding equation (23), (24), (25) and (26), we have

$$\Phi_2 = \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6$$

$$\Phi_2 = P_{11}Y_1eP_6\lambda_1 + efP_{11}P_{31}W_1 + P_{11}P_{12}U_1 + P_{11}P_{12}P_{31}\left(J_{11}J_1 + J_7\lambda_1\right)$$

$$+ \begin{bmatrix} \left(P_{11}Y_2 + Y_1\right)eP_6\lambda_1 + ef\left(P_{31}W_1 + P_{11}\left(P_{31}W_2 + W_1\right)\right) + \left[P_{11}P_{12}U_2 + \left(P_{11} + P_{12}\right)U_1\right] \\ + \left[P_{11}P_{12}P_{31}\left(J_{11}J_1 + J_7\lambda_1\right) + \left(J_{11}J_1 + J_7\lambda_1\right)\left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\right)\right] \end{bmatrix}\lambda$$

$$+ \begin{bmatrix} \left(P_{11}Y_3 + Y_2\right)eP_6\lambda_1 + ef\left(\left(P_{31}W_2 + W_1\right) + P_{11}\left(P_{31}W_3 + W_2\right)\right) + \left[P_{11}P_{12}U_3 + \left(P_{11} + P_{12}\right)U_2 + U_1\right] \\ + P_{11}P_{12}P_{31}\left(J_{11}J_3 + J_2 + J_9\lambda_1\right) + \left(P_{11}P_{12} + P_{31}\left(P_{11} + P_{12}\right)\left(P_{11}P_{11} + P_{11} + P_{11}P_{11}\right)\right) \\ + \left(P_{11} + P_{12} + P_{31}\right)\left(J_{11}J_1 + J_7\lambda_1\right) \end{bmatrix}\lambda^2$$

$$
\begin{aligned}
&+\begin{bmatrix} (P_{11}Y_4+Y_3)eP_6\lambda_1+ef\left((P_{31}W_3+W_2)+P_{11}(P_{31}W_4+W_3)\right)+\left(P_{11}P_{12}U_4+(P_{11}+P_{12})U_3+U_2\right) \\ +P_{11}P_{12}P_{13}(J_{11}J_4+J_3+J_{10}\lambda_1)+(J_{11}J_2+J_1+J_8\lambda_1)(P_{11}+P_{12}+P_{31})+(J_{11}J_1+J_7\lambda_1) \end{bmatrix}\lambda^3 \\
&+\begin{bmatrix} ef\left((P_{31}W_4+W_3)+P_{11}(P_{31}W_5+W_4)\right)+\left[(P_{11}+P_{12})U_4+U_3\right]+eP_6\lambda_1Y_4 \\ +P_{11}P_{12}P_{31}(J_{11}J_5+J_4)+(J_{11}J_4+J_3+J_{10}\lambda_1)(P_{11}P_{12}+P_{31}(P_{11}+P_{12}))+(J_{11}J_2+J_1+J_8\lambda_1) \\ +(J_{11}J_3+J_2+J_9\lambda_1)(P_{11}+P_{12}+P_{31}) \end{bmatrix}\lambda^4 \\
&+\begin{bmatrix} ef\left((P_{31}W_5+W_4)+P_{11}(P_{31}W_6+W_5)\right)+U_4+(J_{11}J_5+J_4)(P_{11}P_{12}+P_{31}(P_{11}+P_{12})) \\ +(J_{11}J_4+J_3+J_{10}\lambda_1)(P_{11}+P_{12}+P_{31})+(J_{11}J_3+J_2+J_9\lambda_1) \end{bmatrix}\lambda^5 \\
&+\begin{bmatrix} ef\left((P_{31}W_6+W_5)+P_{11}\right)+P_{11}P_{12}P_{31}J_6+(J_{11}J_6+J_5)(P_{11}P_{12}+P_{31}(P_{11}+P_{12})) \\ +(J_{11}J_5+J_4)(P_{11}+P_{12}+P_{31})+(J_{11}J_4+J_3+J_{10}\lambda_1) \end{bmatrix}\lambda^6 \\
&+\left[ef+J_6(P_{11}P_{12}+P_{31}(P_{11}+P_{12}))+(J_{11}J_6+J_5)(P_{11}+P_{12}+P_{31})+(J_{11}J_5+J_4)\right]\lambda^7 \\
&+\left[J_6(P_{11}+P_{12}+P_{31})+(J_{11}J_6+J_5)\right]\lambda^8+J_6\lambda^9
\end{aligned}
\tag{27}
$$

Substitute equation (22) and (27) in (21), we have

$$\left|J_{E^*}-\lambda I\right|=\Phi_1+\Phi_2=0$$

$$
\left|J_{E^*}-\lambda I\right|=\left(\begin{aligned}
&P_{11}Y_1eP_6\lambda_1+efP_{11}P_{31}W_1+P_{11}P_{12}U_1+P_{11}P_{12}P_{31}(J_{11}J_1+J_7\lambda_1)+mnP_{12}P_{31}X_1 \\
&+\begin{bmatrix} (P_{11}Y_2+Y_1)eP_6\lambda_1+ef\left(P_{31}W_1+P_{11}(P_{31}W_2+W_1)\right)+\left[P_{11}P_{12}U_2+(P_{11}+P_{12})U_1\right] \\ +\left[P_{11}P_{12}P_{31}(J_{11}J_1+J_7\lambda_1)+(J_{11}J_1+J_7\lambda_1)(P_{11}P_{12}+P_{31}(P_{11}+P_{12}))\right] \\ +mn(P_{12}P_{31}X_2+(P_{12}+P_{31})X_1) \end{bmatrix}\lambda \\
&+\begin{bmatrix} (P_{11}Y_3+Y_2)eP_6\lambda_1+ef\left((P_{31}W_2+W_1)+P_{11}(P_{31}W_3+W_2)\right)+\left[P_{11}P_{12}U_3+(P_{11}+P_{12})U_2+U_1\right] \\ P_{11}P_{12}P_{31}(J_{11}J_3+J_2+J_9\lambda_1)+(P_{11}P_{12}+P_{31}(P_{11}+P_{12})(P_{11}P_{11}+P_{11}+P_{11}P_{11})) \\ +(P_{11}+P_{12}+P_{31})(J_{11}J_1+J_7\lambda_1)+mn(P_{12}P_{31}X_3+(P_{12}+P_{13})X_2+X_1) \end{bmatrix}\lambda^2 \\
&+\begin{bmatrix} (P_{11}Y_4+Y_3)eP_6\lambda_1+ef\left((P_{31}W_3+W_2)+P_{11}(P_{31}W_4+W_3)\right)+\left(P_{11}P_{12}U_4+(P_{11}+P_{12})U_3+U_2\right) \\ +P_{11}P_{12}P_{13}(J_{11}J_4+J_3+J_{10}\lambda_1)+(J_{11}J_2+J_1+J_8\lambda_1)(P_{11}+P_{12}+P_{31})+(J_{11}J_1+J_7\lambda_1) \\ +mn(P_{12}P_{31}X_4+(P_{12}+P_{31})X_3+X_2) \end{bmatrix}\lambda^3 \\
&+\begin{bmatrix} ef\left((P_{31}W_4+W_3)+P_{11}(P_{31}W_5+W_4)\right)+\left[(P_{11}+P_{12})U_4+U_3\right]+eP_6\lambda_1Y_4 \\ +P_{11}P_{12}P_{31}(J_{11}J_5+J_4)+(J_{11}J_4+J_3+J_{10}\lambda_1)(P_{11}P_{12}+P_{31}(P_{11}+P_{12}))+(J_{11}J_2+J_1+J_8\lambda_1) \\ +(J_{11}J_3+J_2+J_9\lambda_1)(P_{11}+P_{12}+P_{31})+mn(P_{12}P_{31}X_5+(P_{12}+P_{31})X_4+X_3) \end{bmatrix}\lambda^4 \\
&+\begin{bmatrix} ef\left((P_{31}W_5+W_4)+P_{11}(P_{31}W_6+W_5)\right)+U_4+(J_{11}J_5+J_4)(P_{11}P_{12}+P_{31}(P_{11}+P_{12})) \\ +(J_{11}J_4+J_3+J_{10}\lambda_1)(P_{11}+P_{12}+P_{31})+(J_{11}J_3+J_2+J_9\lambda_1) \\ +mn(P_{12}P_{31}X_4+(P_{12}+P_{31})X_5+X_4) \end{bmatrix}\lambda^5 \\
&+\begin{bmatrix} ef\left((P_{31}W_6+W_5)+P_{11}\right)+P_{11}P_{12}P_{31}J_6+(J_{11}J_6+J_5)(P_{11}P_{12}+P_{31}(P_{11}+P_{12})) \\ +(J_{11}J_5+J_4)(P_{11}+P_{12}+P_{31})+(J_{11}J_4+J_3+J_{10}\lambda_1)+mn(P_{12}+P_{31}+X_5) \end{bmatrix}\lambda^6 \\
&+\left[ef+J_6(P_{11}P_{12}+P_{31}(P_{11}+P_{12}))+(J_{11}J_6+J_5)(P_{11}+P_{12}+P_{31})+(J_{11}J_5+J_4)+mn\right]\lambda^7 \\
&+\left[J_6(P_{11}+P_{12}+P_{31})+(J_{11}J_6+J_5)\right]\lambda^8 \\
&+J_6\lambda^9
\end{aligned}\right)=0
\tag{28}
$$

Therefore, we used Routh-Hurwitz necessary and sufficient conditions to investigate the stability of the endemic equilibrium of (28) as stated in chapter three. It is given below:

$$a_0\lambda^9+a_1\lambda^8+a_2\lambda^7+a_3\lambda^6+a_4\lambda^5+a_5\lambda^4+a_6\lambda^3+a_7\lambda^2+a_8\lambda^1+a_9=0$$

$$a_0 = 1 > 0$$

$$a_1 = J_6(P_{11} + P_{12} + P_{31}) + (J_{11}J_6 + J_5) = P_{11} + P_{12} + P_{31} + J_{11} + J_5 > 0$$

$$a_2 = ef + J_6(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) + (J_{11}J_6 + J_5)(P_{11} + P_{12} + P_{31}) + (J_{11}J_5 + J_4) + mn > 0$$

$$\begin{aligned} a_3 = {} & ef((P_{31}W_6 + W_5) - P_{11}) + P_{11}P_{12}P_{31}J_6 + (J_{11}J_6 + J_5)(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) + (J_{11}J_5 + J_4)(P_{11} + P_{12} + P_{31}) \\ & + (J_{11}J_4 + J_3 + J_{10}\lambda_1) + mn(P_{12} + P_{31} + X_5) > 0 \end{aligned}$$

$$\begin{aligned} a_4 = {} & ef((P_{31}W_5 + W_4) + P_{11}(P_{31}W_6 + W_5)) + U_4 + (J_{11}J_5 + J_4)(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) + (J_{11}J_3 + J_2 + J_9\lambda_1) \\ & + (J_{11}J_4 + J_3 + J_{10}\lambda_1)(P_{11} + P_{12} + P_{31}) + mn(P_{12}P_{31}X_4 + (P_{12} + P_{31})X_5 + X_4) > 0 \end{aligned}$$

$$\begin{aligned} a_5 = {} & ef((P_{31}W_4 + W_3) + P_{11}(P_{31}W_5 + W_4)) + [(P_{11} + P_{12})U_4 + U_3] + eP_6\lambda_1Y_4 + (J_{11}J_2 + J_1 + J_8\lambda_1) \\ & + P_{11}P_{12}P_{31}(J_{11}J_5 + J_4) + (J_{11}J_4 + J_3 + J_{10}\lambda_1)(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) + (J_{11}J_3 + J_2 + J_9\lambda_1)(P_{11} + P_{12} + P_{31}) \\ & + mn(P_{12}P_{31}X_5 + (P_{12} + P_{31})X_4 + X_3) > 0 \end{aligned}$$

$$\begin{aligned} a_6 = {} & (P_{11}Y_4 + Y_3)eP_6\lambda_1 + ef((P_{31}W_3 + W_2) + P_{11}(P_{31}W_4 + W_3)) + (P_{11}P_{12}U_4 + (P_{11} + P_{12})U_3 + U_2) \\ & + P_{11}P_{12}P_{13}(J_{11}J_4 + J_3 + J_{10}\lambda_1) + (J_{11}J_2 + J_1 + J_8\lambda_1)(P_{11} + P_{12} + P_{31}) + (J_{11}J_1 + J_7\lambda_1) \\ & + mn(P_{12}P_{31}X_4 + (P_{12} + P_{31})X_3 + X_2) > 0 \end{aligned}$$

$$\begin{aligned} a_7 = {} & (P_{11}Y_3 + Y_2)eP_6\lambda_1 + ef((P_{31}W_2 + W_1) + P_{11}(P_{31}W_3 + W_2)) + [P_{11}P_{12}U_3 + (P_{11} + P_{12})U_2 + U_1] \\ & + P_{11}P_{12}P_{31}(J_{11}J_3 + J_2 + J_9\lambda_1) + (P_{11}P_{12} + P_{31}(P_{11} + P_{12})(P_{11}P_{11} + P_{11} + P_{11}P_{11})) + (P_{11} + P_{12} + P_{31})(J_{11}J_1 + J_7\lambda_1) \\ & + mn(P_{12}P_{31}X_3 + (P_{12} + P_{13})X_2 + X_1) > 0 \end{aligned}$$

$$\begin{aligned} a_8 = {} & (P_{11}Y_2 - Y_1)eP_6\lambda_1 + ef(P_{31}W_1 + P_{11}(P_{31}W_2 + W_1)) + (P_{11}P_{12}U_2 + (P_{11} + P_{12})U_1) + mnP_{12}P_{31}X_2 \\ & + [P_{11}P_{12}P_{31}(J_{11}J_1 + J_7\lambda_1) + (J_{11}J_1 + J_7\lambda_1)(P_{11}P_{12} + P_{31}(P_{11} + P_{12}))] + mn(P_{12} + P_{31})X_1 > 0 \end{aligned}$$

$$a_9 = P_{11}Y_1eP_6\lambda_1 - efP_{11}P_{31}W_1 + P_{11}P_{12}U_1 + P_{11}P_{12}P_{31}(J_{11}J_1 + J_7\lambda_1) + mnP_{12}P_{31}X_1 > 0$$

$$\Delta_1 = a_1 = P_{11} + P_{12} + P_{31} + J_{11} + J_5 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} \Rightarrow \Delta_2 = a_1a_2 - a_0a_3 > 0$$

$$\Delta_2 = \begin{pmatrix} (P_{11} + P_{12} + P_{31} + J_{11} + J_5)\times \\ \begin{pmatrix} ef + J_6(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) \\ + (J_{11}J_6 + J_5)(P_{11} + P_{12} + P_{31}) \\ + (J_{11}J_5 + J_4) + mn \end{pmatrix} \end{pmatrix} - \begin{pmatrix} ef((P_{31}W_6 + W_5) + P_{11}) + P_{11}P_{12}P_{31}J_6 \\ + (J_{11}J_6 + J_5)(P_{11}P_{12} + P_{31}(P_{11} + P_{12})) \\ + (J_{11}J_5 + J_4)(P_{11} + P_{12} + P_{31}) \\ + (J_{11}J_4 + J_3 + J_{10}\lambda_1) + mn(P_{12} + P_{31} + X_5) \end{pmatrix} > 0$$

For order three,

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0 \Rightarrow \Delta_3 = a_1(a_2a_3 - a_1a_4) - a_0(a_3^2 - a_1a_5) > 0$$

iff $\Delta_3 = a_1(a_2a_3 - a_1a_4) > a_0(a_3^2 - a_1a_5)$

For order four,

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

$$\Delta_4 = \begin{pmatrix} a_1\{a_2(a_3a_4 - a_2a_5) - a_1(a_4^2 - a_2a_6) + a_0(a_4a_5 - a_3a_6)\} \\ \\ - a_0\{a_3(a_3a_4 - a_2a_5) - a_1(a_4a_5 - a_2a_7) + a_0(a_5^2 - a_3a_7)\} \end{pmatrix} > 0$$

iff $a_1\left\{a_2\left(a_3a_4-a_2a_5\right)-a_1\left(a_4^2-a_2a_6\right)+a_0\left(a_4a_5-a_3a_6\right)\right\}>a_0\left\{a_3\left(a_3a_4-a_2a_5\right)-a_1\left(a_4a_5-a_2a_7\right)+a_0\left(a_5^2-a_3a_7\right)\right\}$

Use the same determinant approach to obtain the remaining results.

The global stability analysis of the endemic equilibrium of the model

We investigated the global asymptotic stability of the endemic equilibrium of shigella using Lyapunov quadratic approach.

Theorem

The endemic equilibrium point Θ^* of system (1) – (9) is globally asymptotically stable whenever $R_0 \geq 1$.

Proof:

Suppose $R_0 \geq 1$then the existence of the endemic equilibrium point is assured. We applied the common quadratic Lyapunov function:

$$F\left(x_1,x_2,x_3,\ldots,x_9\right)=\sum_{i=1}^{9}\frac{c_i}{2}\left(x_i-x_i^*\right)^2$$

as illustrated in (De Le & De Le´on, (2009). We consider the Lyapunov function F with respect to the existing variables; S, V, G, E, A, I, H, R and B.

$$F(S,V,G,E,A,I,H,R,B)=\frac{1}{2}\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)^2$$

Differentiating $F(S,V,G,E,A,I,H,R,B)$ with respect to t resulted to

$$\frac{dF}{dt}=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\times\frac{d}{dt}\left(S+V+G+E+A+I+H+R+B\right)$$

$$\frac{dF}{dt}=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\times\left(\frac{dS}{dt}+\frac{dV}{dt}+\frac{dG}{dt}+\frac{dE}{dt}+\frac{dA}{dt}+\frac{dI}{dt}+\frac{dH}{dt}+\frac{dR}{dt}+\frac{dB}{dt}\right) \quad (29)$$

Substitute equations (1) – (9) in (29), we have

$$\begin{aligned}\frac{dF}{dt}&=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\\&\quad\times\left(\pi-\mu\left(S+V+G+E+A+I+H+R+B\right)-d_1I-d_2H+\varepsilon I+\gamma H+\delta A-\left(\sigma_2+\sigma_3-\sigma_1\right)B\right)\end{aligned} \quad (30)$$

Now setting

$$\pi=\mu\left(S^*+V^*+G^*+E^*+A^*+I^*+H^*+R^*+B^*\right)+d_1I^*+d_2H^*-\varepsilon I^*-\gamma H^*-\delta A^*+\left(\sigma_2+\sigma_3-\sigma_1\right)B^* \quad (31)$$

Substitute equation (31) in (30), we have

$$\begin{aligned}\frac{dF}{dt}&=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\times\\&\left(\begin{array}{l}\mu\left(S^*+V^*+G^*+E^*+A^*+I^*+H^*+R^*+B^*\right)+d_1I^*+d_2H^*-\varepsilon I^*-\gamma H^*-\delta A^*+\left(\sigma_2+\sigma_3-\sigma_1\right)B^*\\ \\-\mu\left(S+V+G+E+A+I+H+R+B\right)-d_1I-d_2H+\varepsilon I+\gamma H+\delta A-\left(\sigma_2+\sigma_3-\sigma_1\right)B\end{array}\right)\end{aligned}$$

$$\begin{aligned}\frac{dF}{dt}&=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\times\\&\left(\begin{array}{l}-\mu\left(S-S^*\right)-\mu\left(V-V^*\right)-\mu\left(G-G^*\right)-\mu\left(E-E^*\right)-\mu\left(A-A^*\right)-\mu\left(I-I^*\right)-\mu\left(H-H^*\right)-\mu\left(R-R^*\right)-\mu\left(B-B^*\right)\\ \\-d_1\left(I-I^*\right)-d_2\left(H-H^*\right)+\varepsilon\left(I-I^*\right)+\gamma\left(H-H^*\right)+\delta\left(A-A^*\right)-\left(\sigma_2+\sigma_3-\sigma_1\right)\left(B-B^*\right)\end{array}\right)\end{aligned}$$

$$\begin{aligned}\frac{dF}{dt}&=\left(\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right)\times\\&\left(\begin{array}{l}-\mu\left(S-S^*\right)-\mu\left(V-V^*\right)-\mu\left(G-G^*\right)-\mu\left(E-E^*\right)-\left(\mu-\delta\right)\left(A-A^*\right)-\left(\mu+d_1-\varepsilon\right)\left(I-I^*\right)-\left(\mu+d_2-\gamma\right)\left(H-H^*\right)\\ \\-\mu\left(R-R^*\right)-\left(\mu+\sigma_2+\sigma_3-\sigma_1\right)\left(B-B^*\right)\end{array}\right)\end{aligned}$$

$$\begin{aligned}\frac{dF}{dt}=&-\mu\left(S-S^*\right)^2-\mu\left(S-S^*\right)\left[\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right]\\&-\mu\left(V-V^*\right)^2-\mu\left(V-V^*\right)\left[\left(S-S^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right]\\&-\mu\left(G-G^*\right)^2-\mu\left(G-G^*\right)\left[\left(S-S^*\right)+\left(V-V^*\right)+\left(E-E^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right]\\&-\mu\left(E-E^*\right)^2-\mu\left(E-E^*\right)\left[\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(A-A^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right]\\&-\left(\mu-\delta\right)\left(A-A^*\right)^2-\left(\mu-\delta\right)\left(A-A^*\right)\left[\left(S-S^*\right)+\left(V-V^*\right)+\left(G-G^*\right)+\left(E-E^*\right)+\left(I-I^*\right)+\left(H-H^*\right)+\left(R-R^*\right)+\left(B-B^*\right)\right]\end{aligned}$$

$$
\begin{aligned}
&-(\mu+d_1-\varepsilon)(I-I^*)^2-(\mu+d_1-\varepsilon)(I-I^*)[(S-S^*)+(V-V^*)+(G-G^*)+(E-E^*)+(A-A^*)+(H-H^*)+(R-R^*)+(B-B^*)] \\
&-(\mu+d_2-\gamma)(H-H^*)^2-(\mu+d_2-\gamma)(H-H^*)[(S-S^*)+(V-V^*)+(G-G^*)+(E-E^*)+(A-A^*)+(I-I^*)+(R-R^*)+(B-B^*)] \\
&-\mu(R-R^*)^2-\mu(R-R^*)[(S-S^*)+(V-V^*)+(G-G^*)+(E-E^*)+(A-A^*)+(I-I^*)+(H-H^*)+(B-B^*)] \\
&-(\mu+\sigma_2+\sigma_3-\sigma_1)(B-B^*)^2-(\mu+\sigma_2+\sigma_3-\sigma_1)(B-B^*)[(S-S^*)+(V-V^*)+(G-G^*)+(E-E^*)+(A-A^*)+(I-I^*)+(H-H^*)+(R-R^*)]
\end{aligned} \tag{32}
$$

From the result of (32), it is obvious that $\frac{dF}{dt}$ is negative$\left(i.e. \frac{dF}{dt} < 0\right)$.
Furthermore, at E^* (i.e. if $S = S^*$, $V = V^*$, $G = G^*$, $E = E^*$,$A = A^*$,$I = I^*$,$H = H^*$, $R = R^*$, $B = B^*$), $\frac{dF}{dt} = 0$ .From La Salle's invariant principle, it follows that all solutions of the system (1) – (9) approaches E^* as $t \to \infty$ if $R_0 > 1$. Therefore, the endemic equilibrium E^* is globally asymptotically stable in Ωwhenever $R_0 > 1$.

RESULTS

Numerical results

In this section, we carried out the numerical solution of the system (1) – (9) using the Runge-Kutta order four scheme. The numerical results are shown in Figure 2 to Figure 5 below. With **data1**, $R_0 = 3.8643 > 1$and with **data 2** (where some of the parameters are varied), $R_0 = 0.0388 < 1$. Figure 2, 3, 4 and 5 represent the graphical behaviour of the asymptomatic, infected, hospitalized and bacteria individuals of a dynamic system respectively.

Data 1: $\pi = 500$,$m = 0.02$,$n = 0.027$,$e = 0.42$,$f = 0.7$,$\beta_1 = 0.0095$,$\beta_2 = 0.0075$, $\beta_B = 0.000039$, $\beta_3 = 0.0055$,$K = 600$,$\omega = 0.35$,$q = 0.9$,$\eta = 0.41$,$\psi = 0.04$,$\theta = 0.03$,$\rho = 0.14$, $\delta = 70$, $d_1 = 0.02$,$d_2 = 0.025$,$r = 0.06$,$\varepsilon = 80$,$\gamma = 90$,$\sigma_1 = 0.73$,$\sigma_2 = 0.83$,$\sigma_3 = 1.60$, $\mu = 0.45$,$\alpha_1 = 0.65$,$\alpha_2 = 0.98$, $\varphi = 0.029$.

Data 2: $\pi = 5$,$m = 0.02$,$n = 0.027$,$e = 0.42$,$f = 0.7$,$\beta_1 = 0.0095$,$\beta_2 = 0.0075$, $\beta_B = 0.000039$, $\beta_3 = 0.0055$,$K = 60$,$\omega = 0.35$,$q = 0.9$,$\eta = 0.41$,$\psi = 0.04$,$\theta = 0.03$,$\rho = 0.14$, $\delta = 70$, $d_1 = 0.02$,$d_2 = 0.025$,$r = 0.06$,$\varepsilon = 80$,$\gamma = 90$,$\sigma_1 = 0.73$,$\sigma_2 = 0.83$,$\sigma_3 = 1.60$, $\mu = 0.45$,$\alpha_1 = 0.65$,$\alpha_2 = 0.98$, $\varphi = 0.029$.

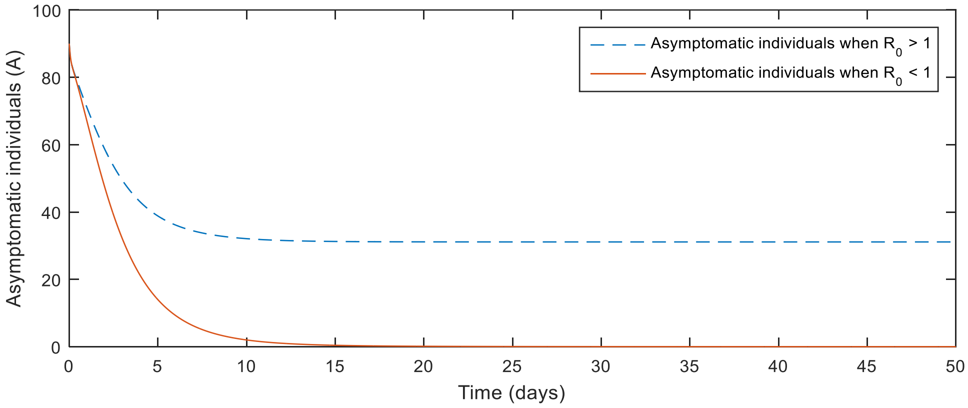

Figure 2: The graphical behavior of the asymptomatic individuals of a dynamic system. With **data1**,$R_0 = 3.8643 > 1$and with **data 2**, $R_0 = 0.0388 < 1$.

It can be seen that when $R_0 > 1$, the number of the asymptomatic individuals drops from 90 at $t = 0$ to its minimum size of 36 after $t = 10$days and remain constant till the final time while when $R_0 < 1$, the number of the asymptomatic individuals drops from 90 at $t = 0$to zero (i.e. eradication point of the disease) after $t = 13$days and remain constant till the final time.

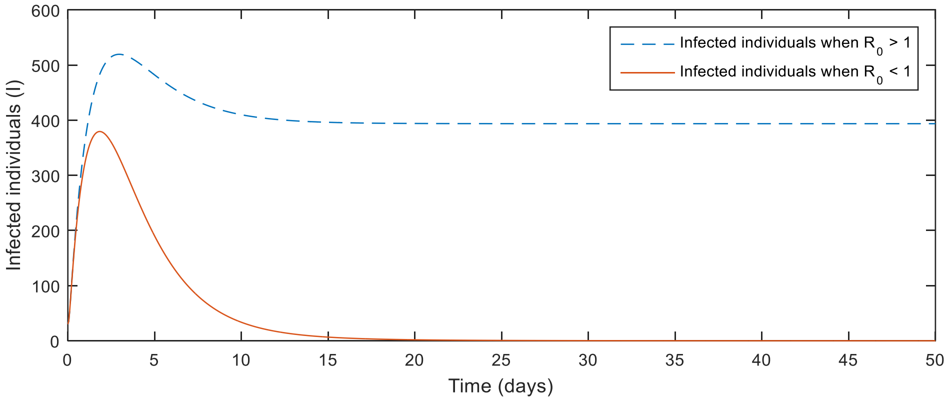

Figure 3: The graphical behaviour of the infected individuals of a dynamic system. With **data1**,$R_0 = 3.8643 > 1$and with **data 2**, $R_0 = 0.0388 < 1$.

It can also be seen that when $R_0 > 1$, the number of the infected individuals rises from 30 at $t = 0$to its maximum size 530 after $t = 4$days and gradually drops to 400 after $t = 11$days before it remains constant till the final time while when $R_0 < 1$, the number of the infected individuals rises from 30 at $t = 0$to its maximum size 380 after $t = 3$days and gradually drops to 0 (i.e. the point of the disease eradication) after $t = 16$days before it remains constant till the final time.

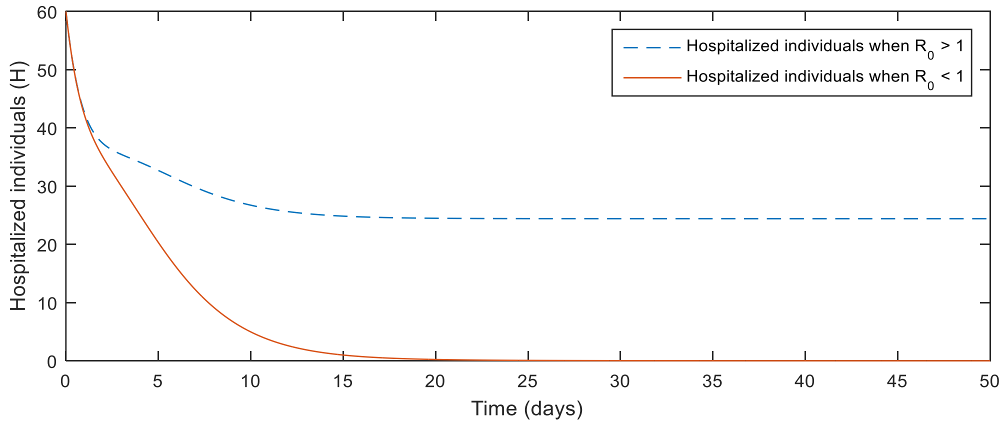

Figure 4: The graphical behaviour of the Hospitalized individuals of a dynamic system. With **data1**,$R_0 = 3.8643 > 1$and with **data 2**, $R_0 = 0.0388 < 1$.

It can also be seen that when $R_0 > 1$, the number of the infected individuals rises from 60 at $t = 0$and gradually drops to 24 after $t = 15$days before it remains constant till the final time while when $R_0 < 1$, the number of the infected individuals drop from 60 at $t = 0$to its minimum size 0 (i.e. the point of the disease eradication) after $t = 17$days before it remains constant till the final time.

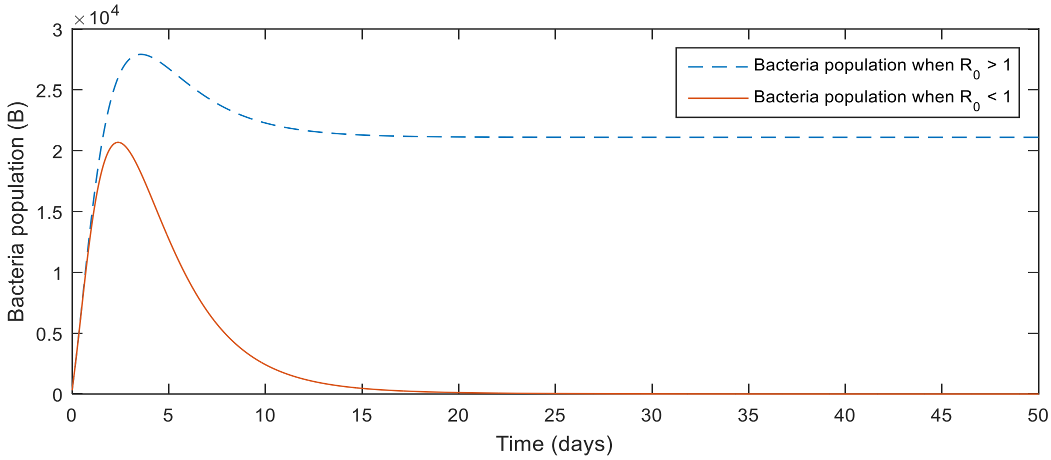

Figure 5: The graphical behaviour of the bacteria individuals of a dynamic system. With **data1**,$R_0 = 3.8643 > 1$and with **data 2**, $R_0 = 0.0388 < 1$.

It can also be seen that when $R_0 > 1$, the number of the bacteria population rises from 300 at$t = 0$to its maximum size 2.8×10^4after $t = 4$days and gradually drops to 2.2×10^4after $t = 15$days before it remains constant till the final time while when $R_0 < 1$, the number of the infected individuals rises from 300 at $t = 0$ to its maximum size 2.1×10^4 after $t = 3$days and gradually drops to 0 (i.e. the point of the disease eradication) after $t = 17$days before it remains constant till the final time.

CONCLUSION

In this Paper, we formulated a mathematical model equation of shigella infection with the aid of system of ordinary differential equations to study the dynamics of shigella infection by incorporating a vaccinated class (V), educated class (G), exposed class (E), asymptomatic (A) hospitalized class (H) and Bacteria class (B) with their corresponding parameters. The next generation matrix approach was used to determine the basic reproduction number R_0.The endemic equilibrium (EE) was obtained. The local and global stability of the endemic equilibrium (EE) were also obtained. The numerical solution of the model system in MATLAB was also obtained. From the simulation, we observed that the shigella infection persist in the environment when $R_0 = 3.8643 > 1$ with the original data and was eradicated with $R_0 = 0.0388 < 1$when some of the original data were varied.

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