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SOLUTION OF FREDHOLM INTEGRO-DIFFERENTIAL EQUATION BY VARIATIONAL ITERATION METHOD

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ABSTRACT

The Variational Iteration Method is applied to solve Fredholm Integro-differential Equation. The initial approximation was selected wisely which satisfies the initial condition of the given Fredholm Integro-differential Equation. The numerical result was analyzed with the aid of Mat-Lab and Maple software's and are compared with the exact solution to show the usefulness and efficiency of the Method. The results show that the Variational Iteration Method is very efficient, reliable and of high accuracy for solving Fredholm Integro-differential Equation when compared with the exact solution.

Keywords: Variational Iteration Method, Fredholm integro-differential, Equation Initial Approximation, Numerical Solution

INTRODUCTION

Most engineering and physical problems are governed by functional Equations, for example, ordinary differential equations, integral equation, Integro-differential equation (IDEs) and stochastic equations. Many mathematical formulations of physical phenomena contain IDEs with proper boundary conditions, these equations arise in fluid dynamics, biological models and chemical kinetics etc. (Abbasbandy, 2009). In most cases, the equation is too complex to allow one to find an exact solution, where solution of such equation is always demanded due to practical interest. Therefore, an efficient, reliable computer simulation is required, it is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in mathematical, physical and engineering problems solving has increased dramatically in recent years. Today, computer and numerical methods provide an alternative for complicated calculations. Using computer power to obtain solution directly, we can approach these calculations without recourse to simplifying assumptions techniques. Although analytical solution is still extremely valuable both for problems solving and for providing insight, numerical methods represent alternatives that greatly enlarge our capabilities to confront and solve problems. Thus, more emphasis has been placed on problem formulation and solution interpretation and incorporation of total system. (Abdou &Soliman, 2005).

The Variational Iteration Method has to be effective and reliable for analytic and numerical purposes. This method established by (Ji-Huan He, 1999) is now used to handle a wide a variety of linear and non-linear homogeneous and non-homogeneous equations. The method provides rapidly convergent successive approximation of the exact solution if such a closed from solution exists, and not component as in Adomian Decomposition method. The Variational Iteration method handles linear and non-linear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for non-linear problems. (Arqub, 2014).

Integro-differential Equations arise in many scientific and engineering applications, especially when we concert Initial value problems (IVPs) or Boundary Value problems (BVPs) to integral equations. An Integro-differential Equation is an equation in which the unknown function $u(x)$ appears under an integral sign and contains an ordinary derivative $u(x)$ as well.

A standard integro-differential is of the form

$$u^n(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k\,(x,r)u(r)dr \qquad (1)$$

Where $g(x) and\ h(x)$the limits of the integration are, λ is a constant parameter $k(x,r)$ is a function of two variables x and r called kernel or the nucleus of the Integro -differential equation. The Integro-differential equations contain both integral and differential operators. The derivatives of the unknown are classified into following:

Fredholm Integro-differential Equation appears when we convert differential equation to integral equations. The Fredholm Integro-differential Equation contains the unknown function $u(x)$ and one of its derivatives $u^n(x), n\ \geq 1$ inside and sign respectively. The limits of integration in this case are fixed as in Fredholm Integro-differential Equations.

The equation is libeled as Integro-differential because it contains differential and integral operators in the same equation. The Fredholm Integro-differential Equation appears in the form:

$$u^n(x) = f(x) + \lambda \int_{a}^{b} k\,(x,r)u(r)dr \qquad (2)$$

Where $u^n(x)$ indicates the nth derivative of $u(x)$

Volterra- Fredholm Integro-differential Equation appears when we convert initial value problems to integral equations. The Volterra- Fredholm Integro-differential Equation contains the unknown function $u(x)$ and one of its derivatives $u^n(x), n\ \geq 1$ inside and outside the integral sign.

At least one of the limits of integration in this case is a variable as in Volterra integral equations. The Volterra Integro-differential Equation appears in the form:

$$u^n(x) = f(x) + \lambda \int_{0}^{x} k\,(x,r)u(r)\,dr \qquad (3)$$

Where $u^n(x)$ indicates the nth derivative of $u(x)$ Volterra-Fredholm Integro-differential Equation arises in the same manner as Volterra-Fredholm Integral equations with one or more of the ordinary derivatives in addition to the integral operator.

A Volterra-Fredholm Integro-differential equation appears in the form:

$$u^n(x) = f(x) + \lambda_1 \int_{0}^{x} k_1\,(x_1,r)u(r)dr + \lambda_2 \int_{a}^{b} k_2\,(x_2,r)u(r)dr \qquad (4)$$

In recent years, there has been an increase interest in the Integro-differential Equation (IDEs). IDEs play an important role in many branches of linear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatic. (Hussein et al, 2016).

The variational iteration method was proposed by Ji-Huan He in 1999 and he successfully applied it to Autonomous ordinary differential equations, to nonlinear polycrystalline solids and other fields. The Method as proposed by Ji-Huan He is a Modification of general Lagrange multiplier method and it has been shown that this procedure is a powerful tool for solving various kinds of nonlinear problems.

Besides its mathematical importance and its links to the other branches of mathematics, it is widely used in all reunifications and of modern sciences. Few years ago, the method received relatively little attention since the number of available materials was limited. A change in this view occurred in Batiha et al, (2008) published a series of publications on Variational Iteration method as a result of the development of some mathematical software such as Mathematical, Mat lab. Some rather extraordinary virtues of the method were exploited and connection to Adomian Method was found.

These merits have enabled a very great number of potential applications. As an advantage of the Variational Iteration Method over decomposition procedure of Adomian, the former method provides the solution of the problem without any need to discretization of the variables. Therefore, it is not affected by computational round of errors and one is not faced with necessity of large computer memory and time.

This scheme provides the approximation of mesh points only. Also, this method is useful for finding an accurate approximation of the exact solution (Tatari &Dehghan , 2007). With rapid development of nonlinear sciences, there appears ever-increasing interest of scientist and engineers in the analytic asymptotic method for nonlinear problems. Though it is easy for us now to find the solution of linear and nonlinear system by means of computer, it is however still very difficult to solve nonlinear problems either numerically or theoretically.

This is possibly due to fact that the various discredited methods or numerical simulation apply iteration methods to find their numerical solution of nonlinear problems and nearly all iterative techniques are sensitive to initial solutions. So, it is very difficult to obtain converged result in case of strong nonlinearity. (Ji-Huan, 2007). Applied the present method to coupled Schrodinger KDV equations and shallow water equations.

Most realistic differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore are used extensively. This new iterative method has proven rather successful in dealing with both linear as well as non-linear problems, as yields analytic solution and offers certain advantages over standard numerical methods. (Hemeda, 2012) extended the analysis of the Variational Iteration Method proposed by He to solve three different types of nonlinear equations, namely the coupled nonlinear RDEs, hirota-Satsuma coupled KDV system and Drinefil'd-Sokolov-Wilson equations. Compared the Variational Iteration Method reduces the volume of calculation by not requiring the Adomian polynomials like the Adomian Decomposition method. Hence the iteration is more accurate. (Hesamedding & Rahimi, 2013). compared the numerical methods for the solution of first and second orders linear Integro-differential equations (Lanlegeet al ,2019), presented a new iterative method of solving IDEs of order n. (Lanlegeet al ,2015), in the computational l method for differential equations presented a new numerical scheme for solving system for solving systems of Integro-differential equations solved the IDE using the modified Laplace Adomian Decomposition Method.(Mohyud-din ,2013) presented a simple numerical method of solving nonlinear Volterra IDEs. (Safavi , 2012)., used the decomposition method for solving nonlinear IDEs. (Soliman &Abdou, 2007). used the method to search for solutions of a class of two-point boundary value problems (BVPs) for fourth order Integro-differential equations showing that the method needs much less computational work compared with traditional methods.

It also gives highly accurate numerical solutions without spatial discretization for fourth order integro-0differential equations. (Sweilam, 2007) Applied the Variational Iteration Methods to solve nonlinear Integro-differential equations and found that the solution obtained by VIM is valid for not only weekly nonlinear equations, but also strong ones.

Also, the method is a powerful tool to search for solutions of various linear and nonlinear problems. (Tatari&Dehghan,2007).extended the analysis of the Variational Iteration Methods to solve the system of generally nonlinear Volterra Integro-differential equations by selecting the initial approximation arbitrarily not in found of the exact solution with unknown constants. The result showed that the Variational Iteration Method is remarkably effective and performing is very easy. In addition, it has more accuracy than Homotopy perturbation method and Adomian decomposition method for this kind of problems. (Taiwo &Jimoh,2014) employed the Variational Iteration Method (VIM) with Adomian decomposition method (ADM) for solving nonlinear Integro-differential equations without using linearization of any restrictive assumptions and it was observed that the Variational Iteration Methods (VIM) reduces the volume of calculations by not requiring the Adomian polynomials, hence the Iteration is direct and straightforward for the solution of Integro-differential equation. (Wazwaz, 2011). used modified Variational Iteration Method (MVIM) for solving Integro-differential equations and coupled system of Integro-differential equations. The proposed modification is made b the elegant coupling of He's polynomials and the correction functional Variational Iteration Method. The proposed (MVIM) was applied without any discretization, Transformation of restrictive assumptions and is free from round off errors and calculations of the Adomian polynomials.

The VIM is a simple and yet a powerful method of solving a wide class of linear and nonlinear Integro-differential equations. The method provides rapidly convergent successive approximations of the exact solutions if such a closed form solution exists. The method was used by he to give approximate solutions for some well-known nonlinear problems.

On the other hand, Golbaba and Javidi solved the nth order Integro-differential equations by transforming to a system of ordinary differential equations and used the Homotopy Method solve. Other methods of solution include the series solutions, the Laplace transform method, the direct computational method and so on.

MATERIALS AND METHOD

Iteration Method

Consider the difference equation

$$Lu + Nu = g(r) \quad (5)$$

Where L and N are linear and nonlinear operators respectively, $g(r)$ is the source inhomogeneous term

The correction functional for the above equation is given by:

$$U_{n+1}(x) = U_n(x) + \int_0^x \lambda(r)(LU_n(r) + NU_n(r) - g(r))\, dr \quad (6)$$

Where λ is the Lagrange's multiplier, it should also be noted that λ can be a constant or function, and Un is a restricted value that means it behaves as a constant, hence $\delta U_n = 0$ where δ is the variational derivate.

The Variational Iteration Method can then be used following these steps:

First, we determine the Lagrange's multiplier $\lambda(r)$ which will be found optimally and, We substitute the result into the correction functional where the restriction should be Omitted Taking the variation of the correction functional with respect to the independent variationUn, we have

$$\frac{\delta U_{n+1}}{\delta U_n} = 1 + \frac{\delta}{\delta Un}\left(\int_o^x \lambda(r)\big(LUn(r) + NUn(r) - g(r)\big)dr\right) \quad (7)$$

Which gives

$$\delta Un + 1 = \delta Un + \delta\left(\int_o^x \lambda(r)(LUn(r)\, dr\right) \quad (8)$$

Integrating by parts to obtain value of the Lagrange multiplier $\lambda(r)$, that is

$$\int_o^x \lambda(r)\, U'n(r)dr = \lambda(r)Un(r) - \int_o^x \lambda'(r)\, Un(r)dr \quad (9)$$

$$\int_o^x \lambda(r)\, U''n(r)dr = \lambda(r)U'n(r) - \lambda'(r)Un(r) + \int_o^x \lambda''(r)Un(r)dr \quad (10)$$

$$\int_o^x \lambda(r)\, U'''n(r)dr = \lambda(r)U''n(r) - \lambda'(r)U'n(r) + \lambda''(r)Un \int_o^x \lambda'''(r)\, Un(r)dr \quad (7) \int_o^x \lambda(r)\, U'^{v}n(r)dr = \lambda(r)U'''n(r) - \lambda'(r)U''n(r) + \lambda''(r)U'n(r) - \lambda'''(r)Un(r) + \int_o^x \lambda'''(r)\, Un(r)dr \quad (11)$$

And so on to the nth derivative. These identities are determined integrating by parts.

For instance, if $LUn(r) = U'n(r)$ in (4) therefore we have,

$$\delta Un + 1 = \delta Un + \delta\left(\int_o^x \lambda(r)(LUn(r)\, dr\right) \quad (12)$$

Or we have by using the identity above

$$\delta Un + 1 = \delta Un(r) + \lambda(r)\delta Un(r) - \int_o^x \lambda'(r)\delta Un(r)dr(10)$$

$$\delta Un + 1 = \delta Un(r) + \lambda(r)\ (1 + \lambda|\tfrac{x}{0}) - \int_o^x \lambda'(r)\delta Un(r)\, dr \quad (13)$$

But the extremism condition of $Un + 1$ requires that $\delta Un + 1 = 0$.

This means that the left-hand side of (11) is zero, and as a result the right-hand side should be zero as well.

This reduces (11) to and thus yield the stationary conditions,

$1 + \lambda\mathrm{r} = 0$,

$\lambda' r = x, \lambda' r = 0$

This gives $\lambda = -1$

Also, if $LUn(r) = U''n(r)$ in (3.4), then it becomes

$$\delta Un + 1 = \delta Un + \delta\left(\int_o^x \lambda(r)(LUn(r)dr)\right) \quad (14)$$

Integrating the integral of (14) by part, we have

$$\delta Un + 1 = \delta Un + \delta\lambda(Un)'\frac{x}{0} - (\lambda' Un)\frac{x}{0} + \int_o^x \lambda''\, \delta Undr \quad (15)$$

Or

$$\delta Un + 1 = \delta Un(r)(1 - \lambda' r = x + \delta\lambda(U'n)r = x^{\mathrm{r}} + \int_o^x \lambda''\, \delta Undr \quad (16)$$

Where $\delta Un + 1 = 0$, and these yields stationary condition

$$1 - \lambda_{\mathrm{r}}' r = x = 0,$$

$$\lambda(r)r = x = 0 \quad (17)$$

$$\lambda = r - x \quad (18)$$

he Lagrange Multiplier can also be determined from the general formula

$$\lambda(r) = \frac{(-1)(r-x)-1}{(-1)!} \quad (19)$$

With the Lagrange multiplier $\lambda(r)$ determined, we now obtain the successive approximation $Un + 1, n \geq 1$, of the solution $U(x)$, which will be obtained using selective function $Uo(x)$.

However, for the solution to converge fast; the function $Uo(x)$ should be selected using the initial conditions as follows:

$U_0(x) = u(0), for\ order\ one.$

$U_0(x) = u(0) + xu'(0), For\ order\ two.$

$U_0(x) = u(0) + xu'(0) + \frac{x2}{2!}u''(0), For\ order\ three$

$U_0(x) = u(0) + xu'(0) + \frac{x2}{2!}u''(0) + \cdots + \frac{X^{n-1}}{(n-1)!}U^{n-1}(0)$, For order n (20)

Consequently, the solution

$U(x) = \frac{\text{Lim}}{n\to\infty}U_n(x)$ (21)

This means that the correction functional (2) will give several approximate solutions which will approach the exact solution. The correction functional (2) is given by

$$U_{n+1}(x) = U_n(x) + \int_o^x \lambda\left(Un(n)(t) - f(t) - \int_a^b k(x,r)Un(r)dr\right)dt$$

For $n = 0, 1, 2 . . .\ , k-1$ then (27) becomes

$$U_1(x) = Uo(x) + \int_0^x \lambda\left[Uo(n)(t) - f(t) - \int_a^b K\,(t,r)Uo(r)dr\right]dt$$

$$U_2(x) = U_1(x) + \int_0^x \lambda\left[U1(n)(t) - f(t) - \int_a^b K\,(t,r)Uo(r)dr\right]dt$$

$Uk(x) = Uk - 1(x) + \int_0^x \lambda\left[Uk - 1(n)(t) - f(t) - \int_a^b K\,(x,r)Uk - 1\,(r)dr\right]dt$ (23)

Numerical examples

Example 1

Consider the third order Fredholm Integro-differential equation

$u'''(x) = -1 + e^x + \int_0^1 lu(l)dl\, u(0) = u'(0) = u''(0) = 1$ (24)

With exact solution $u(x) = e^x$. Following the VIM discussed in above the correctional functional is obtained as:

$$U_{n+1}(x) = U_n(x) + \lambda\int_0^x \left[U'''n(s) + 1\,e^s - \int_0^1 lu\,n(l)dl\right]dr$$

Where $\lambda = -\frac{1}{2}(s-x)^2$ and $Uo(x) = 1 + x + \frac{1}{2}x^2$

Differentiating $Uo\ (x)$ three times we obtain

$U_0'''(x) = 0$

Substituting into correctional functional for n =0 we obtain

$$U_1(x) = 1 + x + \frac{1}{2}x^2 - \int_0^1 \frac{1}{2}(s-x)^2\left[0 + 1 - e^s - \int_0^1 t\left(1 + t + \frac{1}{2}t^2\right)dt\right]ds$$

Simplifying we obtain

$$U_1(x) = -\frac{1}{144}x^3 + e^x$$

For $n = 1$ and substituting the value of $U_1(x)$, we obtain

$$U_1(x) = -\frac{1}{144}x^3 + e^x - \int_0^1 \frac{1}{2}(s-x)^2\left[-\frac{1}{24}s^3 + e^s + 1 - \int_0^1 t\left(-\frac{1}{144}t^3 + e^t\right)dt\right]ds$$

After simplification we obtain

$$U_2(x) = -\frac{1}{4320}x^3 + e^x$$

For $n = 2$ and using the result obtained for $U_2(x)$we have$(x) = -\frac{1}{4320}x^3 + e^x -$

$$\int_0^1 \frac{1}{2}\,(s-x)^2\left[-\frac{1}{720}s^3 + e^s + 1 - e^s - \int_0^1 t\left(-\frac{1}{4320}t^3 + et\right)dt\right]ds$$

$$U_3(x) = -\frac{1}{129600}x^3 + e^x$$

Continuing this process for $n = 3,4,5,6,7,8,9$ we have the results

$$U_4(x) = -\frac{1}{3888000}x^3 + e^x$$

$$U_5(x) = -\frac{1}{116640000}x^3 + e^x$$

$$U_6(x) = -\frac{1}{499200000}x^3 + e^x$$

$$U_7(x) = -\frac{1}{104976000000}x^3 + e^x$$

$$U_8(x) = -\frac{1}{3149280000000}x^3 + e^x$$

$$U_9(x) = -\frac{1}{944784000000000}x^3 + e^x$$

$$U_{10}(x) = -\frac{1}{2834352000000000}x^3 + e^x$$

RESULTS

Table 1: Comparison of the Numerical Results with the Exact Solution $u(x)$ and the Error

X	Exact	VIM Result	Error
1.0	2.718281828459046	2.718281828459045	0.000000000000001
1.1	3.004166023946433	3.004166023946433	0.000000000000000
1.2	3.320116922736547	3.320116922736547	0.000000000000000
1.3	3.669296667619244	3.669296667619244	0.000000000000001
1.4	4.055199966844675	4.055199966844674	0.000000000000001
1.5	4.481689070338065	4.481689070338064	0.000000000000001
1.6	4.953032424395115	4.953032424395113	0.000000000000002
1.7	5.473947391727199	5.473947391727197	0.000000000000002
1.8	6.049647464412947	6.049647464412947	0.000000000000002
1.9	6.685894442279269	6.685894442279266	0.000000000000003
2.0	7.389056098930650	7.389056098930648	0.000000000000002

Graphical Representation of Result as shown below:

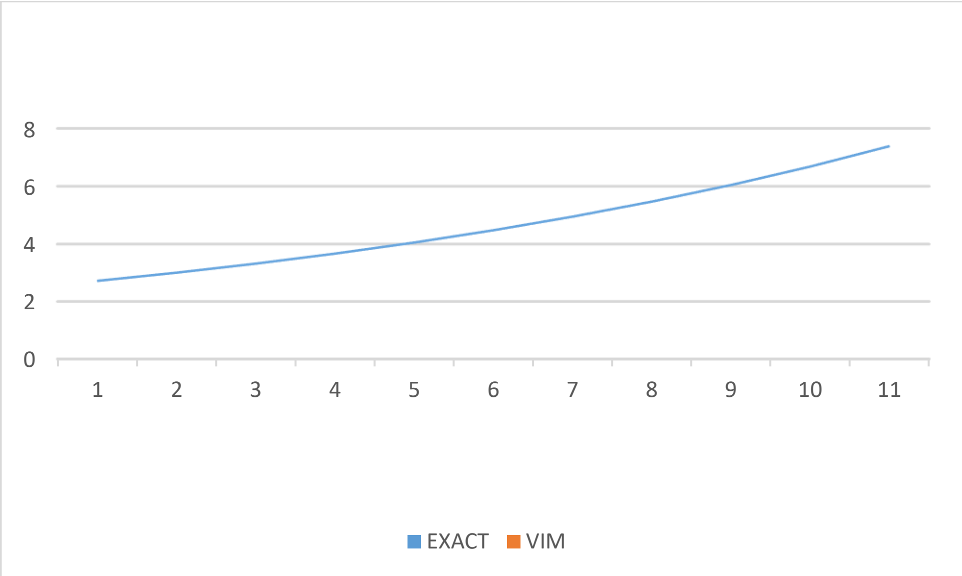

Figure 1: Graphical Illustration of VIM compared to the Exact Solution

Example 2

Consider the second order Fredholm Integro-differential equation

$$u''(x) = -1\sin x \int_0^{\frac{\pi}{2}} tu(t)dt u(0) = 0, u'(0) = 1 \tag{25}$$

With exact solution $u(x) = \sin x$ The correction functional is given by

$$Un + 1\,(x) = Un + \int_0^x \lambda\,[\left[U''n(s) + 1 + \sin s - \int_0^{\frac{\pi}{2}} tun(t)dt\right] ds$$

Where $\lambda = (s - x)$ and $uo''(x) = x$

Differentiating$uo(x) = x$ twice we have $u''o(x) = 0$

Substituting these result in to the correction functional for $n = 0$ we

$$U_1(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3 x^2$$

For $n = 1$ and using the result obtained for $U_1(x)$we have

$$U_2(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3 x^2 + \int_0^x (s - x)(-1 - \sin(s) + \frac{1}{24}\pi^3 + 1 + \sin s - \int_0^{\frac{\pi}{2}} t\left(-\frac{1}{2}t^2 + \sin t + \frac{1}{24}\pi^3 t^2\right) dt$$

After simplification we obtained

$$U_2(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3 x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$

For $n = 2$ and using the result for $U_2(x)$ we have

$$U_2(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$
$$+\int_0^x (s-x)\,[-\sin s + \frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + 1 + \sin s$$
$$-\int_0^{\frac{\pi}{2}} t\left(\left(-\frac{1}{2}t^2 + \sin t\frac{1}{48}\pi^3t^2\right) - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)t^2\right)dt]ds$$

Simplifying we have

$$U_3(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + \frac{1}{16384}\pi^8\frac{1}{393216}\pi^{11}\right)x^2$$

Repeating this procedure for $n = 3,4,5,6,7$ we have

$$U_4(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi 3 - \frac{1}{3072}\pi 7 - 1 + \frac{1}{128}\pi 4\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + \frac{1}{16384}\pi^8\frac{1}{393216}\pi^{11}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{16384}\pi^8 + \frac{1}{393216}\pi^{11} - \frac{1}{50331648}\pi^{15} + \frac{1}{2097152}\pi^{12}\right)x^2$$

$$U_5(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + \frac{1}{16384}\pi^8\frac{1}{393216}\pi^{11}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{16384}\pi^8 + \frac{1}{393216}\pi^{11} - \frac{1}{50331648}\pi^{15} + \frac{1}{2097152}\pi^{12}\right)x^2$$

-12150331648π^{15}- 12097152π^{12}+ 1268435456π^{16}- 16442450944π19x^2

$$U_6(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + \frac{1}{16384}\pi^8\frac{1}{393216}\pi^{11}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{16384}\pi^8 + \frac{1}{393216}\pi^{11} - \frac{1}{50331648}\pi^{15} + \frac{1}{2097152}\pi^{12}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{50331648}\pi^{15} - \frac{1}{2097152}\pi^{12} + \frac{1}{268435456}\pi^{16} - \frac{1}{6442450944}\pi^{19}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{268435456}\pi^{16} - \frac{1}{6442450944}\pi^{19} + \frac{1}{824633720832}\pi^{23} - \frac{1}{39582418599936}\pi^{20}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{549755813888}\pi^{23} - \frac{1}{34359738368}\pi^{20} - \frac{1}{39582418599936}\pi^{26}\right)x^2$$

$$U_7(x) = -\frac{1}{2}x^2 + \sin x + \frac{1}{48}\pi^3x^2 - \frac{1}{2}\left(\frac{1}{24}\pi^3 - \frac{1}{3072}\pi^7 - 1 + \frac{1}{128}\pi^4\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{3072}\pi^7 - \frac{1}{128}\pi^4 + \frac{1}{16384}\pi^8\frac{1}{393216}\pi^{11}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{16384}\pi^8 + \frac{1}{393216}\pi^{11} - \frac{1}{50331648}\pi^{15} + \frac{1}{2097152}\pi^{12}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{50331648}\pi^{15} - \frac{1}{2097152}\pi^{12} + \frac{1}{268435456}\pi^{16} - \frac{1}{6442450944}\pi^{19}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{268435456}\pi^{16} - \frac{1}{6442450944}\pi^{19} + \frac{1}{824633720832}\pi^{23} - \frac{1}{39582418599936}\pi^{20}\right)x^2$$
$$\frac{1}{2}\left(\frac{1}{549755813888}\pi^{23} - \frac{1}{34359738368}\pi^{20} - \frac{1}{39582418599936}\pi^{26}\right)x^2$$
$$-\frac{1}{2}\left(\frac{1}{2638827906624}\pi^{26} - \frac{1}{1649267441664}\pi^{23} - \frac{1}{1899956092796928}\pi^{29}\right)x^2$$

Representation of Result

Table 2: Comparison of the Numerical Results with the Exact Solution $u(x)$ and the Error

X	Exact	VIM Result	Error
1.0	0.000000000000000	0.000000000000000	0.000000000000000
1.1	0.099833416646828	0.099991348565143	0.0001579319191815
1.2	0.198669330795061	0.199301058468320	-0.000631727673258
1.3	0.295520206661340	0.296941593926171	-0.001421387264831
1.4	0.389418342308651	0.391945253001684	-0.002526910693034
1.5	0.479425538604203	0.483373836562068	-0.003948297957865
1.6	0.564642473395035	0.570328022454361	-0.005685549059326

1.7	0.644217687237691	0.651956351235106	-0.007738663997415
1.8	0.717356090899523	0.727463733671657	-0.010107642772134
1.9	0.783326909627483	0.796119395010966	-0.012792485383483
2.0	0.841470984807897	0.857264176639357	-0.015793191831460

Graphical Representation of Numerical Results with the Exact

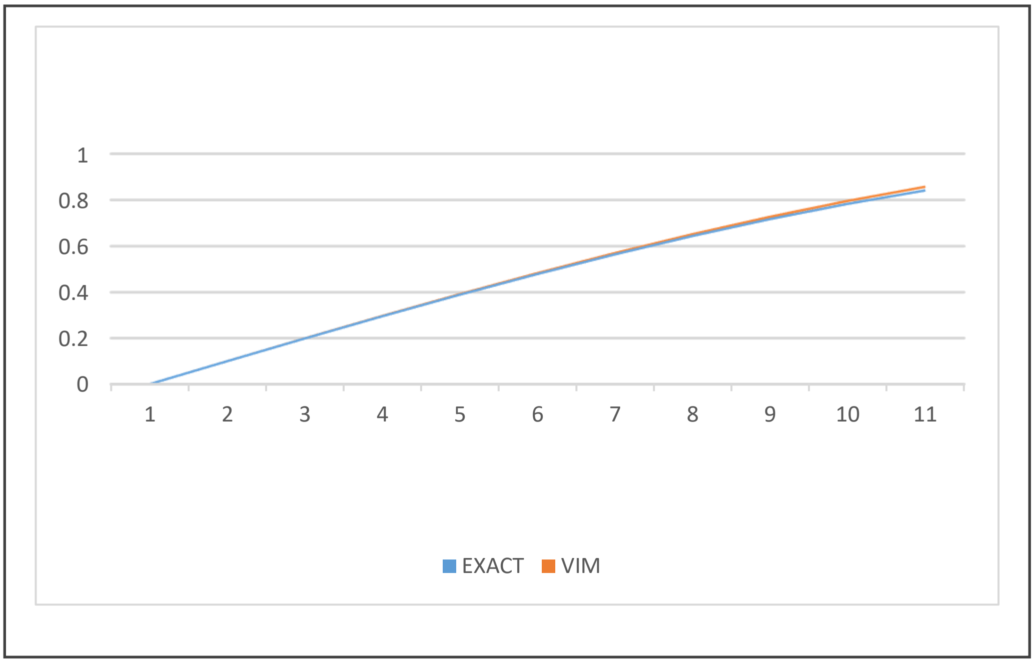

Figure 2: Graphical Illustration of VIM compared to the Exact Solution

It is very clear to observe from the examples considered in this work that the Variational Iteration Method is a powerful tool for solving Fredholm Integro-differential equation. It was also discovered that the result obtained by the Variational Iteration Method are in close agreement with the exact solution, this is evidently seen in the tables above and the obtained by plotting the exact solution with the approximate analytic solutions obtained by the Variational Iteration Method.

From the first example the result obtained by Variational Iteration Method is in close agreement with that of the exact solution with ten (10) Iteration carried out. In the second example, with reduction in the number Iterations to eight (8), there is a slight change in the alignment of the exact solution and the solution obtain by the Variational Iteration Method as shown by the tabulated numerical result and the graph. From the third example and with the number of Iterations increased to nine (9), we see a great agreement of the approximate solution of Variational Iteration Method with the exact solution.

CONCLUSION

The Variational Iteration Method discussed in chapter three has been successfully applied to solve Fredholm Integro-differential equations, three examples were considered to illustrate the procedure. it was observed that the solutions obtained are in closed agreement with those of the exact solutions which can be seen from the tables and the graphs. It was also observed that with more Iterations performed the approximate analytic solution of the Variational Iteration Method will approach the exact solution of the given problem. In this project work, we have studied the solution of Fredholm Integro-differential equations by Variational Iteration Method (VIM). The solution of the initial approximation was done wisely not in form of the exact solution with the unknown constants. Maple package was used to calculate the approximate analytic solution obtained from the Variational Iteration Method. The result showed that the Variational Iteration Method is remarkable effective and is very easy, it may also be concluded that the Variational Iteration Method (VIM) (Arqub, 2014).is very powerful and efficient in finding the analytical solutions of Fredholm Integro-differential equations with increased the number of interactions to performed.

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