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FUZZY SOFT SET AND ITS APPLICATION IN SELECTING BEST CANDIDATE(S) FOR A JOB USING AGGREGATE FUZZY SET APPROACH

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ABSTRACT

Molodtsov in 1999 introduced the concept of soft set theory as a general mathematical tool for handling uncertainties about vague concepts. In this paper, we recalled the definitions of soft set, fuzzy set and some basic operations in soft set. We presented the definition of fuzzy soft set and contributed some related algebraic properties with illustrative examples. We also defined extended intersection, restricted union, AND product, OR product and proved that associative laws holds with respect to AND and OR products. De Morgan's laws and inclusions were stated and proved in the background of fuzzy soft set with respect to various operations with some relevant examples. Finally, we presented the application of fuzzy soft set in multicriteria decision making in choosing a best candidate for a job using aggregate fuzzy set technique.

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INTRODUCTION

Most of the real life problems we encountered in economics, engineering, social and medical sciences do not always involve crisp data. For various kinds of uncertainties presented in these problems, the methods in traditional mathematics are not always successful. To overcome these uncertainties, a number of theories were developed, such as, theory of probability, fuzzy set theory (Prade and Duboise, 1980; Zadeh, 1965; Zimmerman, 1996), intuitionistic fuzzy sets (Attanasov, 1986; Attanasov, 1994), vague sets (Gau and Buehrer, 1993), theory of interval mathematics (Attanasov, 1994; Gorzalzany, 1987), rough set theory (Pawlak, 1982), etc and these may be used as an effective tools for dealing with different kind of uncertainties and imprecision embedded in a system. All these theories however are associated with inherent limitations, which is the inadequacies of the parameterization tools associated with these theories.

Molodtsov (1999), initiated a new concept called soft set theory as a new general mathematical tool for dealing with uncertainties which is free from the above limitations. The soft set introduced in Molodtsov (1999) is a set associated with a set of parameters and has been applied in several directions. After the introduction of soft set by Molodtsov, most of the operations in soft set theory were defined by Maji et al., (2003) and Ali et al.,(2009**).**

Cagman et al., (2011) defined fuzzy soft set, some operations and examine some basic properties. One of the advantages of fuzzy soft set over soft set and fuzzy set is that, it can be applied where the parameters are - fuzzy words or sentences.

In this paper, we contributed to the properties of operations in the fuzzy soft set. We define AND and OR operations in fuzzy soft set context with relevant examples. We state and prove. De Morgan's inclusions and laws with illustrative examples. We also apply fuzzy soft set to decision making in choosing a best candidate for a job.

The rest of the paper is organized as follows: In section 2, we present some basic definitions in fuzzy set and soft set. In section 3, we define fuzzy soft set and present its various properties with some illustrative examples. In section 4, we present the concept of aggregate fuzzy soft set. Section 5 presents the application of fuzzy soft set in multicriteria decision making. Section 6, summarizes the entire paper.

Preliminaries and Basic Definitions

Soft Set

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, $P(U)$ be the power set of U and A ⊆ E.

Definition 2.1.1 (Molodtsov, 1999)**.** A pair (Γ, A) is called a **soft set** over U, where Γ is a mapping given by $\Gamma: A \rightarrow P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $\Gamma(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set (Γ, A).

The soft set (Γ, A) can be represented as a set of ordered pairs as follows:

$(\Gamma, A) = \{(x, \Gamma(x)), x \in A, \Gamma(x) \in P(U)\}$

Definition 2.1.2. Let (Γ, A) and (G, B) be two soft sets over U. Then

(i) (Γ, A) is said to be a **soft subset** of (G, B), denoted by $(\Gamma, A) \tilde{\subseteq} (G, B)$, if $A \subseteq B$ and $\Gamma(x) \subseteq G(x), \forall x \in A$

(ii) (Γ, A) and (G, B) are said to be **soft equal**, denoted by $(\Gamma, A) \cong (G, B)$, if $(\Gamma, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (\Gamma, A)$

Definition 2.1.3. Let (Γ, A) be a soft set over U. Then , the support of (Γ, A) written **supp**(Γ, A) is the set defined as **supp**$(\Gamma, A) = \{x \in A: \Gamma(x) \neq \emptyset\}$.

(ii) (Γ, A) is called a **non-null** soft set if supp $(\Gamma, A) \neq \emptyset$.

(ii) (Γ, A) is called a **relative null** soft set denoted by $\emptyset_A \ if \ \Gamma(x) = \emptyset, \forall x \in A$

(iii) (Γ, A) is called a **relative whole** soft set, denoted by $U_A \ if \ \Gamma(x) = U, \forall x \in A$.

Definition 2.1.4. (Koyuncu and Tanay, 2016**).** Let (Γ, A) be a soft set over U. If $\Gamma(x) \neq \emptyset$ for all $x \in A$, then (Γ, A) is called a non-empty soft set.

Definition 2.1.5. Let (Γ, A) and (G, B) be two soft sets over U. Then the **union** of (Γ, A) and (G, B), denoted by $(\Gamma, A) \tilde{\cup} (G, B)$ is a soft set defined as $(\Gamma, A) \tilde{\cup} (G, B) = (H, C), where \ C = A \cup B \ and \ \forall x \in C$, $H(x) = \begin{cases} \Gamma(x), & if \ x \in A - B \\ G(x), & if x \in B - A \\ \Gamma(x) \cup G(x), & if \ x \in A \cap B \end{cases}$

Definition 2.1. 6. Let (Γ, A) and (G, B) be two soft sets over U. Then the **restricted union** of (Γ, A) and (G, B), denoted by $(\Gamma, A) \tilde{\cup}_R (G, B)$ is a soft set defined as;

$(\Gamma, A) \tilde{\cup}_R (G, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and $\forall x \in C$

$H(x) = \Gamma(x) \cup G(x)$.

Definition 2.1.7. Let (Γ, A) and (G, B) be two soft sets over U. Then the **extended intersection** of (Γ, A) and (G, B), denoted by $(\Gamma, A) \tilde{\cap}_E (G, B)$, is a soft set defined as $(\Gamma, A) \tilde{\cap}_E (G, B) = (H, C)$ where $C = A \cup B$ and

$$\forall x \in C, H(x) = \begin{cases} \Gamma(x), & if \ x \in A - B \\ G(x), & if x \in B - A \\ \Gamma(x) \cap G(x), & if \ x \in A \cap B \end{cases}$$

Definition 2.1. 8. Let (Γ, A) and (G, B) be two soft sets over U. Then the **restricted intersection** of (Γ, A) and (G, B) denoted by $(\Gamma, A) \Cap (G, B)$, is a soft set defined as $(\Gamma, A) \Cap (G, B) = (H, C)$ where $C = A \cap B$ and $\forall x \in C$, $H(x) = \Gamma(x) \cap G(x)$.

Definition 2.1.9. Let (Γ, A) and (G, B) be two soft sets over U. Then the **AND-product** or **AND-intersection** of (Γ, A) and (G, B) denoted by $(\Gamma, A) \tilde{\wedge} (G, B)$ is a soft set defined as

$(\Gamma, A) \tilde{\wedge} (G, B) = (H, C), where \ C = A \times B \ and \ \forall (x, y) \in A \times B$

$H(x, y) = \Gamma(x) \cap G(y)$.

Definition 2.1.10. Let (Γ, A) and (G, B) be two soft sets over U. Then the **OR-product** or **OR-union** of (Γ, A) and (G, B), denoted by $(\Gamma, A) \tilde{\vee} (G, B)$ is a soft set defined as

$(\Gamma, A) \tilde{\vee} (G, B) = (H, C), where \ C = A \times B \ and \ \forall (x, y) \in A \times B$

$H(x, y) = \Gamma(x) \cup G(y)$.

Fuzzy Set

We recall the definition of the notion of fuzzy set by Zadeh (1965):

Definition 2.2.1. Let A be a subset of X, μ_{A} called **indicator function** or **characteristic function** and is define as, $\mu_{\mathrm{A}}: X \longrightarrow \{1, 0\}$ such that $\mu_{\mathrm{A}}(x) = \begin{cases} 1, & \text{if } x \in \mathrm{A} \\ 0, & \text{if } x \notin \mathrm{A} \end{cases}$.

This correspondence between a set and its indicator function is obviously a one-to-one correspondence.

Let U be a universe. A Fuzzy set X over U is a set defined by a function μ_{X} representing a mapping,

$$\mu_{\mathrm{X}}: U \longrightarrow [0, 1]$$

μ_{X} is called the membership function of X, and the value $\mu_{\mathrm{X}}(u)$ is called the grade of **membership** of $u \in U$and represents the degree of u belonging to the fuzzy set X. Thus a fuzzy set X over U, can be represented as follows:

$X = \left\{\frac{u}{\mu_{\mathrm{X}}(u)} : u \in U, \mu_{\mathrm{X}}(u) \in [0, 1]\right\}$ or $X = \left\{\frac{\mu_{\mathrm{X}}(u)}{u} : u \in U, \mu_{\mathrm{X}}(u) \in [0, 1]\right\}$ or

$X = \{\langle u, \mu_{\mathrm{X}}(u) \rangle : u \in U, \mu_{\mathrm{X}}(u) \in [0, 1]\}$.

Example 2.2.1. Let $U = \{h_1, h_2, h_3, h_4\}$. A fuzzy set X over U can be represented by $X = \left\{\frac{h_1}{0.4}, \frac{h_2}{0.6}, \frac{h_3}{0.2}, \frac{h_4}{0.7}\right\}$.

3. Fuzzy Soft Set

Some of the definitions in this section are due to Cagman et al.,(2011**).**

Let U be an initial universe set and E be a set of parameters (which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy subsets of U, and $A \subseteq E$.

Definition 3.1. A pair $(\tilde{\Gamma}, A)$ is called a fuzzy soft set over U, where $\tilde{\Gamma}$ is a mapping given by $\tilde{\Gamma}: A \longrightarrow P(U)$. In other words, a fuzzy soft set over U is a parameterized family of fuzzy subsets of the universe U. For $x \in A$, $\tilde{\Gamma}(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the fuzzy soft set $(\tilde{\Gamma}, A)$. Therefore, a fuzzy soft set $(\tilde{\Gamma}, A)$ over U can be represented by the set of ordered pairs

$(\tilde{\Gamma}, A) = \left\{\left(x, \tilde{\Gamma}(x)\right): x \in A, \tilde{\Gamma}(x) \in P(U)\right\}$.

Let the set of all fuzzy soft sets over U be denoted by $FS(U)$.

Example 3.1. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe set and $E = \{a_1, a_2, a_3, a_4\}$ be a set of parameters. $A = \{a_1, a_2, a_3\} \subseteq E$, $\tilde{\Gamma}(a_1) = \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}$, $\tilde{\Gamma}(a_2) = U$ and $\tilde{\Gamma}(a_3) = \left\{\frac{h_1}{0.3}, \frac{h_3}{0.4}, \frac{h_5}{0.9}\right\}$, then the fuzzy soft set $(\tilde{\Gamma}, A)$ written as

$(\tilde{\Gamma}, A) = \left\{\left(a_1, \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}\right), (a_2, U), \left(a_3, \left\{\frac{h_1}{0.3}, \frac{h_3}{0.4}, \frac{h_5}{0.9}\right\}\right)\right\}$.

Definition 3.2. Let $(\tilde{\Gamma}, A) \in FS(U)$. If $\tilde{\Gamma}(x) = \emptyset$, $\forall a \in E$, then $(\tilde{\Gamma}, A)$ is called an **empty fuzzy soft set,** denoted by $(\tilde{\Gamma}, A)_{\emptyset}$.

Definition 3.3. Let $(\tilde{\Gamma}, A) \in FS(U)$. If $\tilde{\Gamma}(a) = U$, $\forall a \in A$, then $(\tilde{\Gamma}, A)_{\tilde{A}}$ is called ***A* −universal fuzzy soft set** . If $A = E$, then the A −universal fuzzy soft set is called **universal fuzzy soft set,** denoted by $(\tilde{\Gamma}, E)_{\tilde{E}}$.

Example 3.2. Assume that $U = \{h_1, h_2, h_3, h_4, h_5\}$ is a universal set and $E = \{a_1, a_2, a_3, a_4\}$ is the set of all parameters. If $A = \{a_1, a_3, a_4\} \subseteq E$, $\tilde{\Gamma}(a_1) = \left\{\frac{h_2}{0.5}, \frac{h_4}{0.9}\right\}$, $\tilde{\Gamma}(a_3) = \emptyset$ and $\tilde{\Gamma}(a_4) = U$, then the fuzzy soft set $(\tilde{\Gamma}, A)$ is written as $(\tilde{\Gamma}, A) = \left\{\left(a_1, \left\{\frac{h_2}{0.5}, \frac{h_4}{0.9}\right\}\right), (a_4, U)\right\}$.

If $B = \{a_1, a_3\}$ and $\tilde{G}(a_1) = \emptyset$, $\tilde{G}(a_3) = \emptyset$, then the fuzzy soft set $(\tilde{G}, B)$ is an empty fuzzy soft set, that is $(\tilde{G}, B) = (\tilde{G}, B)_{\emptyset}$.

If $C = \{a_1, a_2\}$ and $\tilde{H}(a_1) = U$, $\tilde{H}(a_2) = U$, then the fuzzy soft set $(\tilde{H}, C)$ is a C −Universal fuzzy soft set, that is $(\tilde{H}, C) \cong (\tilde{H}, C)_{\tilde{C}}$.

If $D = E$ and $\tilde{J}(a_i) = U, \forall\, a_i \in E$, where $i = 1, 2, 3, 4$ then the fuzzy soft set $(\tilde{J}, D)$ is a universal fuzzy soft set, that is, $(\tilde{J}, D) \cong (\tilde{J}, D)_{\tilde{E}}$.

Definition 3.4. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. Then $(\tilde{\Gamma}, A)$ is a **fuzzy soft subset** of $(\tilde{G}, B)$, denoted by $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{G}, B)$, if

(i) $A \subseteq B$, (ii) $\tilde{\Gamma}(a) \leq \tilde{G}(a), \forall\, a \in A$.

Definition 3.5. The **complement of the fuzzy soft set** $(\tilde{\Gamma}, A)$ is denoted by $(\tilde{\Gamma}, A)^C$ and is defined $(\tilde{\Gamma}, A)^C \cong (\tilde{\Gamma}^C, \neg A)$, where $\tilde{\Gamma}^C: \neg A \longrightarrow P(U)$ is a mapping given by $\tilde{\Gamma}^C(\neg e) = (\tilde{\Gamma}(e))^C, \forall \neg e \in \neg A$.

Theorem 3.1. Let $(\tilde{\Gamma}, A) \in FS(U)$. Then

(i) $\left((\tilde{\Gamma}, A)^C\right)^C \cong (\tilde{\Gamma}, A)$,

(ii) $(\tilde{\Gamma}, A)^{\tilde{C}}_{\emptyset} \cong (\tilde{\Gamma}, A)_{\tilde{E}}$.

Proof. By using the fuzzy approximate functions of the fuzzy sets, the proof is straight forward.

Theorem 3.2. (Cagman et al.,2011**).** Let $(\tilde{\Gamma}, A), (\tilde{\Gamma}, B), (\tilde{\Gamma}, C) \in FS(U)$. Then,

(i) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, A)_{\tilde{E}}$,

(ii) $(\tilde{\Gamma}, A)_{\emptyset} \tilde{\subseteq} (\tilde{\Gamma}, A)$,

(iii) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, A)$,

(iv) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, B)$ and $(\tilde{\Gamma}, B) \tilde{\subseteq} (\tilde{\Gamma}, C) \Longrightarrow (\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, C)$.

Proof: The proof is straight forward by using the fuzzy approximate function of the fuzzy soft sets.

Definition 3.6. Let $(\tilde{\Gamma}, A), (\tilde{\Gamma}, B) \in FS(U)$. Then, $(\tilde{\Gamma}, A)$ and $(\tilde{\Gamma}, B)$ are **fuzzy soft equal**, written as $(\tilde{\Gamma}, A) \cong (\tilde{\Gamma}, B)$ if and only if $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, B)$ and $(\tilde{\Gamma}, B) \tilde{\subseteq} (\tilde{\Gamma}, A)$.

Theorem 3.3. Let $(\tilde{\Gamma}, A), (\tilde{\Gamma}, B), (\tilde{\Gamma}, C) \in FS(U)$. Then,

(i) $(\tilde{\Gamma}, A) \cong (\tilde{\Gamma}, B)$ and $(\tilde{\Gamma}, B) \cong (\tilde{\Gamma}, C) \Longrightarrow (\tilde{\Gamma}, A) \cong (\tilde{\Gamma}, C)$,

(ii) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, B)$ and $(\tilde{\Gamma}, B) \tilde{\subseteq} (\tilde{\Gamma}, A)$ if and only if $(\tilde{\Gamma}, A) \cong (\tilde{\Gamma}, B)$.

Proof The proof is straight forward, hence omitted.

Definition 3.7. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. Then, the **Union** of $(\tilde{\Gamma}, A)$ and $(\tilde{G}, B)$, denoted by $(\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)$, is defined by $(\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B) = (\tilde{H}, C)$, where $C = A \cup B$ and $\forall\, x \in C$,

$$\tilde{H}(x) = \begin{cases} \tilde{\Gamma}(x), & if\ x \in A/B \\ \tilde{G}(x), & if\ x \in B/A\ . \\ \tilde{\Gamma}(x) \cup \tilde{G}(x), & if\ x \in A \cap B \end{cases}$$

Proposition 3.1. (Cagman et al.,2011**).** Let $(\tilde{\Gamma}, A), (\tilde{G}, B), (\tilde{H}, C) \in FS(U)$. Then the following properties hold.

(i) $(\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B) \cong (\tilde{G}, B) \tilde{\cup} (\tilde{\Gamma}, A)$,

(ii) $(\tilde{\Gamma}, A) \tilde{\cup} \left((\tilde{G}, B) \tilde{\cup} (\tilde{H}, C)\right) \cong \left((\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)\right) \tilde{\cup} (\tilde{H}, C)$,

(iii) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)$ then $(\tilde{G}, B) \tilde{\subseteq} (\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)$,

(iv) $(\tilde{\Gamma}, A) \tilde{\subseteq} (\tilde{G}, B) \Longrightarrow (\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B) \cong (\tilde{G}, B)$.

Proofs. The proof is straight forward.

Definition 3.8. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$ be two fuzzy soft sets. Then the **restricted intersection** of $(\tilde{\Gamma}, A)$ and $(\tilde{G}, B)$, denoted by $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B)$, is defined as $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B) = (\tilde{K}, C)$, where $C = A \cap B \neq \emptyset$, and for all $x \in C$, $\tilde{K}(x) = \tilde{\Gamma}(x) \cap \tilde{G}(x)$.

Example 3.3. Suppose $U = \{h_1, h_2, h_3, h_4\}$ and $E = \{\, very\ costly, costly, beautiful, in\ green\ surrounding, cheap\}$. Consider the fuzzy soft set $(\tilde{\Gamma}, A)$ which describes the cost of the houses and the soft set $(\tilde{G}, B)$which describes the attractiveness of the houses. Let $A, B \subseteq E$ as $A = \{\, very\ costly, costly, cheap\}$, $B = \{beautiful, in\ green\ surrounding, cheap\}$. Suppose that,

$\tilde{\Gamma}(very\ costly) = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.6}, \frac{h_3}{0.1}, \frac{h_4}{0.9}\right\}$,

$\tilde{\Gamma}(costly) = \left\{\frac{h_1}{0.6}, \frac{h_2}{0.3}, \frac{h_3}{0.7}, \frac{h_4}{0.2}\right\}$,

$\tilde{\Gamma}(cheap) = \left\{\frac{h_1}{0.4}, \frac{h_2}{0.2}, \frac{h_3}{0.7}, \frac{h_4}{0.8}\right\}$,

$\tilde{G}(beautiful) = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.6}, \frac{h_3}{0.3}, \frac{h_4}{0.4}\right\}$,
$\tilde{G}(in\ green\ surrounding) = \left\{\frac{h_1}{0.8}, \frac{h_2}{0.4}, \frac{h_3}{0.4}, \frac{h_4}{0.7}\right\}$,
$\tilde{G}(cheap) = \left\{\frac{h_1}{0.6}, \frac{h_2}{0.7}, \frac{h_3}{0.2}, \frac{h_4}{0.8}\right\}$,
$\tilde{K}(cheap) = \tilde{\Gamma}(cheap) \cap \tilde{G}(cheap) = \left\{\frac{h_1}{0.4}, \frac{h_2}{0.2}, \frac{h_3}{0.2}, \frac{h_4}{0.8}\right\}$.

Proposition 3.2. (Cagman et al.,2011**).** Let $(\tilde{\Gamma}, A), (\tilde{G}, B), (\tilde{H}, C) \in FS(U)$. Then the following properties hold.

(i) $(\tilde{\Gamma}, A) \Cap (\tilde{\Gamma}, A) \cong (\tilde{\Gamma}, A)$,

(ii) $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B) \cong (\tilde{G}, B) \Cap (\tilde{\Gamma}, A)$,

(iii) $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B) \subseteq (\tilde{\Gamma}, A)$ and $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B) \subseteq (\tilde{G}, B)$,

(iv) $(\tilde{\Gamma}, A) \subseteq (\tilde{G}, B) \Longrightarrow (\tilde{\Gamma}, A) \Cap (\tilde{G}, B) \cong (\tilde{\Gamma}, A)$,

(v) $\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, B)\right) \Cap (\tilde{H}, C) \cong (\tilde{\Gamma}, A) \Cap \left((\tilde{G}, B) \Cap (\tilde{H}, C)\right)$.

Proof. The proof is straight forward, hence omitted.

De Morgan's Inclusion and Laws

We shall prove the following De Morgan's inclusions and laws:

Theorem 3.4. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. Then the following hold.

(i) $\left((\tilde{\Gamma}, A) \Cup (\tilde{G}, B)\right)^C \subseteq (\tilde{\Gamma}, A)^C \Cup (\tilde{G}, B)^C$,

(ii) $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, B)^C \subseteq \left((\tilde{\Gamma}, A) \Cap (\tilde{G}, B)\right)^C$

Proof

(i) Let $(\tilde{\Gamma}, A) \Cup (\tilde{G}, B) = (\tilde{H}, A \cup B)$. Therefore,

$\left((\tilde{\Gamma}, A) \Cup (\tilde{G}, B)\right)^C = (\tilde{H}, A \cup B)^C$,

$= \left(\tilde{H}^C, \neg(A \cup B)\right)$,

Take $\neg\boldsymbol{\alpha} \in \neg(A \cup B)$,

$$\tilde{H}^C(\neg\boldsymbol{\alpha}) = \left(\tilde{H}(\alpha)\right)^C,$$

$$= \begin{cases} (\tilde{\Gamma}(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ (\tilde{G}(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ (\tilde{\Gamma}(\alpha) \cup \tilde{G}(\alpha))^C, & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

$$= \begin{cases} \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ \tilde{G}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Consider,

$(\tilde{\Gamma}, A)^C \Cup (\tilde{G}, B)^C = (\tilde{\Gamma}^C, \neg A) \Cup (\tilde{G}^C, \neg B)$.

$= (J, \neg A \cup \neg B)$, (say), where

$$J(\neg\boldsymbol{\alpha}) = \begin{cases} \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ \tilde{G}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Obviously, $\tilde{H}^C(\neg\boldsymbol{\alpha}) \subseteq J(\neg\boldsymbol{\alpha})$, hence, (i) holds.

(ii) Consider $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, B)^C \cong (\tilde{\Gamma}^C, \neg A) \Cap (\tilde{G}^C, \neg B)$

$= (\widetilde{K}, \neg A \cap \neg B)$, (say), where

$\widetilde{K}(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha}), \forall \neg\boldsymbol{\alpha} \in \neg A \cap \neg B.$

On the other hand

$\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, B)\right)^C = (M, A \cap B)^C$, (say)

$= (M^C, \neg(A \cap B)$.

Now for $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$,

$M^C(\neg\boldsymbol{\alpha}) = (M(\boldsymbol{\alpha}))^C,$

$= (\tilde{\Gamma}(\boldsymbol{\alpha}) \cap \tilde{G}(\boldsymbol{\alpha}))^C,$

$= \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha}).$

Clearly, $\widetilde{K}(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha}) \subseteq \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha}) = M^C(\neg\boldsymbol{\alpha}).$

It is well known from standard set that De Morgan's Law interrelate union and intersection via complements.

Theorem 3.5. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. Then the following De Morgan's inclusions hold.

(i) $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, B)^C \,\tilde{\subseteq}\, \left((\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)\right)^C.$

(ii) $\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, B)\right)^C \,\tilde{\subseteq}\, (\tilde{\Gamma}, A)^C \tilde{\cup} (\tilde{G}, B)^C,$

Proof:

(i) Consider $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, B)^C = (\tilde{\Gamma}^C, \neg A) \Cap (\tilde{G}^C, \neg B)$

$= (\widetilde{H}^C, \neg A \cap \neg B)$, (say), where

$\widetilde{H}^C(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha}), \forall \neg\boldsymbol{\alpha} \in \neg A \cap \neg B.$

Again, let $(\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B) = (V, A \cup B)$

$\left((\tilde{\Gamma}, A) \tilde{\cup} (\tilde{G}, B)\right)^C = (V, A \cup B)^C$, (say)

$= (V^C, \neg(A \cup B)).$

For $\neg\boldsymbol{\alpha} \in \neg(A \cup B)$, we have

$$V^C(\neg\boldsymbol{\alpha}) = (V(\alpha))^C = \begin{cases} (\tilde{\Gamma}(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ (\tilde{G}(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ (\tilde{\Gamma}(\alpha) \cup \tilde{G}(\alpha))^C, & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

$$= \begin{cases} \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ \tilde{G}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Obviously, $\widetilde{H}^C(\neg\boldsymbol{\alpha}) \subseteq (V^C(\neg\boldsymbol{\alpha})$.

(ii)Suppose that $(\tilde{\Gamma}, A) \Cap (\tilde{G}, B) = (D, A \cap B)$, where

$D(\boldsymbol{\alpha}) = \tilde{\Gamma}(\alpha) \cap \tilde{G}(\alpha)$, for all $\alpha \in A \cap B$.

Therefore, $\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, B)\right)^C = (D, A \cap B)^C$

$= \left(D^C, \neg(A \cap B)\right).$

Let us take $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, then

$D^C(\neg\boldsymbol{\alpha}) = (D(\boldsymbol{\alpha}))^C = (\tilde{\Gamma}(\alpha) \cap \tilde{G}(\alpha))^C$

$= (\tilde{\Gamma}(\boldsymbol{\alpha}))^C \cup (\tilde{G}(\boldsymbol{\alpha}))^C$

$D^C(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha}).$

Now consider, $(\tilde{\Gamma}, A)^C \tilde{\cup} (\tilde{G}, B)^C = (\tilde{\Gamma}^C, \neg A) \tilde{\cup} (\tilde{G}^C, \neg B)$

$= (T, \neg A \cup \neg B)$, (say)

For $\neg\boldsymbol{\alpha} \in \neg A \cup \neg B$, we have

$$T(\neg\boldsymbol{\alpha}) = \begin{cases} \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ \tilde{G}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Clearly, $D^C(\neg\boldsymbol{\alpha}) \subset T(\neg\boldsymbol{\alpha})$. Hence the result.

It is natural to ask when the De Morgan's inclusions in theorem 3.5 become De Morgan's Laws, as can be seen in theorem 3.6.

Theorem 3.6. Let $(\tilde{\Gamma}, A), (\tilde{G}, A) \in FS(U)$. Then the following De Morgan's law holds.

(i) $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, A)^C \cong \left((\tilde{\Gamma}, A) \,\tilde{\cup}\, (\tilde{G}, A)\right)^C$.

(ii) $\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, A)\right)^C \cong (\tilde{\Gamma}, A)^C \,\tilde{\cup}\, (\tilde{G}, A)^C$.

Proof (i) Consider $(\tilde{\Gamma}, A)^C \Cap (\tilde{G}, A)^C = (\tilde{\Gamma}^C, \neg A) \Cap (\tilde{G}^C, \neg A)$

$= (\tilde{F}^C, \neg A)$, (say), where for all $\neg\boldsymbol{\alpha} \in \neg A$

$\tilde{F}^C(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha})$.

Again suppose that $(\tilde{\Gamma}, A) \,\tilde{\cup}\, (\tilde{G}, A) = (\widetilde{M}, A)$. Therefore,

$\left((\tilde{\Gamma}, A) \,\tilde{\cup}\, (\tilde{G}, A)\right)^C = (\widetilde{M}, A)^C$, (say)

$= (\widetilde{M}^C, \neg A)$.

Where, $\widetilde{M}^C(\neg\boldsymbol{\alpha}) = (\widetilde{M}(\boldsymbol{\alpha}))^C = (\tilde{\Gamma}(\alpha) \cup \tilde{G}(\alpha))^C$

For all $\neg\boldsymbol{\alpha} \in \neg\boldsymbol{A}$, we have

$\widetilde{M}^C(\neg\boldsymbol{\alpha}) = (\tilde{\Gamma}(\alpha) \cup \tilde{G}(\alpha))^C = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cap \tilde{G}^C(\neg\boldsymbol{\alpha})$

Clearly, $\tilde{F}^C(\neg\boldsymbol{\alpha}) = \widetilde{M}^C(\neg\boldsymbol{\alpha})$. Therefore, (i) has been established.

(ii) Suppose that $(\tilde{\Gamma}, A) \Cap (\tilde{G}, A) = (\tilde{T}, A)$. Therefore,

$$\left((\tilde{\Gamma}, A) \Cap (\tilde{G}, A)\right)^C = (\tilde{T}, A)^C, \\ = (\tilde{T}^C, \neg A),$$

Where, $\tilde{T}^C(\neg\alpha) = (\tilde{T}(\alpha))^C = (\tilde{\Gamma}(\alpha) \cap \tilde{G}(\alpha))^C$

For all $\neg\boldsymbol{\alpha} \in \neg A$, we have

$\tilde{T}^C(\neg\alpha) = (\tilde{\Gamma}(\alpha) \cap \tilde{G}(\alpha))^C = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha})$.

Let $(\tilde{\Gamma}, A)^C \,\tilde{\cup}\, (\tilde{G}, A)^C = (V, \neg A)$

$(\tilde{\Gamma}^C, \neg A) \,\tilde{\cup}\, (\tilde{G}^C, \neg A) = (V, \neg A)$, (say),

Where for all $\neg\boldsymbol{\alpha} \in \neg\boldsymbol{A}$.

$V(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^C(\neg\boldsymbol{\alpha}) \cup \tilde{G}^C(\neg\boldsymbol{\alpha})$.

Obviously, $V(\neg\boldsymbol{\alpha}) = \tilde{T}^C(\neg\alpha)$, hence (ii) has been established.

Definition 3.9. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. The **extended intersection** of $(\tilde{\Gamma}, A)$ and $(\tilde{G}, B)$ denoted by $(\tilde{\Gamma}, A) \,\tilde{\cap}_E\, (\tilde{G}, B)$, is defined by $(\tilde{\Gamma}, A) \,\tilde{\cap}_E\, (\tilde{G}, B) = (\tilde{V}, C)$, where $C = A \cup B$, and for all $x \in C$,

$$\tilde{V}(x) = \begin{cases} \tilde{\Gamma}(x), & if\ x \in A/B \\ \tilde{G}(x), & if\ x \in B/A \\ \tilde{\Gamma}(x) \cap \tilde{G}(x), & if\ x \in A \cap B \end{cases}.$$

Definition 3.10. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. The **restricted union** of $(\tilde{\Gamma}, A)$ and $(\tilde{G}, B)$ denoted by $(\tilde{\Gamma}, A) \,\tilde{\cup}_R\, (\tilde{G}, B)$, is defined by $(\tilde{\Gamma}, A) \,\tilde{\cup}_R\, (\tilde{G}, B) = (\tilde{T}, C)$, where $C = A \cap B \neq \emptyset$, and for all $x \in C$, $\tilde{T}(x) = \tilde{\Gamma}(x) \cup \tilde{G}(x)$.

Theorem 3.7. Let $(\tilde{\Gamma}, A), (\tilde{G}, A) \in FS(U)$. Then the following De Morgan laws hold:

(i) $\left((\tilde{\Gamma}, A) \,\tilde{\cap}_E\, (\tilde{G}, A)\right)^C \cong (\tilde{\Gamma}, A)^C \,\tilde{\cup}_R\, (\tilde{G}, A)^C$.

(ii) $\left(\tilde{\Gamma},A\right)^{C} \tilde{\cap}_{\mathrm{E}} \left(\tilde{G},A\right)^{C} \cong \left(\left(\tilde{\Gamma},A\right) \tilde{\cup}_{\mathrm{R}} \left(\tilde{G},A\right)\right)^{C}$.

Proof (i) Let $\left(\tilde{\Gamma},A\right) \tilde{\cap}_{\mathrm{E}} \left(\tilde{G},A\right) = (\tilde{T}, A)$. Thus,

$\left(\left(\tilde{\Gamma},A\right) \tilde{\cap}_{\mathrm{E}} \left(\tilde{G},A\right)\right)^{C} = (\tilde{T},A)^{C} = (\tilde{T}^{C},\neg A)$.

Where for all $\neg\boldsymbol{\alpha} \in \neg A$,

$\tilde{T}^{C}(\neg\boldsymbol{\alpha}) = (\tilde{T}(\alpha))^{C} = (\tilde{\Gamma}(\alpha) \cap \tilde{G}(\alpha))^{C}$,

$= \tilde{\Gamma}^{C}(\neg\boldsymbol{\alpha}) \cup \tilde{G}^{C}(\neg\boldsymbol{\alpha})$.

Also, let $\left(\tilde{\Gamma},A\right)^{C} \tilde{\cup}_{\mathrm{R}} \left(\tilde{G},A\right)^{C} = (\tilde{\Gamma}^{C},\neg A) \tilde{\cup}_{\mathrm{R}} (\tilde{G}^{C},\neg A)$

$= (D,\neg A)$, (say).

For all $\neg\boldsymbol{\alpha} \in \neg A$, we have

$D(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^{C}(\neg\boldsymbol{\alpha}) \cup \tilde{G}^{C}(\neg\boldsymbol{\alpha})$.

It is obvious that $D(\neg\boldsymbol{\alpha}) = \tilde{T}^{C}(\neg\boldsymbol{\alpha})$. Hence, (i) has been established.

(ii) Let $\left(\tilde{\Gamma},A\right)^{C} \tilde{\cap}_{\mathrm{E}} \left(\tilde{G},A\right)^{C} = (\tilde{\Gamma}^{C},\neg A) \tilde{\cap}_{\mathrm{E}} (\tilde{G}^{C},\neg A)$.

$= (\widetilde{W},\neg A)$, (say),

For all $\neg\boldsymbol{\alpha} \in \neg A$, we have,

$\widetilde{W}(\neg\boldsymbol{\alpha}) = \tilde{\Gamma}^{C}(\neg\boldsymbol{\alpha}) \cap \tilde{G}^{C}(\neg\boldsymbol{\alpha})$.

Also, let $\left(\tilde{\Gamma},A\right) \tilde{\cup}_{\mathrm{R}} \left(\tilde{G},A\right) = (\tilde{Q}, A)$, (say),

$\left(\left(\tilde{\Gamma},A\right) \tilde{\cup}_{\mathrm{R}} \left(\tilde{G},A\right)\right)^{C} = (\tilde{Q}, A)^{C} = (\tilde{Q}^{C},\neg A)$.

Therefore, for all $\neg\boldsymbol{\alpha} \in \neg A$,

$\tilde{Q}^{C}(\neg\boldsymbol{\alpha}) = (\tilde{Q}(\boldsymbol{\alpha}))^{C} = (\tilde{\Gamma}^{C}(\neg\boldsymbol{\alpha}) \cup \tilde{G}^{C}(\neg\boldsymbol{\alpha}))^{C}$,

$= \tilde{\Gamma}^{C}(\neg\boldsymbol{\alpha}) \cap \tilde{G}^{C}(\neg\boldsymbol{\alpha})$.

Since, $\widetilde{W}(\neg\boldsymbol{\alpha}) = \tilde{Q}^{C}(\neg\boldsymbol{\alpha})$. Hence (ii) has been proved.

$\tilde{T}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) = \tilde{\Gamma}(\boldsymbol{\alpha}_1) \cap \tilde{G}(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) \in A \times B$.

Definition 3.11. Let $\left(\tilde{\Gamma},A\right),\left(\tilde{G},B\right) \in FS(U)$. The **OR Operation** denoted by $\left(\tilde{\Gamma},A\right)\tilde{\vee}\left(\tilde{G},B\right)$ is defined by $\left(\tilde{\Gamma},A\right)\tilde{\vee}\left(\tilde{G},B\right) = (\tilde{N}\ A \times B)$, where

$\tilde{N}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) = \tilde{\Gamma}(\boldsymbol{\alpha}_1) \cup \tilde{G}(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) \in A \times B$.

Definition 3.12. Let $\left(\tilde{\Gamma},A\right),\left(\tilde{G},B\right) \in FS(U)$. The **AND Operation** denoted by $\left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\tilde{G},B\right)$ is defined by $\left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\tilde{G},B\right) = (\widetilde{M},A \times B)$, where

$\widetilde{M}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) = \tilde{\Gamma}(\boldsymbol{\alpha}_1) \cap \tilde{G}(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2) \in A \times B$.

Proposition 3.3. Let $\left(\tilde{\Gamma},A\right),\left(\tilde{G},B\right),\left(\widetilde{H},C\right) \in FS(U)$, then

(i) $\left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\left(\tilde{G},B\right)\tilde{\wedge}\left(\widetilde{H},C\right)\right) \cong \left(\left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\tilde{G},B\right)\right)\tilde{\wedge}\left(\widetilde{H},C\right)$.

(ii) $\left(\tilde{\Gamma},A\right)\tilde{\vee}\left(\left(\tilde{G},B\right)\tilde{\vee}\left(\widetilde{H},C\right)\right) \cong \left(\left(\tilde{\Gamma},A\right)\tilde{\vee}\left(\tilde{G},B\right)\right)\tilde{\vee}\left(\widetilde{H},C\right)$,

Proof (i) By using definition 3.12,

$\left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\left(\tilde{G},B\right)\tilde{\wedge}\left(\widetilde{H},C\right)\right) = \left(\tilde{\Gamma},A\right)\tilde{\wedge}\left(\widetilde{M},B \times C\right) = \left(\tilde{N},A \times B \times C\right)$, where for all $(b,c) \in B \times C$,

$\widetilde{M}(b,c) = \tilde{G}(b) \cap \widetilde{H}(c)$ and for all $(a,b,c) \in A \times B \times C$,

$\tilde{N}(a,b,c) = \tilde{\Gamma}(a) \cap \widetilde{M}(b,c) = \tilde{\Gamma}(a) \cap \left(\tilde{G}(b) \cap \widetilde{H}(c)\right) = \left(\tilde{\Gamma}(a) \cap \tilde{G}(b)\right) \cap \widetilde{H}(c) = \tilde{Q}(a,b) \cap \widetilde{H}(c)$ with $\tilde{Q}(a,b) = \tilde{\Gamma}(a) \cap \tilde{G}(b)$.

$\left(\tilde{Q},A \times B\right)\tilde{\wedge}\left(\widetilde{H},C\right) = \left(\left(\tilde{G},B\right)\tilde{\wedge}\left(\widetilde{H},C\right)\right)\tilde{\wedge}\left(\widetilde{H},C\right)$. Hence (i) has been proved.

(ii) Similar to proof of (i), (ii) can be proved.

Generalized De Morgan's Inclusion

We first define arbitrary union and intersection of a family of fuzzy soft set in $FS(U)$ as follows.

Definition 3.13. Let ▯ $= \{(\tilde{\Gamma}_i, A_i): i \in I\}$ be a family of fuzzy soft set in $FS(U)$. Then the **union of fuzzy soft sets** in ▯ is a fuzzy soft set $(\widetilde{H}, C)$, where $C = \bigcup_{i\in I} A_i$ and for all $x \in C$, $\widetilde{H}(x) = \cup_{i\in I} \tilde{\Gamma}_i(x, A_i)$, where $\tilde{\Gamma}_i(x, A_i) = \begin{cases} \tilde{\Gamma}_i(x), & if\ x \in A_i, \\ \emptyset, & if\ x \notin A_i \end{cases}.$

Let us illustrate this with a simple example as follows:

Example 3.5. Let $(\tilde{\Gamma}_1, A_1)$, $(\tilde{\Gamma}_2, A_2)$, $(\tilde{\Gamma}_3, A_3) \in FS(U)$ be given as

$(\tilde{\Gamma}_1, A_1) = \left\{a_1 = \left\{\frac{h_1}{0.8}, \frac{h_2}{0.4}, \frac{h_3}{0.2}\right\}, a_3 = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.2}, \frac{h_3}{0.8}\right\}, a_4 = \left\{\frac{h_1}{0.2}, \frac{h_2}{0.1}, \frac{h_3}{0}\right\}\right\}$,

$(\tilde{\Gamma}_2, A_2) = \left\{a_1 = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.5}, \frac{h_3}{0.7}\right\},\ a_4 = \left\{\frac{h_1}{0.1}, \frac{h_2}{0.8}, \frac{h_3}{0.6}\right\}\right\}$,

$(\tilde{\Gamma}_3, A_3) = \left\{a_1 = \left\{\frac{h_1}{0.7}, \frac{h_2}{0.5}, \frac{h_3}{0.2}\right\}, a_2 = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.8}, \frac{h_3}{0.7}\right\}\right\}$.

Calculations give

$(\tilde{\Gamma}_1, A_1)\ \tilde{\cup}\ (\tilde{\Gamma}_2, A_2)\ \tilde{\cup}\ (\tilde{\Gamma}_3, A_3) = \left\{a_1 = \left\{\frac{h_1}{0.8}, \frac{h_2}{0.5}, \frac{h_3}{0.7}\right\}, a_2 = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.8}, \frac{h_3}{0.7}\right\}, a_3 = \left\{\frac{h_1}{0.5}, \frac{h_2}{0.2}, \frac{h_3}{0.8}\right\}, a_4 = \left\{\frac{h_1}{0.2}, \frac{h_2}{0.8}, \frac{h_3}{0.6}\right\}\right\}$.

Definition 3.14. Let ▯ $= \{(\tilde{\Gamma}_i, A_i): i \in I\}$ be a family of fuzzy soft set in $FS(U)$, with $\bigcap_{i\in I} A_i \neq \emptyset$. Then the **intersection** of fuzzy soft set in ▯ is a fuzzy soft set $(\widetilde{H}, C)$, where $C = \bigcap_{i\in I} A_i$ and for all $x \in C$, $\widetilde{H}(x) = \bigcap_{i\in I} \tilde{\Gamma}_i(x)$.

Theorem 3.8. Let **S** $= \{(\tilde{\Gamma}_i, A_i): i \in I\}$ be a family of fuzzy soft set in $FS(U)$. Then the following holds:

(i) $\Cap_{i\in I} (\tilde{\Gamma}_i, A_i)^C \,\tilde{\subseteq}\, \left(\tilde{\cup}_{i\in I} (\tilde{\Gamma}_i, A_i)\right)^C$.

(ii) $\Cap_{i\in I} (\tilde{\Gamma}_i, A_i)^C \,\tilde{\subseteq}\, \tilde{\cup}_{i\in I} (\tilde{\Gamma}_i, A_i)^C$.

Proof. The proof follows from the proof of theorem 3.5.

In a special case where $A_{i\in I} = A$, then theorem 3.9 holds

Theorem 3.9. Let ▯ $= \{(\tilde{\Gamma}_i, A): i \in I\}$ be a family of fuzzy soft set in $FS(U)$. Then the following holds:

(i) $\Cap_{i\in I} (\tilde{\Gamma}_i, A)^C \,\tilde{\subseteq}\, \left(\tilde{\cup}_{i\in I} (\tilde{\Gamma}_i, A)\right)^C$.

(ii) $\left(\Cap_{i\in I} (\tilde{\Gamma}_i, A)\right)^C \,\tilde{\subseteq}\, \tilde{\cup}_{i\in I} (\tilde{\Gamma}_i, A)^C$.

Proof. The proof follows from the proof of theorem 3.5.

Definition 3.14. Let $(\tilde{\Gamma}, A), (\tilde{G}, B) \in FS(U)$. Such that $A \cap B \neq \emptyset$. The **restricted difference** of $(\tilde{\Gamma}, A)$ and $(\tilde{G}, B)$ is denoted by $\left((\tilde{\Gamma}, A) -_R (\tilde{G}, B)\right)$ and is defined by $\left((\tilde{\Gamma}, A) -_R (\tilde{G}, B)\right) = (\widetilde{K}, P)$, where $P = A \cap B$ and $\forall x \in P$, $\widetilde{K}(x) = \tilde{\Gamma}(x) -_R \tilde{G}(x)$.

The fuzzy difference of two fuzzy sets $\tilde{\Gamma}(x)$ and $\tilde{G}(x)$ is denoted by $\tilde{\Gamma}(x) -_R \tilde{G}(x)$ and is defined as $\tilde{\Gamma}(x) -_R \tilde{G}(x) = \tilde{\Gamma}(x) \cap \tilde{G}^C(x)$.

Definition 3.15. The **relative complement** of fuzzy soft set $(\tilde{\Gamma}, A)$ is denoted by $(\tilde{\Gamma}, A)^r$, where $\tilde{\Gamma}^r: A \longrightarrow F(U)$ is a mapping given by $\tilde{\Gamma}^r(\alpha) =$ Fuzzy complement of $\tilde{\Gamma}(\alpha)$, for all $\alpha \in A$. Consequently, $\left((\tilde{\Gamma}, A)^r\right)^r$.

It is important to note that in the above definition of complement, the parameter set of the complement $\left((\tilde{\Gamma}, A)^r\right)^r$ is still the original parameter set A, instead of $\neg A$. To emphasize this difference, the complement $(\tilde{\Gamma}, A)^C$ is called neg-complement (or Pseudo-complement).

Fuzzy Soft Aggregation

In this section, we present a fuzzy soft aggregation operator that produces an aggregate fuzzy set from a fuzzy soft set and its cardinal set. The approximate functions of a fuzzy soft set are fuzzy. A fuzzy soft aggregation operator on the fuzzy sets is an operation by which several approximate functions of a fuzzy soft set are combined to produce a single fuzzy set which is the aggregate fuzzy set of the fuzzy soft set. Once an aggregate fuzzy set has been arrived at, it may be necessary to choose the best single crisp alternative from this set.

Definition 4.1. Let $(\tilde{\Gamma}, A) \in FS(U)$. Assume that $U = \{u_1, u_2, \ldots, u_m\}$, $E = \{x_1, x_2, \ldots, x_n\}$ and $A \subseteq E$, then $(\tilde{\Gamma}, A)$ can be presented by the following table.

$(\tilde{\Gamma}, A)$	x_1	x_2	…	x_n.
u_1	$\mu_{\tilde{\Gamma}(x_1)}(u_1)$	$\mu_{\tilde{\Gamma}(x_2)}(u_1)$	…	$\mu_{\tilde{\Gamma}(x_n)}(u_1)$.
u_2	$\mu_{\tilde{\Gamma}(x_1)}(u_2)$	$\mu_{\tilde{\Gamma}(x_2)}(u_2)$	…	$\mu_{\tilde{\Gamma}(x_n)}(u_2)$.
.	……………………………………………………			
.	……………………………………………………			
.	……………………………………………………..			
u_m	$\mu_{\tilde{\Gamma}(x_1)}(u_2)$	$\mu_{\tilde{\Gamma}(x_2)}(u_2)$	…	$\mu_{\tilde{\Gamma}(x_n)}(u_2)$.

Where $\mu_{\tilde{\Gamma}(x)}$ is a membership function of $(\tilde{\Gamma}, A)$.

If $a_{ij} = \mu_{\tilde{\Gamma}(x_j)}(u_i)$ for $i = 1,2, \ldots, m$ and $j = 1,2, \ldots, n$, then the fuzzy soft set $(\tilde{\Gamma}, A)$ is uniquely characterized by a matrix,

$$[a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

is called an $m \times n$ fuzzy soft matrix of the fuzzy soft set $(\tilde{\Gamma}, A)$ over U.

Definition 4.2. Let $(\tilde{\Gamma}, A) \in FS(U)$. Then, the cardinal set of $(\tilde{\Gamma}, A)$, denoted by $c(\tilde{\Gamma}, A)$ and is defined by $c(\tilde{\Gamma}, A) = \left\{\frac{x}{\mu C(\tilde{\Gamma}, A)(x)} : x \in E\right\}$,

is a fuzzy set over E. The membership function $\mu c(\tilde{\Gamma}, A)$ of $c(\tilde{\Gamma}, A)$ is defined by

$\mu c(\tilde{\Gamma}, A): E \longrightarrow [0, 1], \mu c(\tilde{\Gamma}, A)(x) = \frac{|(\tilde{\Gamma}, A)(x)|}{|U|}$.

Where $|U|$ is the cardinality of the universe U and $|(\tilde{\Gamma}, A)(x)|$ is the scalar cardinality of fuzzy set $(\tilde{\Gamma}, A)(x)$.

It is worthy to note that, the set of all cardinal sets of the fuzzy sets over U will be denoted by $cFS(U)$. It is clear that $cFS(U) \subseteq F(E)$.

Definition 4.3. Let $(\tilde{\Gamma}, A) \in FS(U)$ and $c(\tilde{\Gamma}, A) \in cFS(U)$. Assume that $E = \{x_1, x_2, \ldots, x_n\}$ and $A \subseteq E$, then $c(\tilde{\Gamma}, A)$ can be presented by the following table

E	x_1	x_2	…	x_n.
$\mu c(\tilde{\Gamma}, A)$	$\mu c(\tilde{\Gamma}, A)(x_1)$	$\mu c(\tilde{\Gamma}, A)(x_2)$	…	$\mu c(\tilde{\Gamma}, A)(x_n)$.

If $a_{ij} = \mu c(\tilde{\Gamma}, A)(x_j)$ for $j = 1, 2, \ldots, n$, then the cardinal set $c(\tilde{\Gamma}, A)$ is uniquely characterized by a matrix

$$[a_{1j}]_{1 \times n} = [a_{11} \quad a_{12} \quad \ldots \quad a_{1n}]$$

which is called the cardinal matrix of the cardinal set $c(\tilde{\Gamma}, A)$ over E.

Definition 4.4. Let $(\tilde{\Gamma}, A) \in FS(U)$ and $c(\tilde{\Gamma}, A) \in CFS(U)$. Then the fuzzy soft aggregation operator, denoted by $FS_{agg}: cFS(U) \times FS(U) \longrightarrow F(U)$, $FS_{agg}\left(c(\tilde{\Gamma}, A), (\tilde{\Gamma}, A)\right) = (\tilde{\Gamma}, A)^*$

Where $(\tilde{\Gamma}, A)^* = \left\{\frac{u}{\mu_{(\tilde{\Gamma}, A)^*}(u)} : u \in U\right\}$

is a fuzzy set over U. $(\tilde{\Gamma}, A)^*$ is called the aggregate fuzzy set of the fuzzy soft set $(\tilde{\Gamma}, A)$. The membership function $\mu_{(\tilde{\Gamma}, A)^*}$ of $(\tilde{\Gamma}, A)^*$ is defined as follows:

$\mu_{(\tilde{\Gamma}, A)^*} : U \longrightarrow [0, 1]$, $\mu_{(\tilde{\Gamma}, A)^*}(u) = \frac{1}{|E|} \sum_{x \in E} \mu\, c(\tilde{\Gamma}, A)(x) \mu_{(\tilde{\Gamma}, A)(x)}(u)$, where $|E|$ is the cardinality of E.

Definition 4.5. Let $(\tilde{\Gamma}, A) \in FS(U)$. Assume that $U = \{u_1, u_2, \ldots, u_m\}$, $E = \{x_1, x_2, \ldots, x_n\}$ and $A \subseteq E$, then $(\tilde{\Gamma}, A)$ can be presented by the following table.

$(\tilde{\Gamma}, A)$	$\mu_{(\tilde{\Gamma},A)^*}$
u_1	$\mu_{(\tilde{\Gamma},A)^*}(u_1)$
u_2	$\mu_{(\tilde{\Gamma},A)^*}(u_2)$
.	. . .
.	. . .
.	. . .
u_m	$\mu_{(\tilde{\Gamma},A)^*}(u_m)$

If $a_{i1} = \mu_{(\tilde{\Gamma},A)^*}(u_i)$ for $i = 1, 2, \dots, m$ then $(\tilde{\Gamma}, A)^*$ is uniquely characterized by the matrix,

$$[a_{i1}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

which is called the aggregate matrix of $(\tilde{\Gamma}, A)^*$ over U.

Theorem 4.1. Let $(\tilde{\Gamma}, A) \in FS(U)$ and $A \subseteq E$. If $M(\tilde{\Gamma}, A), Mc(\tilde{\Gamma}, A)$ and $M(\tilde{\Gamma}, A)^*$ are representation matrices of $(\tilde{\Gamma}, A), c(\tilde{\Gamma}, A)$ and $(\tilde{\Gamma}, A)^*$, respectively, then

$|E| \times M(\tilde{\Gamma}, A)^* = M(\tilde{\Gamma}, A) \times M^T{}_{c(\tilde{\Gamma},A)}$,

Where $M^T{}_{C(\tilde{\Gamma},A)}$ is the transposition of $Mc(\tilde{\Gamma}, A)$ and $|E|$ is the cardinality of E.

Proof It is sufficient to consider $[a_{i1}]_{m\times 1} = [a_{ij}]_{m\times n} \times [a_{1j}]^T{}_{1\times n}$.

The theorem above is applicable to computing the aggregate fuzzy set of a fuzzy soft set.

Application in Decision Making Problems

Once an aggregate fuzzy set has been arrived at, it is necessary to choose the best alternative from this set. Therefore, we can make a decision by the following algorithm.

Step I: Construct a fuzzy soft set $(\tilde{\Gamma}, A)$ over U.

Step II: Find the cardinal set $c(\tilde{\Gamma}, A)$ of $(\tilde{\Gamma}, A)$.

Step III: Find the aggregate fuzzy set $(\tilde{\Gamma}, A)^*$ of $(\tilde{\Gamma}, A)$.

Step IV: Find the best alternative from this set that has the largest membership grade by $max\ \mu_{(\tilde{\Gamma},A)^*}(C)$.

Example 5.1. Suppose a new generation Bank X. wants to fill a vacant position in the bank. There are eight candidates who applied for the vacancy, $U = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8\}$. The recruitment team considers a set of parameters, $E = \{e_1, e_2, e_3, e_4, e_5\}$. For $i = 1,2,3,4,5$, the parameters x_i stand for experience, computer knowledge, young age, good speaking, friendly, respectively.

After serious deliberations, each candidate is evaluated from the goals and constraint point of view according to a chosen subset $A = \{e_2, e_3, e_4\}$ of E. Finally, the team applies the following steps:

Step I: The recruitment team constructs a fuzzy soft set $(\tilde{\Gamma}, A)$ over U.

$(\tilde{\Gamma}, A) = \left\{\left(e_2, \left\{\frac{C_2}{0.6}, \frac{C_3}{0.5}, \frac{C_4}{0.4}, \frac{C_5}{0.8}, \frac{C_7}{0.9}\right\}\right), \left(e_3, \left\{\frac{C_1}{0.5}, \frac{C_2}{0.4}, \frac{C_3}{0.9}, \frac{C_4}{0.6}\right\}\right), \left(e_4, \left\{\frac{C_1}{0.5}, \frac{C_2}{0.6}, \frac{C_5}{0.4}, \frac{C_7}{0.7}, \frac{C_8}{1}\right\}\right)\right\}$.

Step II: The cardinal is computed as,

$c(\tilde{\Gamma}, A) = \left\{\frac{e_2}{0.4}, \frac{e_3}{0.3}, \frac{e_4}{0.4}\right\}$.

Step III: The aggregate fuzzy set obtained by using theorem 4.1

$$M(\tilde{\Gamma},A)^* = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0.6 & 0.4 & 0.6 & 0 \\ 0 & 0.5 & 0.9 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.8 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0.4 \\ 0.3 \\ 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.12 \\ 0.094 \\ 0.068 \\ 0.096 \\ 0.000 \\ 0.128 \\ 0.08 \end{bmatrix}.$$

That means,

$$(\tilde{\Gamma},A)^* = \left\{\frac{C_1}{0.07},\frac{C_2}{0.12},\frac{C_3}{0.094},\frac{C_4}{0.068},\frac{C_5}{0.096},\frac{C_6}{0.000},\frac{C_7}{0.128},\frac{C_8}{0.08}\right\}.$$

Step IV: Finally, the largest membership is chosen by $max\ \mu_{(\tilde{\Gamma},A)^*}(C) = 0.128$, which means that, the candidate C_7 has the largest membership grade, hence he /she will be selected for the job. If more than one candidate is needed C_2 will be chosen, because it has second highest membership grade of 0.12.

CONCLUSION

A soft set initiated by Molodtsov is a mapping from a parameter set to subsets of the universe. But the situation may be more complicated in the real life situation, because of the fuzzy characters of the parameters. To further develop the theory, we proved some theorems including the De Morgan's inclusions and laws with relevant examples. We defined AND and OR operations in the fuzzy soft set context. Finally, application of fuzzy soft set to decision making problem was presented with concrete example.

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