FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 3 No. 4, December, 2019, pp 360 –365

NUMERICAL SOLUTION OF SINGULAR VOLTERRAINTEGRAL EQUATION VIA MIDPOINT METHOD

[*1]Yahaya, M. and **[2]Lawan, S. B.**

[1]Department of Mathematics, Faculty of Science, Federal University Dutse, Jigawa, Nigeria,
[2]Department of Mathematical Sciences, Faculty of Science, Bayero University Kano, Kano, Nigeria

[*]Corresponding author's email: ymustapha830@fud.edu.ng

ABSTRACT
This paper presents numerical scheme for solving singular Volterra integral equations via midpoint rule. The functions were approximated under the integrals by considering the non-variable subinterval. The convergence analysis of the error bound of the scheme is established. The numerical results show that the scheme has less number of iterations to obtain the best errors compared with other method.

Keywords: singular Volterra integral equation, error bound analysis, midpoint rule and iterations.

INTRODUCTION

Singular Volterra integral equations are of high applicability in different areas of applied mathematics, physics and chemistry, Rahbar and Hashemizadeh (2008). Singular Volterra integral equation can be viewed as the form

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds + g(t), \quad t \in (0,T] \qquad (1)$$

With $\mu > 0$, g(t) $\in C[0,T]$ is a given function and the kernel of the equation is of weaklytype that has been considered in several works. The precise analysis of (1) depends particularly on the value of μ, if $\mu > 1$ the kernel has singularity at t = 0 and a function g leads to the smooth solution u. However, if $0 < \mu < 1$ then, thekernel has singularities at both t = 0 and s = 0 for all value of t. Lima and Diogo (1997) developed an extrapolation algorithm, based on Eulers method. Also Lima and Diogo in (2002) proved that the Eulers method converges to a particular solution and showed that the asymptotic error expansion converges to order μ. Diogo and Lima (2004) investigated the application of product integration method for the numerical solutions. In (2005) Diogo et al. obtained the analytic results for the existence and uniqueness solutions of (1). Diogo et al. (2006) obtained the numerical schemes of Eulers and Trapezoidal methods, also Proved the convergence of the error bound analysis of the methods. Also in (2006) Diogo et al. used the product Eulers and classical Trapezoidal methods over an initial time interval of the split-interval method. However Diogo and Lima (2007) analyzed discrete supperconvergence properties of spline collocation solutions of (1). Diogo and Lima (2008) proved that a higher orderattained at the meshes points by special choice of the collocation methods. Also Diogo (2009) Used iterated collocation methods. Al-Jawary and Shehan (2015) implemented an efficient method for the exact solutions of (1). Wazwaz et al. (2013) used systematic modified Adomian decomposition method for solving singularproblems. Prajapati et al. (2012) used friendly algorithm based on the variational iteration method of singular integral equations.

In this paper we consider the work of Diogo et al. (2006) where we used the Midpoint method, in the case $0 < \mu < 1$. However, for these values of μ (1) has a family of solutions in thespace of continuous class of functions $C[0,T]$. The work has been organized as follows; In section 2, we derived the scheme by the used of Midpoint method. In section 3, we estimated the convergence of error bound analysis of the scheme obtained. Also, in section 4, we used some examples and tested the scheme and finally in section 5 the conclusion was presented.

Derivation of the Scheme by Midpoint Rule Approach

Definitions of the basic concepts

We start by presenting some definitions, theorems and lemmas;

Definition1 A kernel is called separable if it can be expressed as the outer product of two variables (vectors). For examples

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds + g(t), \quad t \in (0,T]$$

where $k(t,s) = \frac{s^{\mu-1}}{t^\mu}$ that can be expressed as $k(t,s) = \frac{1}{t^\mu} s^{\mu-1}$ otherwise, it is non-separable.

Theorem2 Mean Value Theorem: Let u(x) be a function which is continuous onthe closed interval [a, b] and which is differentiable at every point of (a, b). Then there is a point c ∈ (a, b) such that

$$u'(c) = \frac{u(b) - u(a)}{b - a}$$

Lemma3 Special Gronwall lemma: Let (e_n) and (e_j) be nonnegative sequences and C a nonnegative constant if

$$u_n \leq C + \sum_{k=0}^{n-1} g_k u_k \qquad for \quad n \geq 0$$

then

$$u_n \leq Ce^{\sum_{j=0}^{n-1} g_j} \qquad for \quad n \geq 0$$

Lemma4 (i) If $0 < \mu \leq 1$ and $g \in C^1[0,T]$ with$g(0) = 0$ if$\mu = 1$, then (1) has a family of solutions $u \in C[0,T]$ given by the formula

$$u(t) = c_0 t^{\mu-1} + g(t) + \gamma + t^{\mu-1} \int_0^t s^{\mu-2}\,(g(s) - g(0))ds \qquad (*)$$

where

$$\gamma = \begin{cases} \frac{1}{\mu-1} g(0) & if\ \mu<1, \\ 0 \ \ if & \mu=1, \end{cases}$$

and $c_0$is an arbitrary constant. Out of this family of solutions there is one particular solution$u \in C^1[0,T]$

(ii) If $\mu \leq 1$and $g \in C^m[0,T]$,$m \geq 0$, then the unique solution$u \in C^m[0,T]$ of (1) is given by

$$u(t) = g(t) + t^{\mu-1} \int_0^t s^{\mu-2}\, g(s)ds$$

We note that (3) can be obtained from (2) with $c_0 = 0$. Indeed; it follows from (2) that

$$c_0 = \lim_{t\to 0} t^{\mu-1} u(t)$$

and this limit is zero when$\mu > 1$.

Derivation of the scheme

Let us reformulate (1) into a new form by choosing some fixed real number α> 0.

Substituting t by $t + \alpha$in (1) we have

$$u(t+\alpha) = \int_0^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^\mu} u(s)ds + g(t+\alpha), \ \ t \in [0,T] \qquad (2)$$

by splitting of the interval we have

$$u(t+\alpha) = \frac{1}{(t+\alpha)^\mu} \int_0^\alpha s^{\mu-1} u(s)ds + \int_\alpha^{t+\alpha} \frac{s^{\mu-1}}{(t+\alpha)^\mu} u(s)ds + g(t+\alpha), \ \ t \in [\alpha, T] \qquad (3)$$

or, equivalently,

$$u(t+\alpha) = \frac{I_\alpha}{(t+\alpha)^\mu} + \int_0^t \frac{(s+\alpha)^{\mu-1}}{(t+\alpha)^\mu} u(s+\alpha)ds + g(t+\alpha) \qquad (4)$$

where

$$I_\alpha := \int_0^\alpha s^{\mu-1} u(s)ds \qquad (5)$$

Since $I_\alpha$is known exactly for a chosen the exact solution from using the solution formula

then we can apply the numerical method to (4) and obtain the approximation. Now, let us define a uniform grid $X_h$with stepsize $h = \frac{t}{n}$

$$X_h := \{t_i = ih + \alpha, \qquad 0 \leq i \leq N\}$$

Setting $t_i = nh$ in (4) we have

$$u(t_n) = \frac{I_\alpha}{{t_n}^\mu} + \frac{1}{{t_n}^\mu} \int_0^{nh} (s+\alpha)^{\mu-1}\, u(s+\alpha)ds + g(t_n) \qquad (6)$$

In the Midpoint method, we approximated the integral on the right-hand side of equation (9) by considering each subinterval using:

$$u(s+\alpha) \approx u\left(\tfrac{jh+(j+1)h}{2}\right)[(j+1)h - jh] \qquad (7)$$

on each subinterval $s \in [jh, (j+1)h]$ Defining

$$D_J := \int_{jh}^{(j+1)h} (s+\alpha)^{\mu-1} ds$$

which can be obtain analytically.

Hence the scheme

$$u(t_n)_n^h = \frac{I_\alpha}{t_n{}^\mu} + \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_J u_j^h + g(t_n), \quad n = 1,2,\dots,N \tag{8}$$

Algorithm: Midpoint rule approach

Step1: Given $n = 1,\ t \in [0,T],\ \mu \in (0,1],\ \alpha > 0,\ u(t),\ g(t),\ I_\alpha$.

Step2: Set $h = \frac{t}{n}$

Step3: Compute

$$t_n = nh + \alpha$$
$$t_n{}^\mu = (nh+\alpha)^\mu$$
$$u_n^h = \frac{I_\alpha}{t_n{}^\mu} + \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_J u_j^h + g(t_n)$$

where $D_J := \frac{h\left(\left((j+1)h+\alpha\right)^\mu - (jh+\alpha)^\mu\right)}{\mu}$ and $u_j^h = u\left(\frac{jh+(j+1)h}{2}\right)$

and check $|u(t) - u_j^h| < \epsilon$ stop

Step4: Set $n = n+1$ and go to step3.

Error Bound of the Scheme in Midpoint Rule Approach

In this section we present the error bound for the convergence of the scheme.

Theorem 3.1 Consider (1) with $0 < \mu \le 1$ and $u \in C^1[0,T]$. Let $\alpha \ne 0$ be fixed in the equivalent (4) and assume the integral $I_\alpha$is known exactly for a chosen particular solution (corresponding to a certain value of the parameter c_0). Then the approximatesolution obtained by the product Midpoint method converges with order 2 to the exactsolution.

Proof

The solution u of the exact solution satisfies

$$u(t_n)^h = \frac{I_\alpha}{t_n{}^\mu} + \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_j u(t_j) + g(t_n) + \eta(h,t_n), \quad n \ge 1 \tag{9}$$

where $\eta(h,t_n)$ is the consistency error given by

$$\eta(h,t_n) = \int_0^{t_n} \frac{s^{\mu-1}}{t_n{}^\mu} u(s)ds - \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_j u(t_j) \tag{10}$$

but the exact solution is

$$u(t_n) = \frac{I_\alpha}{t_n{}^\mu} + \frac{1}{t_n{}^\mu}\int_\alpha^T s^{\mu-1}u(s)ds + g(t_n) \tag{11}$$

Setting $e_n = u(t_n) - u(t_n)^h$ for $n \ge 1$and by utilizing (9) and (11) this gives

$$e_n = \frac{1}{t_n{}^\mu}\int_\alpha^T s^{\mu-1}u(s)ds - \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_j u_j^h + \eta(h,t_n)$$
$$= \frac{1}{t_n{}^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}u(t_j)ds - \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} s^{\mu-1}u(t_j)ds + \eta(h,t_n)$$

$$= \frac{1}{t_n{}^\mu}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(u(t_j) - u(t_j)^h\right)s^{\mu-1}ds + \eta(h,t_n) \tag{12}$$

Let $e_j = \left(u(t_j) - u(t_j)^h\right)$

$$e_n = \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} e_j \int_{t_j}^{t_{j+1}} s^{\mu-1}ds + \eta(h,t_n), \quad n \ge 1 \tag{13}$$

but

$$\frac{1}{t_n{}^\mu}\int_{t_j}^{t_{j+1}} s^{\mu-1}u(s)ds \le \frac{t_j{}^{\mu-1}}{t_n{}^\mu}\int_{t_j}^{t_{j+1}} ds = h\frac{t_j{}^{\mu-1}}{t_n{}^\mu} \le h\left(\frac{t_j}{t_n}\right)^\mu \frac{1}{t_j} \le \frac{h}{\alpha} \tag{14}$$

Since $\alpha \ne 0$ and $\alpha > 0$ choose $\alpha \le t_j\left(\frac{t_n}{t_j}\right)^\mu$. By utilizing (14) in (13) we have

$$e_n \le \frac{h^2}{\alpha}\sum_{j=0}^{n-1} e_j + \eta(h,t_n), \quad n \ge 1 \tag{15}$$

Taking the modulus in (15) we have

$$|e_n| \le \frac{h^2}{\alpha}\sum_{j=0}^{n-1} |e_j| + |\eta(h,t_n)|, \quad n \ge 1 \tag{16}$$

On the other hand from (10) we have

$$|\eta(h,t_n)| = \left|\int_0^{t_n}\frac{s^{\mu-1}}{t_n{}^\mu}u(s)ds - \frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_j u(t_j)\right|$$
$$= \left|\frac{h}{t_n{}^\mu}\sum_{j=0}^{n-1} D_j\left(u(s) - u(t_j)\right)\right|$$

but

$$D_j := \int_{t_j}^{t_{j+1}} s^{\mu-1} ds$$

Therefore,

$|\eta(h,t_n)| \le \frac{h}{t_n{}^{\mu}} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} s^{\mu-1} |\left(u(s) - u(t_j)\right)| ds$ (17)

by applying the mean value theorem in (17), we have

$$|\eta(h,t_n)| \le \frac{h^2}{t_n{}^{\mu}} \max_{s\in[\alpha,T]} |u'(s)| \int_{\alpha}^{t_n} s^{\mu-1} ds$$

Defining $M(\alpha) := \max_{s\in[\alpha,T]} |u'(s)|$

$$|\eta(h,t_n)| \le \frac{M(\alpha)h^2}{t_n{}^{\mu}} \int_{\alpha}^{t_n} s^{\mu-1} ds$$
$$= \frac{M(\alpha)h^2}{\mu}\left(\frac{t_n{}^{\mu} - \alpha^{\mu}}{t_n{}^{\mu}}\right)$$
$$= \left(1 - \frac{\alpha^{\mu}}{t_n{}^{\mu}}\right)\frac{M(\alpha)h^2}{\mu}$$

we obtained the following bound

$|\eta(h,t_n)| \le \left(1 - \frac{\alpha^{\mu}}{t_n{}^{\mu}}\right)\frac{M(\alpha)h^2}{\mu}$ (18)

substitute (18) into (16) we have

$|\, e_n| \le \frac{h^2}{\alpha} \sum_{j=0}^{n-1} |e_j| + \left(1 - \frac{\alpha^{\mu}}{t_n{}^{\mu}}\right)\frac{M(\alpha)h^2}{\mu}, \quad n \ge 1$ (19)

by applying the special Gronwall lemma for the discrete in (19) we have

$$|\, e_n| \le \left(1 - \frac{\alpha^{\mu}}{t_n{}^{\mu}}\right)\frac{M(\alpha)h^2}{\mu} \prod_{j=0}^{n-1}\left(1 + \frac{n-1}{\alpha}\right)$$

we obtained the error bound as

$$|\, e_n| \le \left(1 - \frac{\alpha^{\mu}}{t_n{}^{\mu}}\right)\frac{M(\alpha)h^2}{\mu} e^{\left(\frac{T-1}{\alpha}\right)}$$

Hence, a second order convergence follows.

NUMERICAL RESULTS

In this section we tested the scheme using Maple13 version 10 with the stopping rule as

$$|u_n^h - u(t)| \le 10^{-4}$$

Problem 1 Given $g(t) = 1 + t + t^2$and $0 < \mu \le 1$ in (1), then using (*) we obtained the general form of its family of solutions:

$u(t) = c_0 t^{\mu-1} + \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t + \frac{\mu+2}{\mu+1} t^2$ (20)

where $c_0$is an arbitrary constant. The exact solution (20) when $\mathrm{t} = 0.99$ is compared with numerical solution (8) and errors are presented in Table (1)

Table 1: The results obtained by the numerical scheme (8) on problem1.

n	u_n^h (8)	$\lvert u_n^h - u(t)\rvert$
80	2.4241	5.759E − 1
82	2.4817	5.183E − 1
84	2.5401	4.599E − 1
86	2.5991	4.009E − 1
88	2.6588	3.412E − 1
90	2.7193	2.807E − 1
92	2.7804	2.196E − 1
94	2.8422	1.578E − 1
96	2.9048	9.520E − 2
98	2.9681	3.190E − 2
99	2.9999	1.000E − 4

Table (1) Shows that the results of problem1 obtained from scheme (10) is a good resultsat $\mathrm{n} = 99$ with an error of1.000E − 4 compared with Euler's method in Diogo et al. (2006).

Problem 2 Given $g(t) = 1 + t$and $0 < \mu \le 1$ in (1), then using (*) we obtainedthe general form of its family of solutions:

$$u(t) = c_0 t^{\mu-1} + \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu} t \tag{21}$$

where $c_0$is an arbitrary constant. The exact solution (21) when $t = 0.99$ is compared with numerical solution (8) and errors are presented in Table (2)

Table 2: The results obtained by the numerical scheme (8) on problem2.

n	u_n^h (8)	$\lvert u_n^h - u(t)\rvert$
80	1.7679	2.321E − 1
82	1.7928	2.072E − 1
84	1.8175	1.825E − 1
86	1.8422	1.578E − 1
88	1.8667	1.333E − 1
90	1.8911	1.089E − 1
92	1.9154	8.460E − 2
94	1.9397	6.030E − 2
96	1.9638	3.620E − 2
98	1.9879	1.210E − 2
99	1.9999	1.000E − 4

Table (2) shows that the numerical results of problem2 obtained from the scheme (8) is the best result at $n = 99$ with an error of1.000E − 4 when compared with Euler's method in Diogo et al. (2006).

The comparison of the numerical schemes

Here we presented the scheme (8) derived from Midpoint's rule when compared with Euler's method in Diogo et al. (2006).

Table 3: The comparison of scheme (8) and Euler's methods in Diogo et al. (2006) using errors of problem 1 and 2.

N	scheme (8) Error1	scheme (8) Error2	Euler's method Errors
80	5.759E − 1	2.321E − 1	3.919E − 1
82	5.183E − 1	2.072E − 1	4.173E − 1
84	4.599E − 1	1.825E − 1	4.423E − 1
86	4.009E − 1	1.578E − 1	4.671E − 1
88	3.412E − 1	1.333E − 1	4.817E − 1
90	2.807E − 1	1.089E − 1	5.159E − 1
92	2.196E − 1	8.460E − 2	5.400E − 1
94	1.578E − 1	6.030E − 2	5.638E − 1
96	9.520E − 2	3.620E − 2	5.874E − 1
98	3.190E − 2	1.210E − 2	6.108E − 1
99	1.000E − 4	1.000E − 4	6: 224E − 1

Table (3) shows that the errors obtained from the scheme (8) is an improvement when compared with the work of Diogo et al. (2006), which uses Euler's method. Since the error decreases when the number of iterations are increased. This shows that the scheme obtained has a better result when compared with the Euler's method with number of iterations up to 1600 corresponding to an error of 4.82E − 2.

CONCLUSION

We presented new numerical scheme for solving singular Volterra integral equations, where the functions under the integrals were approximated by means of Midpoint rule.The error bound estimation were established for the convergence of the new scheme obtained. The some problems were used to test the effectiveness and accuracy of the scheme and compared with other existing method.

REFERENCES

P. Lima and T. Diogo (2002) Numerical solution of a nonuniquely solvable Volterra integral equation using extrapolation methods, Journal of Computational and Applied Mathematics 140 537-557.

P. Lima, T. Diogo (1997) An extrapolation method for a Volterra integral equation withweakly singular kernel, Applied Numerical Mathematics vol. 24, 131-148.

A.M. Wazwaz (2011) Linear and Nonlinear Integral Equations Methods and Applications, Springer Heidelberg Dordrecht London New York, Library of congress control no. 201192916.

T. Diogo, J.T Edwards, N.J Ford and S.M Thomas (2005) Numerical analysis of a singular integral equation, Applied Mathematics and Computation, 167, 372-382.

Teresa Diogo, Neville J. Ford, Pedro Lima and SvilenValtcheva (2005) Numerical methods for a Volterra integral equation with non-smooth solutions, Journal of Computational and Applied Mathematics, 189, 412423.

M. Rahman (2007) Integral Equations and their Applications, Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK, Library of Congress Catalog Card Number: 2007922339.

M.T. Diogo, P.M. Lima and M.S. Rebelo (2004) Comparative study of numerical methodsfor a nonlinear weakly singular Volterra integral equation, Journal of mathematics and mathematical sciences, vol.8.

T. Diogo, N.J. Ford, P.M. Lima and S.M. Thomas (2006) Solution of a Singular Integral Equation by a Split-Interval Method, International Journal of Numerical Analysis and Modelling. Number 1,Vol. 4 Pages 63-73.

T. Diogo, P. Lima (2007) Collocation Solutions of a Weakly Singular Volterra Integral Equation, TEMA Tend. Mat. Apl. Comput, 8, No. 2, 229-238.

T. Diogo, P. Lima (2008) Superconvergence of collocation methods for a class of weakly singular Volterra integral equations Journal of Computational and Applied Mathematics, Vol. 218, pp.307316.

C. Yang and J. Hou (2013) Numerical method for solving Volterra integral equations with a convolution kernel, Vol. IJAM, 43403..

T. Diogo (2009) Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations, Journal of Computational and Applied Mathematics Vol.229, pp.363372.

R.N. Prajapati, R. Mohan and P. Kumar (2012) Numerical Solution of Generalized Abels Integral Equation by Variational Iteration Method, American Journal of Computational Mathematics, Vol. 2, 312-315

S. Rahbar and E. Hashemizadeh (2008) A computational approach to the Fredholm Integral Equations of the second kind, Vol.2 pp 978-988.

Fawzi Abdelwahid (2010) Adomian Decomposition Method Applied to Nonlinear IntegralEquations, Alexandria Journal of Mathematics Vol. 1, No. 1, 2090-4320.

M.A Al-Jawary and A.MShehan (2015) Exact solutions for weakly singular Volterra integral equations by using an e_cient iterative method, IOSR Jour nal of Mathematics(IOSR-JM), vol.11, pp, 67-76.

A. Wazwaz, R. Rach and J.S. Duan (2013) The modified Adomian decomposition methodand the noise terms phenomenon for solving nonlinear weakly-singular Volterra and Fredholm integral equations, Central European Journal of Engineering Vol.4 pp 669-678.