SET-BASED ORDERING ON SIMPLE MULTISETS OF INCOMPARABLE OBJECTS

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ABSTRACT

It is convincing that there exists an ordering on simple multisets of incomparable objects by means of the Jouannaud-Lascanne set-based multiset ordering. A successful attempt to show that a singleton multiset is dominated by a simple pair multiset despite the incomparability of their objects is made with respect to the ordering. Furthermore, an existence of the ordering among simple multisets of higher cardinalities where the cardinality of the preceding multiset is a unit or more less than that of the succeeding multiset is observed. Thus, we obtain an extension of the ordering to simple multisets of incomparable objects. No stronger set-based multiset ordering may be found to exist.

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INTRODUCTION

Set-based multiset ordering was originally defined as a partition-based ordering by Jouannaud and Lascanne and was proved to be stronger than the Dershowitz-Manna multiset ordering. The ordering was redefined via the concept of *grid* alongside its submultiset-based counterpart for an efficient implementation of the orderings. Until now, only the Dershowitz-Manna multiset ordering has been found to be the strongest equipped with monotonicity property. See Dershowitz and Manna (1979), Jouannaud and Lescanne (1982), Huet and Oppen (1980) and Peter and Singh (2013) for further details.

One may ask if there exist a multiset ordering stronger than Jouannaud-Lascanne's (which includes an extension of it). The results obtained in this paper seem to answer the question in the affirmative. Some extensions of the ordering may exist and would be seen to accommodate simple multisets of incomparable objects. Specifically, the set-based multiset ordering may exist on simple multisets of incomparable objects and would be more powerful than presumed.

There is a need to recall the definition of *set ordering* since it is the ordering that exists among the *references* in the set-based grids of multisets. Moreover, we recall the Dershowitz-Manna ordering and most importantly the concept of grid, *difference grid* and the set-based ordering that depends on them. Thereafter, we investigate the existence of the ordering on simple multisets via some proved results.

SOME PRELIMINARY CONCEPTS

Many concepts on multiset and multiset ordering exist in the literature and have been captured in many published texts. We explain only those which are of interest to the current assertion. The reader is referred to Singh et. al (2007) for an overview of multisets.

Multiset

A multiset is an unordered collection of *objects* in which, unlike the elements of the Cantorian set, the objects are allowed to repeat where each occurrence of an object is referred to as *an element* of the multiset. The total number of occurrences of all the objects of a multiset is known as its *cardinality*.

Objects and elements

Further to their roles in the definition of multiset above, it is necessary to lay emphasis on the use of the words "element" and "object" when dealing with multisets. This is especially with respect to the standard use of "element" for the members of a set whereas the role of "element" for multiset is quite different – the word object is used to distinguish the members of a multiset, while an element is each individual duplicate of an object. Thus, the multiset $\{a, a, a, b, b, c\}$ has three objects and six elements (Singh et. al, 2007).

Simple multiset

A *simple multiset* is a multiset that has only one object. For example, the multisets $\{a, a, a, a, a\}$ and $\{b\}$ are simple multisets (Singh et. al, 2007).

Incomparability relation

Suppose $\prec$ is a strict partial order on a set S and let x and y be elements of S such that x is not equal to y. The elements x and y are *incomparable* with respect to $\prec$ if and only if x is not less than y and y is not less than x. Symbolically, we write

$$x \# y = \neg(x \prec y \text{ or } y > x \text{ or } x = y)$$

Set ordering

A set S is greater than a set T (written $S \gg T$) if and only if $\forall x \in T \backslash S, \exists y \in S \backslash T$ such that $y > x$ (Peter and Singh, 2013).

It is important to observe the differences between *set ordering* and *set-based ordering*. Set ordering is any basic basic ordering that exists among sets (including the references of the set-based grid of a multiset, since they are sets as well). On the other hand, set-based ordering (or set-based multiset ordering) is an ordering that exists among multsets but which has been defined using sets. As the name implies, set-based multiset ordering is defined using set ordering as we shall see in Subsection 3 of Section 3.

Dershowitz-Manna Multiset ordering

Let $\prec$ be a partial ordering defined on multisets M and N. Then $M \ll_{DM} N$ if there exist two multisets X and Y satisfying the following:

    i. $\emptyset \neq Y \subseteq N$,
    ii. $M = (N \backslash Y) \uplus X$, and
    iii. $(\forall x \in X)(\exists y \in Y)[y > x]$.

In other words, $M \ll_{DM} N$ if M is obtained from N by removing at least one element (those in Y) from N, and replacing each such element y by zero or any finite number of elements (those in X), each of which is strictly less than (in the ordering $\prec$) one of the elements y that have been removed (Dershowitz and Manna, 1979).

SET-BASED MULTISET ORDERING

We recall the notions of *set-based* grid and multiset ordering based on set-based grid.

Set-based grid

Let $\preceq$ be a partial order defined on a set S and let M be a multiset of cardinality n over S. The permutation $[M_i]$ of subsets $M_1, M_2, \ldots, M_m$ of M, $m \leq n$, is called the *set-based grid* of M if the following properties are satisfied:

    i. $\forall(x, y, i)[x \in M_i, \ y \in M_i] \Longrightarrow$
        $[x \# y]$.(*Incomparability property*)

    ii. $\forall(i < j)[x \in M_j] \iff \exists y(y \in M_i) \ [x \preccurlyeq y]$. (*Order property*)

Each subset M_i above is referred to as a *reference* in the grid. From Property (i), only incomparable objects belong to a reference. From Property (ii), an element x does not belong to a reference preceding the reference it belongs to if and only if an element greater than or equal to x belongs to the preceding reference.

Difference grid

Let $[M_i]$ and $[N_j]$ be the set based grids of multisets M and N for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, respectively. Let $q = \max\{m, n\}$ and let $k = 1, 2, \ldots, q$. We construct the set–based difference grid $[M_k, N_k]$ of M and N as follows:

    i. $M_q \neq \emptyset$ or $N_q \neq \emptyset$,
    ii.   If $M_k \neq \emptyset$ and $N_k \neq \emptyset$ for a given k then $M_{k-1} \neq \emptyset$ and $N_{k-1} \neq \emptyset$.

The M_k's are the references of M in the grid $[M_k, N_k]$. Correspondingly, the N_k's are the references of N in the grid $[M_k, N_k]$. Unlike the references in the grid of a multiset, the references in the difference grid of two multisets are empty up to the number of references with which the grid with more references exceeds the grid with fewer references. Also, the difference grid of two multisets is a collection of all the references from the individual grids of the multisets. The different grid of multisets can also be represented in the form of a table. For instance, consider the multisets $M = \{5, 4, 2, 2, 1, a, a, a, a, b, b\}$ and $N = \{6, 3, 3, a, a, a, b, b, c\}$ where the alphabets are incomparable with one another as well as with the integers. The difference grid of M and N is given in Table 3.1.

Table 3.1: The difference grid of $\{5, 4, 2, 2, 1, a, a, a, a, b, b\}$ and $\{6, 3, 3, a, a, a, b, b, c\}$

$\{5, a, b\}$	$\{4, a, b\}$	$\{2, a\}$	$\{2, a\}$	$\{1\}$
$\{6, a, b, c\}$	$\{3, a, b\}$	$\{3, a\}$	$\emptyset$	$\emptyset$

Set-based multiset ordering

Given that $[M_i]$ and $[N_j]$ are the respective set-based grids of the partially ordered multisets M and N, we say that M is *less than N* in the sense of set-based grid, written $M \ll_\mathbb{s} N$ if and only if for all k such that $M_k \neg\ll N_k$ there exists $l < k$ such that $M_l \ll N_l$, where $\ll$ is the set ordering.

The multiset M is said to "prececde" the multiset N (or N succeeds M). Henceforth, we shall refer to the set-based multiset ordering as simply set-based ordering or set-based dominance except where necessary, with the understanding that the ordering is on multisets.

SOME EXTENSIONS OF SET-BASED ORDERING TO SIMPLE MULTISETS OF INCOMPARABLE OBJECTS

We begin with a simple corollary which establishes a relationship between pair sets and simple multisets of finite cardinalities with respect to set-based ordering. That is, if S is a pair set and M is a simple multiset such that an object a belongs to both S and M, then S dominates M with respect to set-based ordering.

Corollary 4.1
Every pair set dominates a finite simple multiset with respect to set-based ordering where the set and the multiset have a common object.

Proof
Let S be the set $\{a, b\}$ and let M be a simple multiset of an object a. Since a belongs to both S and M then a belongs to each of the first set-based references of S and M as illustrated in Table 4.1.

Table 4.1: The set-based difference grid of the set $\{a, b\}$ and the multiset $\{a, a, …, a\}$

$\{a, b\}$	$\emptyset$	…	$\emptyset$
$\{a\}$	$\{a\}$	…	$\{a\}$

Since the first and only non-empty reference of S dominates the first reference of M by set ordering (or set inclusion in particular), then S dominates M by the set-based ordering defined above.   □

The two multisets $\{e, e\}$ and $\{a\}$ cannot be shown to be comparable using the conventional set-based approach. In fact, their difference grid has no basis for comparability. To see this, consider placing them in a difference grid in the following table:

Table 4.2: Set-based difference grid of $\{e, e\}$ and $\{a\}$

$\{e\}$	$\{e\}$
$\{a\}$	$\emptyset$

For two multisets to be comparable, one reference must be *greater* than the other in the last unequal pair of references in their difference grid. Such is not the case here. Thus, the method fails to determine the dominance between the multisets. The ordering between such multisets is better established in the following proposition which says that if e and a are incomparable objects with respect to a basic ordering $\succ$, then the relation $\{e, e\} \gg_{\scriptscriptstyle S} \{a\}$ holds.

Proposition 4.2
Every simple pair multiset dominates a singleton multiset with respect to set-based ordering where both multisets have incomparable objects.

Proof
Choose multisets $M = \{a, a, e\}$, $N = \{a, a, e, e\}$ and $X = \{a, a, a, e\}$. Now N is greater than M in the sense of set-based ordering. To see this, consider the difference grid of M and N in Table 4.3 below.

Table 4.3: Set-based difference grid of the multisets $\{a, a, e\}$ and $\{a, a, e, e\}$

$\{a, e\}$	$\{a\}$
$\{a, e\}$	$\{a, e\}$

The relation $N \gg_{\scriptscriptstyle S} M$ holds since $\{a, e\} \gg \{a\}$ holds from the second pair of references of M and N. Similarly, the relation $N \gg_{\scriptscriptstyle S} X$ holds. The proof is completed by the following multiset operations:

$$N \gg_{\scriptscriptstyle S} M$$

$$N \uplus \{a, e\} \gg_{\scriptscriptstyle S} M \uplus \{a, e\}$$

$$N \uplus \{a, e\} - X \gg_{\scriptscriptstyle S} M \uplus \{a, e\} - N$$

$$\{e, e\} \gg_{\scriptscriptstyle S} \{a\}$$

hence the proof.   □

It follows that the set-based ordering has an extension and is more powerful than one thought. Similar to the multisets in Proposition 4.2, the set-based ordering between the multisets $\{e, e, e, e\}$ and $\{a, a, a\}$ cannot be determined using the conventional difference grid approach. Consider then the following proposition, which is an extension of Proposition 4.2. It says that if e and a are incomparable objects with respect to a basic ordering $\succ$, then the relation $\{e, e, e, …, e\}$ ($n + 1$ times) $\gg_{\scriptscriptstyle S} \{a, a, a, …, a\}$ (n times) holds where n is a non-negative integer.

Proposition 4.3
Every simple multiset dominates another simple multiset with respect to set-based ordering where both multisets have incomparable objects and the cardinality of the preceding multiset is a unit less than that of the succeeding multiset.

Proof

The proof is by induction on n. The statement is true for $n = 1$, since $\{e, e\} \gg_s \{a\}$ holds by Proposition 4.2. Suppose it is true for k. That is,

$$\{e, e, e, \dots, e\}(k + 1 \text{ times}) \gg_s \{a, a, a, \dots, a\}(k \text{ times})$$

(4.1)

We need to show from (4.1) that it is true for $n = k + 1$. Now the relation $\{a, e\} \gg_s \{a, a\}$ holds by Corollary 4.1. Taking the additive union of $\{a, e\}$ with the L.H.S of (4.1) and that of $\{a, a\}$ with the R.H.S of (4.1) we obtain:

$$\{a, e\} \uplus \{e, e, e, \dots, e\} \, (k + 1 \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (k \text{ times}) \uplus \{a, a\}$$

This implies

$$\{e, e, e, \dots, e\}(k + 2 \text{ times}) \uplus \{a\} \gg_s \{a, a, a, \dots, a\} \, (k + 2 \text{ times})$$

Subtract $\{a\}$ from both sides to get

$$\{e, e, e, \dots, e\} \, (k + 2 \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (k + 1 \text{ times})$$

That is,

$$\{e, e, e, \dots, e\} \, ((k + 1) + 1 \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (k + 1 \text{ times})$$

It follows that

$$\{e, e, e, \dots, e\} \, (n + 1 \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (n \text{ times})$$

Since being true for $n = k$ implies being true for $n = k + 1$, then the statement is true for all $n \in \mathbb{N}$. □

We take the extension further to multisets where the cardinality of the succeeding multiset is more than a unit greater than the cardinality of the preceding multiset in Proposition 4.4 which says that if e and a are incomparable objects with respect to a basic ordering $\succ$, then the relation $\{e, e, e, \dots, e\} \, (n + m \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (n \text{ times})$ holds where m and n are non-negative integers with $m \neq 0$.

Proposition 4.4

Every simple multiset dominates another simple multiset with respect to set-based ordering where both multisets have incomparable objects and the cardinality of the preceding multiset is less than that of the succeeding multiset.

Proof

The proof is by induction on m, where m is the number by which the multiplicity of the elements of $\{e, e, e, \dots, e\}$ exceeds the multiplicity of the elements of $\{a, a, a, \dots, a\}$. The statement is true for $m = 1$ since $\{e, e, e, \dots, e\}$ $(n + 1$ times) is greater than $\{a, a, a, \dots, a\}$ (n times) by Proposition 4.3. Suppose the statement is true for k. That is,

$$\{e, e, e, \dots, e\} \, (n + k \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (n \text{ times})$$

(4.2)

Consider the relation $\{e\} \gg_s \emptyset$ and take the additive union of $\{e\}$ with the L.H.S. and that of $\emptyset$ with the R.H.S. of (4.2).

$$\{e\} \uplus \{e, e, e, \dots, e\} \, (n + k \text{ times}) \gg_s \emptyset \uplus \{a, a, a, \dots, a\} \, (n \text{ times}).$$

This implies

$$\{e, e, e, \dots, e\} \, (n + k + 1 \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (n \text{ times}).$$

It follows that

$$\{e, e, e, \dots, e\} \, (n + m \text{ times}) \gg_s \{a, a, a, \dots, a\} \, (n \text{ times}).$$

where m is the number by which the multiplicity of the elements of $\{e, e, e, \dots, e\}$ exceeds the multiplicity of the elements of $\{a, a, a, \dots, a\}$. □

CONCLUSION

The existence of a set-based ordering stronger than the one introduced by Jouanaud-Lascanne was investigated. Such an ordering is obtained by an extension of the said Jouannaud-Lascanne ordering to accommodate simple multisets of incomparable objects, hence stronger than one thought. It is also easy to see that no stronger multiset ordering may be found to exist on simple multisets.

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