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APPLICATION OF POWER NUMERICAL METHOD FOR THE STATIONARY DISTRIBUTION OF MARKOV CHAIN

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ABSTRACT

The evolution of this model is represented by transitions from one state to the next. Also, the physical or mathematical behavior of this system can also be illustrated by identifying all of the possible states and explaining how it transitions between them. The iterative solution approaches for the stationary distribution of Markov chains, which begin with an initial estimate of the solution vector and it becomes closer and closer to the true solution with each iteration are investigated. Our goal is to compute solutions of stationary distribution of Markov chain by utilizing the power iterative method which leaves the transition matrices unchanged and saves time by considering the discretization effect, and the convergency. Matrices operations such as multiplication with one or more vectors, lower, diagonal and upper concepts of matrix, with the help of several existing Markov chain laws, theorems, formulas, and the normalization principle are applied. For the illustrative examples, the stationary distribution vectors $\pi_i,\ i = 1, 2, \dots, n$ and table of convergence are obtained.

Keywords: Eigenvalue, infinitesimal generator**,** normalization, power method, stationary distribution

INTRODUCTION

The best method to determine the stationary distribution of a finite, ergodic, discrete-time Markov chain is numerical power solution approach, and there are two types of solution methods in numerical analysis: iterative solution methods and direct solution methods. Iterative techniques begin with a rough estimate of the solution vector, which is subsequently adjusted until it gets closer to the true solution with each step or iteration. It eventually finds the correct solution. If an initial approximation is unknown, a guess is made or an arbitrary initial vector is utilized instead. When a certain number of well-defined steps have been performed, the solution must be computed.

Iterative approaches in one form or another are the most extensively used methods for obtaining the stationary probability vector from either the stochastic transition probability matrix or the infinitesimal generator. Conventional iterative procedures indicate that the matrices are only used for one operation: multiplication with one or more vectors, leaving the transition matrices untouched. Compact storage strategies can thus be easily developed, reducing the amount of memory required to store the matrix while still being well suited to matrix multiplication, such approaches can save a lot of time because the matrices involved are often large and sparse. When employing direct equation solving methods, eliminating one nonzero matrix element during the reduction phase often results in the generation of numerous nonzero elements in previously zero positions.

This is called fill-in, and it complicates the construction of a compact storage system because it requires provisions for object deletion and insertion. Furthermore, the amount of fill-in might frequently be so big that it consumes all available memory. Other advantage of iterative approaches is that it is possible to utilize good initial approximations to the solution vector, which is particularly advantageous when doing a sequence of connected test. As a result, it is reasonable to expect the new experiment's result to be similar to the old one, and it is preferable to use the old result as the new initial approximation.

The new result is computed in a few iterations if the difference is not significant. Furthermore, if a pre-specified tolerance criterion is met, an iterative process may be suspended, which could be a rather lax criterion. For example, if a mathematical model contains errors of the order of 5–10 percent, computing the solution to full machine accuracy may be inefficient.

Moreover, using iterative techniques, the accumulation of rounding error is almost non-existent because the matrix is never modified. For these reasons, iterative procedures have generally been recommended to direct ones. In the implementation of direct approaches, the data structure employed to represent the coefficient matrix is a source of concern. Markov models generate matrices that are frequently too large to be stored in computer memory using ordinary two-dimensional arrays. Because these matrices are often sparse, a packing method that saves only the nonzero elements and their locations in the matrix is cost-effective, if not required. Direct approaches are generally not recommended when the transition matrix is large and not banded, because the volume of fill-in can quickly surpass available storage capacity.

Romanovsky (1970) pioneered the use and simulation of discrete Markov Chains, while Ramaswami and Neuts (1980) expressed some explicit formulas and computational methods for infinite server queues with phase type arrivals and Ramaswami (1988) introduced the stable recursion for the steady state vector in Markov chains of M/G/1 type, while Philippe and Sidje (1993) introduced the transient solution of Markov processes by Krylov Subspaces and this was followed by Dayar (1998) that Permuted Markov chains to nearly completely decomposable of reducible and irreducible forms. Stewart (1994, 2009) expanded the numerical solutions of Markov chains, and Pesch et al ((2015) demonstrated the Markov chain techniques suitability for wind energy in Germany. Uzun and Kiral (2017) utilized the Markov chain model of fuzzy state to forecast gold price movement and

calculate the probabilistic transition matrix of gold price closing returns, whereas Aziza et al. (2019) used the Markov chain model of fuzzy state to forecast monthly rainfall data.
Clement (2019) applied Markov chain to show how Hepatitis B became more infectious over time than tuberculosis and HIV, whereas Vermeer and Trilling (2020) used Markov chain to show how it may be used in journalism. Agboola (2021) computed the solutions and algorithms for lower – upper triangular matrix approach and the Grassmann–Taksar–Heyman which is an extension of Gaussian elimination, while Agboola and Ayoade (2021) used matrix geometric and analytical block numerical iterative approaches. Agboola and Ayinde (2021) analysed the performance measure analysis on the states classification of Markov chain into reducible, irreducible states, and Agboola and Badmus (2021a) performed the application of Runge-kutta and backward differentiation methods for solving transient distribution in Markov chains, while Agboola and Badmus (2021b) analysed the distribution function of the renewal process and sequence $\{X_n,\ n \geq i\}$ using the concept of discrete time Markov chain to obtain the probability of exactly $n$renewals by time t.
Agboola (2022a) considered the left-hand eigenvector u_i of length n_i corresponding to the eigenvalue λ_{i_1} closest to 1 in each block $i,\ 1 \leq i \leq N$ and the weights ξ_i, an approximate solution to the stationary probability vector π to computed the global solution $\pi^* = \xi_i u_i = (\xi_1 u_1,\ \xi_2 u_2,\ \xi_3 u_3,\ \ldots,\ \xi_N u_N)$, whereas Agboola (2022b) developed the states of the Markov chain with the integers $0, \pm 1, \pm 2, \ldots$(the drunkard's straight line) where the only transitions from any state k are to neighbouring states $(k+1)$; a step to the right with probability p and $(k-1)$a step to the left with probability $q = (1-p)$ to determining whether the gambler is ruined. Agboola and Ayinde (2022) demonstrated the use of successive overrelaxation algorithmic and block numerical iterative solutions for the stationary distribution in Markov chain.
Agboola and Ayoade (2022) discussed the computation of the elements of the reachability matrix *F* which *are* separated into different categories depending on the classification of the initial and terminal states, such as, when both states are recurrent and belong to the same closed communicating class, when both states are recurrent but belong to different closed communicating classes, when state i is recurrent and state *j* is transient and when both states are transient, while Agboola, and Nehad (2022) applied matrix scaling and powering methods of small state spaces for solving transient distribution in Markov chain. However, in this research, the Power technique numerical iteration is used to compute the stationary distribution of a Markov chain which involved the use of concept of Eigen values and Eigen vectors, matrices formulae, discretization effect and the normalization principle. Also, for the illustrative examples, the stationary distribution vectors $\pi_i,\ i = 1, 2, \ldots, n$ and table of convergence are obtained.

Notation

P, the finite irreducible discrete-time Markov chain's transition probability matrix; π, the left-hand eigenvector corresponding to P's dominant (and simple) unit eigenvalue; Q, the infinitesimal generator**;** Δt, time interval; $0(\Delta t)$, big order of Δt; k, number of transition or a normalization factor; z, equation un-known variable; P^T, the transpose of the transition probability matrix; A, Probability matrix; $\lambda_i\ \forall\ i = 1, 2, \ldots, n$, eigen values; $x_i\ \forall\ i = 1, 2, \ldots, n$, eigen vectors; $\sum_{i=1}^{n} \alpha_i x_i$, linear combination of matrix A;$q_i \forall\ i = 1, 2, \ldots, n$, element of the infinitesimal generator Q;

MATERIALS AND METHODS

This research focused on power solution method analysis for stationary Markov chain distributions. This refers to iterative methods for solving systems of equations that begin with an estimate of the solution, or a guess, and then use numerical operations to bring the approximation closer to the true solution. Because the coefficient matrix is not modified throughout the execution of the algorithm, iterative algorithms are well suited to compacted storage methods. Iterative techniques rate of convergence, or the rate at which the initial approximation approaches the solution, is a constant source of concern.We will begin with finite, irreducible Markov chains, in which the elements have a single stationary probability distribution that is strictly greater than zero. When the Markov chain is similarly aperiodic, this unique stationary distribution is also the steady-state distribution. Let P be the transition probability matrix of a finite, irreducible discrete-time Markov chain. Then, πnormalized to 1, is the left-hand eigenvector corresponding to the dominant of P unit eigenvalue.

$$\pi P = \pi \quad with\ \pi e = 1\,. \tag{1}$$

The system of linear equations can be used to find the stationary distribution of a Markov chain that evolves in continuous time rather than discrete time and has an infinitesimal generator indicated by Q.

$$\pi Q = 0 \quad with\ \pi e = 1. \tag{2}$$

Both of these equations (1) and (2) can be stated in the same manner. The first, $\pi P = \pi$, can be represented as $\pi(P - I) = 0$, which is equivalent to Equation (2). Note that $(P - I)$ possesses all of the characteristics of an infinitesimal generator, including nonnegative off-diagonal elements, zero row sums, and diagonal elements equal to the negated sum of off-diagonal row elements. On the other hand, a continuous-time Markov chain can be discretized.
From Equation (2), we may write

$$\pi(Q\Delta t + I) = \pi \quad with\ \pi e = 1. \tag{3}$$

As a result, it is given in the form of an Equation (1). In the discretized Markov chain, transitions occur at intervals t, with t being short enough that the chance of two transitions occurring in time t is trivial, i.e., of order $0(\Delta t)$. Taking a class is one possibility.

$$\Delta t = \frac{1}{Max_i \|q_{ii}\|} \tag{4}$$

The stationary probability vector of the continuous-time Markov chain produced from $\pi Q = 0$is similar to that of the discretized chain obtained from $\pi(Q\Delta t + I) = \pi$; however, the matrix $(Q\Delta t + I)$ is stochastic in this instance. As a result, numerical methods for determining the stationary distributions of discrete-time Markov chains can now be used to the stationary distributions of continuous-time Markov chains and vice versa.
Estimating the probability distribution at each time step until no further changes are seen is one of the most easy approaches for calculating the stationary distribution of a discrete-time Markov chain.

The Power Method

The first approach that comes to mind when we need to determine the stationary distribution of a finite, ergodic, discrete-time Markov chain is to let the chain evolve over time, step by step, until it reaches its stationary distribution. Because $zp = z$ at that point, we can call the probability

vector stationary when it does not change while the process advances from step n to step $(n+1)$.

Let A be a square matrix with n dimensions. The power approach is described by the iterative procedure.

$$z^{(k+1)} = \frac{1}{\varepsilon_k} Az^{(k)} \quad (5)$$

Where $\varepsilon_k = \left\|Az^{(k)}\right\|_\infty$, $z^{(0)}$ is an arbitrary starting vector and k is a normalization factor. Although a normalizing phase is included in this version of the power technique at each iteration, in which each element of the newly formed iterate is divided by ε_k.

To investigate the rate of convergence of the power approach, let A have an eigen solution.

$$Ax_i = \lambda_i x_i, \quad i = 1, 2, \dots, n \quad (6)$$

and suppose that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|. \quad (7)$$

By Considering the possibility of expressing the initial vector as a linear combination of A's eigenvectors, i.e.

$$z^{(0)} = \sum_{i=1}^{n} \alpha_i x_i.$$

The rate of convergence of the power technique can then be calculated using the relationship.

$$z^{(k)} = A^{(k)} z^{(0)} = A^{(k)} \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \alpha_i A^{(k)} x_i = \sum_{i=1}^{n} \alpha_i \lambda_i^k x_i = \lambda_1^k \left\{ \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i \left(\frac{\lambda_i}{\lambda_1}\right)^{\boldsymbol{k}} x_i \right\} \quad (8)$$

It is possible to see how the process converges to the dominant eigenvector x_i. The ratios $\frac{|\lambda_i|}{|\lambda_1|}$ determine the rate of convergence for $i = 2, 3, \dots, n$. The faster the right-hand side summation approaches zero, the smaller these ratios get. The convergence rate is determined by the magnitude of the subdominant eigenvalueλ_2. The power method will not work properly when $|\lambda_2| \approx |\lambda_1|$ is used. There are obviously significant challenges when $|\lambda_2| \approx |\lambda_1|$.

RESULTS AND DISCUSSIONS

This section discusses the derivation of results such as probability distribution, stationary distribution, eigen values, eigen vectors, rate of convergence of the power technique and table of convergence in power method and this is demonstrated by the illustrative examples.

Illustrative example 1: Consider a Markov chain in discrete time with a transition probability matrix, and if the system starts in state 1, the initial probability vector is

$$P = \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.0 & 0.9 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$

If the system starts in state 1, the initial probability vector is

$$\pi^{(0)} = (1 \quad 0 \quad 0)$$

After the initial transition, the system will be in state 2 with a probability of.8, or state 3 with a probability of.2. The vector $\pi^{(1)}$ denotes the probability distribution after one transition (or step), and this result can be obtained by creating the product $\pi^{(0)}P$. The probability of being in state 1 after two time steps is calculated by multiplying the likelihood of being in state i after one step (given by π_i^1) by the probability of making a transition from state $i$to state 1. We have

$$\sum_{i=1}^{3} \pi_i^1 P_{i1} = (\pi_1^1 \times 0) + (\pi_2^1 \times 0) + (\pi_1^1 \times 0.6) = 0.12.$$

Similarly, after two steps, the system will be in state 2 with a probability of

$$0.08 = (0.0 \times 0.8) + (0.8 \times .1) + (0.2 \times .0),$$

and in state 3 with a probability of

$$0.8 = (0.0 \times 0.2) + (0.8 \times 0.9) + (0.2 \times 0.4).$$

After two steps, we have the following probability distribution, given that the system starts in state 1:

$$\pi^{(2)} = (0.12 \quad 0.08 \quad 0.8)$$

Also, $\pi^{(2)}$can be obtained by multiplying $\pi^{(1)}$and P together.

$$\pi^{(2)} = (0.12 \quad 0.08 \quad 0.8)$$
$$= (0 \quad 0.8 \quad 0.2) \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.0 & 0.9 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$
$$= \pi^{(1)} P.$$

Therefore, for any integer k, multiply the probability vector acquired after $(k-1)$ transitions by P to get the state of the system after k transitions. Thus

$$\pi^{(k)} = \pi^{(k-1)} P = \pi^{(k-2)} P^2 = \cdots = \pi^{(O)} P \quad (9)$$

At step k = 25, we find the probability distribution to be

$$\pi = (0.2813 \quad 0.2500 \quad 0.4688)$$

Correct to four decimal places after that.

$$(0.2813 \quad 0.2500 \quad 0.4688) \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.0 & 0.9 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}$$
$$= (0.2813 \quad 0.2500 \quad 0.4688)$$

which is now thought to be the stationary distribution (correct to four decimal places). When the Markov chain is finite, aperiodic, and irreducible (as in Example 1), the vectors $\pi^{(k)}$ converge to the stationary probability vector, regardless of the starting vector. Thus

$$\lim_{k \to \infty} \pi^{(k)} = \pi \quad (10)$$

The stationary probability vector is obtained using the power method, also known as power iteration. The convergence aspects of the power techniqueis then looked at in the context of determining the right-hand eigenvector corresponding to a dominating eigenvalue of a matrix, A. In a Markov chain setting, however, the matrix A must be substituted by P^T, the transpose of the transition probability matrix, because the left-hand eigenvector produces the stationary distribution of a Markov chain.

Illustrative example 2: Returning to the 3x3 matrix from illustrative example 1, The eigenvalues of P in the 3x3 matrix from illustrative example 1 are $\lambda_1 = 1$ and $\lambda_{2,3} = -0.25 \pm .5979i$. As a result, $|\lambda_2| \approx 0.65$ is obtained. It is worth noticing that $0.65^{10} \approx .01, 0.65^{25} \approx 2 \times 10^{-5}$ and $0.65^{100} \approx 2 \times 10^{-19}$. Table 1 displays the probability distribution of the states of this example at defined steps for each of three alternative starting configurations. After 25 cycles, no further changes in the first four values are detected for either of the starting settings.

The table shows that two decimal places of accuracy were acquired after 10 iterations and four decimal places were obtained after 25 iterations, which matches to convergence criterion calculated from the subdominant eigenvalue. Furthermore, for 25 iterations, the solution is confirmed to be correct to full machine accuracy, as represented by the value of 0.65^{25} and, the magnitude of $|\lambda_2|^k$ does not ensure a definite number of decimal places of accuracy in the answer. A relative error norm of 10^{-j} yields approximately j decimal places of accuracy; and because both matrices and vectors have unit 1-norms, the Markov chain context allows for a lot of flexibility.

Table 1: Table of Convergence in Power Method

Step	Initial state			Initial state			Initial state		
	1.0	0	0	0	1	0	0	0	1
1	0.0	0.8	0.2	0.0	0.9	0.1	0.6	0.0	0.4
2	0.12	0.08	0.8	0.54	0.01	0.45	0.24	0.48	0.28
3	0.48	0.104	0.416	0.27	0.443	0.297	0.168	0.24	0.592
4	0.2496	0.3944	0.356	0.1782	0.26930	0.5626	0.3552	0.1584	0.4864
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.2860	0.2555	0.4584	0.2731	0.2573	0.4696	0.2827	0.2428	0.4745
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	0.2813	0.2500	0.4688	0.2813	0.2500	0.4688	0.2813	0.2500	0.4688

In the general formulation of the power technique, it is frequently necessary to normalize following iterates, because otherwise the term λ_1^K may cause succeeding approximations to become too high (if $\lambda_1 > 1$) or too small (if $\lambda_1 < 1$), resulting in overflow or underflow. This normalization is also necessary for providing a standardized vector for convergence testing. However, in Markov chain problems, the coefficient matrix has 1 as the dominating eigenvalue ($\lambda_1 = 1$), thus the power method's requirement for periodic normalization of iterates disappears. If the first approximation is a probability vector, all subsequent approximations will be probability vectors. When utilizing the power approach in the context of a Markov chain, the left-hand eigenvector corresponding to a unit eigenvalue is required, therefore the matrix to which the method is applied is P^T, and the above iteration takes the form

$$z^{(k+1)} = P^T z^{(k)} \quad (11)$$

A stochastic matrix's unit eigenvalue is a dominating eigenvalue, and if the matrix is irreducible, there are no additional unit eigenvalues. Other eigenvalues on the unit circle that are not 1, but have the same modulus as 1, exist when the matrix is periodic. A straightforward application of the power approach will fail in this case. This difficulty can be avoided by making a little adjustment to the unit eigenvalue and its associated eigenvector. The connection $|\lambda_2| \approx |\lambda_1|$ is commonly used to obtain the matrix P from the infinitesimal generator Q such that

$$P = (Q\Delta T + I), \quad (12)$$

Where $\Delta T \leq \frac{1}{Max|q_{ii}|}$ is used. If T is set in such a way that$\Delta T < \frac{1}{Max|q_{ii}|}$, the resulting stochastic matrix has diagonal elements $P_{ii} < 0$ and so cannot be periodic. Under these conditions, the power technique is certain to converge (irreducible and aperiodic). The rate of convergence is determined by the ratio $\frac{|\lambda_2|}{|\lambda_1|}$, i.e. $|\lambda_2|$. Unfortunately, there may be a large disconnect between theoretical convergence criteria for iterative methods and observed behavior in practice. What will converge in principle may require so many iterations that the method should be considered unfeasible in all practical applications. The power method occurs when the modulus of the subdominant eigenvalue, $|\lambda_2|$, is near to unity. For example, nearly completely decomposable (NCD) stochastic matrices are often used in the modeling of physical and mathematical systems; such matrices must have subdominant eigenvalues close to 1. In these situations, the power technique will converge very slowly. Given that convergence frequently necessitates multiple iterations, it may be assumed that squaring the matrix P repeatedly is a more cost effective technique. Let $c = 2^m$ for some integer $\pi^{(k)} = \pi^{(k-1)}P$),the basic iterative formula is P which requires k iterations to get $\pi^{(k)}$ with each iteration involving a matrix-vector product. P^{2m} can be quickly computed by iteratively squaring the matrix with only m matrix products, resulting in $\pi^{(c)} = \pi^{(0)}P^c$. Because a matrix vector product requires $n^{(2)}$ multiplications and a matrix product requires $n^{(3)}$, the squaring approach should be employed when $mn^{(3)} < 2^m n^{(2)}$, i.e. when $nm < 2^m$.Unfortunately, in this approach, the fact that the matrix P is typically large and sparse is simply ignored. As a result, a matrix-vector product requires only n_Z multiplications, where n_Z is the number of nonzero items in P. The number of nonzero members in the matrix will be increased by the matrix-squaring procedure. In reality, for an irreducible matrix, convergence will not be achieved until all of the elements are nonzero, which will increase both the number of multiplications and the amount of memory required. Memory constraints, rather than temporal constraints, may limit the application of matrix powering and the effect of discretization may be find when the values of $\Delta t > 0$ for which the matrix $P = Q\Delta t + I$ is stochastic.

Illustrative example 3: Consider a two-state Markov chain whose infinitesimal generator is

$$Q = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

with $q_1, q_2 \geq 0$. The transition probability matrix is then

$$P = Q\Delta t + I = \begin{pmatrix} 1 - q_1\Delta t & q_1\Delta t \\ q_2\Delta t & 1 - q_2\Delta t \end{pmatrix}.$$

The row sums are obviously equal to one. We need $0 \leq q_1\Delta t \leq 1$ and $0 \leq q_2\Delta t \leq 1$ to verify that $0 \leq \Delta t \leq {q_1}^{-1}$ and $0 \leq \Delta t \leq {q_2}^{-1}$.Assuming that $q_1 \geq q_2$. Then $0 \leq \Delta t \leq {q_1}^{-1}$fulfills both requirements. To ensure that $0 \leq 1 - q_1\Delta t \leq 1$ and $0 \leq 1 - q_2\Delta t \leq 1$ are same, $\Delta t \leq {q_1}^{-1}$ must be equal. As a result, the highest value may be given to under the assumption that P is stochastic and $\Delta t = 1/\mathrm{Max_i}|q_1|$|. Given that Q is an infinitesimal generator, same results hold for a general stochastic matrix $P = Q\Delta t + I$to be stochastic. The row sums of P are unity for any value of t, andsince the row sums of Q are 0 by definition, as a result, we must be concerned with the values of t that ensure that the members of P fall within the interval [0,1]. Let q be the biggest off diagonal element size:

$$q = max_{i,j\ i\neq j}(q_{ij}) \text{ and } q_{ij} \geq 0 \quad \text{for all i, j} \quad (13)$$

$0 \leq p_{ij} \leq 1$ holds if $0 \leq q_{ij}\Delta t \leq 1$, which is true if $\Delta t = {q_1}^{-1}$.

Now consider a diagonal element $p_{ii} = q_{ii}\Delta t + 1$. We have

$$0 \leq q_{ii}\Delta t + 1 \leq 1 \quad (14)$$

Or

$$-1 \leq q_{ii}\Delta t \leq 0 \quad (15)$$

Because q_{ii} is negative, the right-hand inequality holds for every $\Delta t \geq 0$, $\Delta t \leq -q_{ii}^{-1}$, i.e., the left-hand inequality $q_{ii}\Delta t \geq -1$ is true,as a result, if $0 \leq \Delta t \leq (\mathrm{Max_i}|q_{ii}|)^{-1}$, then the matrix P is stochastic. We have $\mathrm{Max_i}|q_{ii}| \geq max_{i,j\ i\neq j}(q_{ij})$ because the diagonal elements of Q equal the negated sum of the off-diagonal elements in a row,as a result, Δt must be less than or equal to the reciprocal of the absolute value of Q 's largest diagonal element andsome iterative approaches for calculating the stationary probability vector from Equation (3) is used. The magnitude of the eigenvalues of P is intimately connected to the velocity of convergence, therefore choosing an appropriate value for Δt is critical.The magnitudes of the subdominant eigenvalues are closer to 1 as the convergence rate slows.As a result, we seek to maximize the distance between$\lambda_1 = 1$ and the subdominant eigenvalue (the eigenvalue that in modulus is closest to 1). As$\Delta t \to 0$approaches, all of P's eigenvalues converge to unity. This indicates that we set t to the largest possible value, with the restriction that P must be a stochastic matrix. Using $\Delta t = (\mathrm{Max_i}|q_{ii}|)^{-1}$ does not ensure that the dominant and subdominant eigenvalues are separated as much as feasible.

Illustrative example 4: Consider the (2 × 2) case as an example. The eigenvalues of *P* are the roots of the characteristic equation $|P - \lambda I| = 0$, i.e.,

$$\begin{vmatrix} 1 - q_1\Delta t - \lambda & q_1\Delta t \\ q_2\Delta t & 1 - q_2\Delta t - \lambda \end{vmatrix} = 0.$$

These roots are $\lambda_1 = 1$ and $\lambda_2 = 1 - \Delta t(q_1 + q_2)$, as $\Delta t \to 0$, $\lambda_1 \to \lambda_2 = 1$.
Also notice that theleft-hand eigenvector corresponding to the unit eigenvalue λ_1 is independent of the choice of Δt.
We have

$$\frac{q_2}{(q_1+q_2)}\ \frac{q_2}{(q_1+q_2)}\begin{pmatrix} 1 - q_1\Delta t & q_1\Delta t \\ q_2\Delta t & 1 - q_2\Delta t \end{pmatrix} = \frac{q_2}{(q_1+q_2)}\ \frac{q_2}{(q_1+q_2)}$$

This eigenvector is the stationary probability vector of the Markov chain, and as such, it must be independent of Δt. The parameter Δtaffects only the pace at which matrix iterative algorithms converge to this vector. With the exception that the matrix P must be a stochastic matrix, it is generally advantageous to pick t to be as large as possible. On the surface, it appears like selecting a large value for Δt will get us closer to the stationary distribution.

CONCLUSION

Iterative solution approaches for the stationary distribution of Markov chains that start with an initial estimate of the solution vector and then adjust it so that it gets closer and closer to the true solution with each step or iteration, which leaves the transition matrices unaltered and saves time, have been researched in attempt to provide some insight into the solutions of stationary Markov chain distributions which are investigated in this work, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our goal is to compute the answers utilizing the Power iterative method which leaves the transition matrices unchanged and saves time, by considering the discretization effect, and the convergency. Matrices operations such as multiplication with one or more vectors, lower, diagonal and upper concepts of matrix, with the help of several existing Markov chain laws, theorems, and formulas, the normalization principle are applied. For the illustrative examples, the stationary distribution vectors $\pi_i,\ i = 1, 2, \ldots, n$ and table of convergence are obtained.

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