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A STUDY ON SOME SUBSTRUCTURES OF ORDERED MULTISETS

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ABSTRACT
In this work, the concepts of chains and antichains of partially ordered sets are studied on multisets where repetition is significant. A partial multiset ordering, $\preccurlyeq\leq_{\mathcal{M}}$, is defined on a finite multiset M in order to introduce hierarchies between its points. Properties of the structure, $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$, and its substructures obtained via this partial multiset ordering are presented. In the sequel, the concept of *semimset chain* is introduced and some results are outlined.
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INTRODUCTION

In a classical set, two mathematical objects are either equal or different. Howbeit, in real life applications, mathematical objects are not necessarily distinct, and each occurrence of an object in an event is significant for computations in order to obtain accurate and exact results or outcomes. Instances where repetition of objects is momentous include, the prime factorization of an integer $n > 0$, repeated roots of polynomial equations, repeated observations in a statistical sample and repeated hydrogen atom in a water molecule, to mention a few. A mathematical model or structure that allows and deals with repeated occurrences of any object is known as a multiset (or mset). The concept of multiple membership has become well-established and has found applications in various areas such as logic, linguistic and computer science among others.

Many researches have appeared on extending notions and related results on orderings from the classical or Cantorian sets to multisets (Dershowitz and Manna, 1979; Conders *et al.*, 2007; Singh *et al.*, 2012; Balogun and Tella, 2017, Balogun and Singh, 2017). A lot of previous works have been concerned with the study of mset orderings on the class $M(S)$, of finite msets defined over a given set S. Dershowitz and Manna (1979) proposed an mset ordering on $M(S)$, this is usually referred to as the standard mset ordering. Several definitions of multiset orderings have been proposed by building on the Dershowitz-Manna mset ordering, each embodying certain limitations and advantages over others (see Tella *et al.*, 2013 for a comparative study of mset orderings). Well-founded partial orderings on $M(S)$ are employed in proving termination of production systems, programs defined in terms of rewriting rules (Bachmair *et al.*, 1986; Dershowitz and Manna, 1979). These orderings are also employed in the theory of partitions (Brandt, 1982).

The theory of partially ordered structures is relatively well developed in the classical setting (see the survey article Trotter, 1995, and monograph Trotter, 1992 for details on partially ordered sets (or posets)). In this work, the notion of a partially ordered multiset (or pomset) is studied using a finite mset M. In order to introduce hierarchies between the points of the mset M, we build on an idea of ordered multisets suggested in Girish and Sunil (2009). Our proposed hierarchy is achieved by defining an ordering on M using the ordering induced by the base set S, and an ordering on the set of the multiplicities of the elements of S. Pomsets have applications in database systems, distributed computing and geographical information system to mention a few. To make the article self-contained, we present some basic definitions and notations on msets in section 2. In the next section, we define a pomset $\mathcal{M}$, and present some properties of the ordered multiset structure $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$. Substructures of $\mathcal{M}$ are presented in section 4. In section 5, the concept of semimset chain is introduced and some results are outlined.

MULTISETS

Formally, an mset is a mapping from some ground or generic set say S, into some set of numbers. The root set of an mset M (denoted by M^*), is the set $\{x \in S | M(x) > 0\}$, where $M(x)$ represents the multiplicity of the element x in M. An mset is completely described by its root set and the multiplicities of each element of the root set. Each individual occurrence of an object in an mset is called its *element,* while *objects* of an mset are the distinguishable elements. For instance, in the mset $M = [1,2,3,4,1,3,1,4]$, the objects of M are; 1,2,3,4. While the elements of M are; 1,2,3,4,1,3,1,4. The *cardinality* of an mset is the sum of the multiplicities of all its objects. In most of the known application areas, objects are allowed to repeat in an mset finitely. Infinite and negative multiplicities of objects have also been studied in the literature (Blizard, 1990). In this work, we consider finite msets whose objects have nonnegative integral multiplicities. The use of square brackets to represent an mset has become almost standard (Meyer and McRobbie, 1982). An mset containing one occurrence of x_1, two occurrences of x_2 and three occurrences of x_3 can be represented using any of the following: $[[x_1, x_2, x_2, x_3, x_3, x_3]]$, $[x_1, x_2, x_2, x_3, x_3, x_3]$, $[x_1, x_2, x_3]_{1,2,3}$, $[x_1{}^1, x_2{}^2, x_3{}^3]$, or $[x_1 1, x_2 2, x_3 3]$, depending on one's preference. We shall adopt the representation, $M = [m_1 x_1, m_2 x_2, \ldots, m_r x_r]$. A *point* in a finite mset M shall be denoted by $m_i x_i$, where m_i is the multiplicity of the element x_i in M, for $1 \leq i \leq r$, i.e., a point represents an object in M together with its multiplicity. Given two msets M and N in $M(S)$, the mset M is a *submset* of N, denoted by $M \subseteq N$, if $M(x) \leq N(x)$ for all $x \in S$, and M is a *proper* submset of N if and only if $M(x) < N(x)$ for at least one x. A submset of a given mset that contains all multiplicities of common elements is called a *whole submset*. A *full submset* contains all objects of the parent mset. The *union* of two msets M and N is the mset given by $(M \cup N)(x) = max\{m, n\}$ such that, $mx \in M$ and $nx \in N$ for all $x \in S$. The *intersection* of M and N is the mset given by $(M \cap N)(x) = min\{m, n\}$ such that, $mx \in M$ and $nx \in N$ for all $x \in S$. In Blizard (1989), a deep survey and axiomatic introduction of msets is presented (see also Singh et al., 2007; Singh and Isah, 2016 for further details on msets).

The Structure $(M, \preccurlyeq\leq_{\mathcal{M}})$

We present the following definition to introduce hierarchies between the points of a finite mset M, using a partially ordered base set and an ordered set of multiplicities. This definition gives a non-trivial involvement of the multiplicities in ordering the points of the mset M.

Definition 1

Let $(S, \preccurlyeq)$ and $(\mathbb{N}, \leq)$ be ordered sets, where the ordering on $\mathbb{N}$ could be (but not necessarily) the natural ordering. For any pair of points $m_i x_i$ *and* $m_j x_j$ in $M \in M(S)$, $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$ if and only if $(x_i \preccurlyeq x_j) \wedge (\ m_i \leq m_j)$. The points $m_i x_i$ *and* $m_j x_j$ coincide i.e., $m_i x_i == m_j x_j$ if and only if $x_i = x_j$ ($m_i = m_j$ follows from the principle of uniqueness of the multiplicity of an object in an mset). The points $m_i x_i$ *and* $m_j x_j$ in M are said to be comparable in $\mathcal{M}$ if and only if $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j \vee m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_i x_i$, otherwise they are incomparable, denoted by $m_i x_i || m_i x_j$.

Remark 1

By definition 1, a point $m_i x_i$ does not precede another point $m_j x_j$ under $\preccurlyeq\leq_{\mathcal{M}}$ if any of the following conditions hold:

i. $m_i x_i \npreccurlyeq\leq_{\mathcal{M}} m_j x_j$
ii. $m_i x_i \preccurlyeq\nleq_{\mathcal{M}} m_j x_j$
iii. $m_i x_i \npreccurlyeq\nleq_{\mathcal{M}} m_j x_j$

The points $m_i x_i$ and $m_j x_j$ are incomparable if any of conditions i or ii holds, or $[(m_i x_i \npreccurlyeq\nleq_{\mathcal{M}} m_j x_j) \wedge (m_i x_i \nsucccurlyeq\ngtr_{\mathcal{M}} m_j x_j)]$.

Definition 2

The relation $\preccurlyeq\leq_{\mathcal{M}}$ on M is *reflexive* if and only if $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_i x_i$ for any point $m_i x_i \in M$. It is *antisymmetric* if and only if $(m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j) \wedge (m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_i x_i)$ implies $m_i x_i == m_j x_j$ for all $m_i x_i, m_j x_j$ in M, and *transitive* if and only if $(m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j) \wedge (m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_k x_k)$ implies $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_k x_k$. The relation $\preccurlyeq\leq_{\mathcal{M}}$ is a *partial mset order* if it is reflexive, antisymmetric and transitive, and a *strict partial mset order* (denoted by $\prec<_{\mathcal{M}}$) if it is irreflexive and transitive.

Definition 3

A pomset $\mathcal{M}$ is a pair $(M, \preccurlyeq\leq_{\mathcal{M}})$, where M is an mset and $\preccurlyeq\leq_{\mathcal{M}}$ is a partial mset order on M.

We have the following results:

Theorem 1

Let S be a partially ordered set. For any $M \in M(S)$, $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ is a pomset.

Proof

Let $x_i \in S$ and $m_i \in \mathbb{N}$, for $i = 1,2, \dots, n$. In particular $x_i \in M^*$ for any point $m_i x_i \in M$, where $M^* \subseteq S$ is the root set of M.
For any $m_i x_i$ in M, $x_i \preccurlyeq x_i$ and $m_i \leq m_i$. Thus, $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_i x_i$ and $(M, \preccurlyeq\leq_{\mathcal{M}})$ is reflexive.
For antisymmetry, suppose that $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j$ and $m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_i x_i$ in $\mathcal{M}$. Then, $x_i \preccurlyeq x_j$ and $x_j \preccurlyeq x_i$
Thus, $x_i = x_j$ (since S is partially ordered) (1)
Also, $m_i \leq m_j$ and $m_j \leq m_i$ implies
$m_i = m_j$ for any $m_i, m_j \in \mathbb{N}$ (2)
(1) and (2) imply $m_i x_i == m_j x_j$, therefore $\preccurlyeq\leq_{\mathcal{M}}$ is antisymmetric.
Let $m_i x_i, m_j x_j, m_k x_k$ be points in M such that $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_j x_j \preccurlyeq\leq_{\mathcal{M}} m_k x_k$
Thus, $x_i \preccurlyeq x_j \preccurlyeq x_k$
In particular, $x_i \preccurlyeq x_k$ (3)
Similarly,
$m_i \leq m_j \leq m_k$ implies
$m_i \leq m_k$ (4)
(3) and (4) imply $m_i x_i \preccurlyeq\leq_{\mathcal{M}} m_k x_k$, thus $\preccurlyeq\leq_{\mathcal{M}}$ is transitive.
Therefore $(M, \preccurlyeq\leq_{\mathcal{M}})$ is a pomset. □

Theorem 2

Let $(S, \preccurlyeq_1)$ and $(S, \preccurlyeq_2)$ be two posets. If $\mathcal{M} = (M, \preccurlyeq_1\leq_1)$ and $\mathcal{N} = (M, \preccurlyeq_2\leq_2)$ are pomsets with $M \in M(S)$ then, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq\leq_{\mathcal{M}\cap\mathcal{N}})$ is a pomset. Where $\preccurlyeq\leq_{\mathcal{M}\cap\mathcal{N}} = \preccurlyeq_1\leq_1 \cap \preccurlyeq_2\leq_2$.

Proof

For any point $m_i x_i$ in M, clearly $m_i x_i \preccurlyeq_1\leq_1 m_i x_i$ and $m_i x_i \preccurlyeq_2\leq_2 m_i x_i$ since $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ are partial mset orders.
Thus, $m_i x_i \preccurlyeq\leq_{\mathcal{M}\cap\mathcal{N}} m_i x_i$ (reflexive property)
Next, let $m_i x_i$ and $m_j x_j$ be points in M such that
$m_i x_i \preccurlyeq\leq_{\mathcal{M}\cap\mathcal{N}} m_j x_j$ and $m_j x_j \preccurlyeq\leq_{\mathcal{M}\cap\mathcal{N}} m_i x_i$ (1)
From (1) we have,
$m_i x_i \preccurlyeq_1\leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1\leq_1 m_i x_i$ (2)
Since $\preccurlyeq_1\leq_1$ is antisymmetric, we have
$m_i x_i =_1=_1 m_j x_j$ (3)
Similarly, from (1) we have,
$m_i x_i \preccurlyeq_2\leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2\leq_2 m_i x_i$ (4)

imply
$m_i x_i =_2 =_2 m_j x_j$ (5)
From (2) - (5) we have,
$m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_j x_j$ and $m_j x_j \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_i x_i$ imply $m_i x_i == m_j x_j$
Therefore, $\preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}}$ is antisymmetric.
For transitivity,
let $m_i x_i, m_j x_j$ and $m_k x_k$ be points in M such that,
$m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_j x_j$ and $m_j x_j \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_k x_k$
Now,
$m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_j x_j$ and $m_j x_j \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_k x_k$ imply
$m_i x_i \preccurlyeq_1 \leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1 m_k x_k$
Since $\preccurlyeq_1 \leq_1$ is transitive, we have
$m_i x_i \preccurlyeq_1 \leq_1 m_k x_k$ (6)
Similarly,
$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_k x_k$ imply,
$m_i x_i \preccurlyeq_2 \leq_2 m_k x_k$ (7)
From (6) and (7) we have,

$$m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_k x_k$$

Since $m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_j x_j$ and $m_j x_j \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_k x_k$ imply $m_i x_i \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}} m_k x_k$, then $\preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}}$ is transitive.
Therefore, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq \leq_{\mathcal{M} \cap \mathcal{N}})$ is a pomset. □

Lemma 3
A total ordering on S does not necessarily induce a total ordering on $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$.
Proof
Let $x_1, x_2, \ldots, x_n$ be elements of S ordered such that $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n$, and suppose $\leq$ is the natural ordering on $\mathbb{N}$, we have $(m_i \leq m_j) \vee (m_j \leq m_i)$ for all m_i, m_j in $\mathbb{N}$. The case where $x_i = x_j$ is trivial since $m_i = m_j$ follows by the principle of uniqueness of the multiplicity of an object.
Suppose $m_i \leq m_j$ for $i < j$, it follows that $m_i x_i \preccurlyeq \leq_{\mathcal{M}} m_j x_j$. If this condition holds for all i, j then $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$ is totally ordered.
Now, suppose $m_j < m_i$ for some i, j with $i < j$, this implies $m_i x_i \prec \nless_{\mathcal{M}} m_j x_j$ for some i, j, hence, $\mathcal{M}$ cannot be totally ordered.
Again, suppose $\leq$ is some other partial ordering on the set $\mathbb{N}$ (say $m_i \leq m_j$ if and only if m_i divides m_j, hence, we could have $m_i || m_j$ for some i, j). Then, M could have points $m_i x_i$ and $m_j x_j$ with $m_i x_i \preccurlyeq \nleq_{\mathcal{M}} m_j x_j$. Therefore, a total ordering on S does not necessarily induce a total ordering on $\mathcal{M}$.
□

Definition 4
Let $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$ be a pomset. A point $m_i x_i$ in M is *maximal* in $\mathcal{M}$ if for any other point $m_j x_j \in M$ with $m_i x_i \preccurlyeq \leq_{\mathcal{M}} m_j x_j$ we have $m_i x_i == m_j x_j$. Similarly, a point $m_i x_i$ in M is *minimal* if for any other point $m_j x_j \in M$ with $m_j x_j \preccurlyeq \leq_{\mathcal{M}} m_i x_i$ we have $m_i x_i == m_j x_j$. If such points are unique, they are called *maximum* and *minimum* points, respectively.

Remark 2
Observe that with the definition of the ordering $\preccurlyeq \leq_{\mathcal{M}}$, a point $m_i x_i$ being maximal in $(M, \preccurlyeq \leq_{\mathcal{M}})$ does not imply that the generating object x_i is maximal in $(S, \preccurlyeq)$. This is illustrated in the following example:

Example 1
Let $\mathcal{M}$ be the pomset with only two points say $[10x_1, 5x_2]$, where $x_1 \preccurlyeq x_2$ in $(S, \preccurlyeq)$ and $\leq$ is the natural ordering on $\mathbb{N}$.
Now, $x_1 \preccurlyeq x_2$ and $10 > 5$, thus by definition 1, we have, $10x_1 \preccurlyeq \nleq_{\mathcal{M}} 5x_2$ and $5x_2 \not\preccurlyeq \leq_{\mathcal{M}} 10x_1$ in $\mathcal{M}$, and hence $10x_1$ and $5x_2$ are both maximal in $(M, \preccurlyeq \leq_{\mathcal{M}})$. The point $10x_1$ is maximal in $(M, \preccurlyeq \leq_{\mathcal{M}})$ but its generating object x_1 is not maximal in $(S, \preccurlyeq)$.

Definition 5
Let $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$ be a pomset, and N a submset of M. A suborder $\preccurlyeq \leq_{\mathcal{K}}$ is the restriction of $\preccurlyeq \leq_{\mathcal{M}}$ to pairs of points in the submset N of M such that,
$n_i x_i \preccurlyeq \leq_{\mathcal{K}} n_j x_j \Leftrightarrow n_i x_i \preccurlyeq \leq_{\mathcal{M}} n_j x_j$, where $n_i \leq m_i$. The pair $(N, \preccurlyeq \leq_{\mathcal{K}})$ is called a subpomset of $\mathcal{M}$.

Definition 6
A subpomset $\mathcal{C} = (N, \preccurlyeq \leq_{\mathcal{C}})$ of a pomset $\mathcal{M} = (M, \preccurlyeq \leq_{\mathcal{M}})$ is called an *mset chain* if $\mathcal{C}$ is linearly (or totally) ordered. A subpomset $\mathcal{A}$ of $\mathcal{M}$ is called an *mset antichain* if no two points are comparable in $\mathcal{A}$. The pomset $\mathcal{M}$ is *connected* (or is an mset chain) if $(m_i x_i \preccurlyeq \leq_{\mathcal{M}} m_j x_j) \vee (m_j x_j \preccurlyeq \leq_{\mathcal{M}} m_i x_i)$ holds for all pairs of points $m_i x_i, m_j x_j \in M$. Also, $\mathcal{M}$ is an mset antichain if $m_i x_i || m_j x_j$ for all distinct pairs $m_i x_i, m_j x_j$ in M

Example 2
Let $M = [x_1, 3x_2, 5x_3, 3x_4, 8x_5, 2x_6]$, where the root set is partially ordered as follows: $x_1 \preccurlyeq x_3 \preccurlyeq x_5$, $x_1 \preccurlyeq x_4$, and $x_2 \preccurlyeq x_4 \preccurlyeq x_6$, and $\leq$ is the natural ordering on $\mathbb{N}$.
The following are mset chains in $\mathcal{M}$:

$$\mathcal{C}_1 = [x_1, 5x_3, 8x_5]$$
$$\mathcal{C}_2 = [x_1, 3x_4]$$
$$\mathcal{C}_3 = [3x_2, 3x_4]$$
$$\mathcal{C}_4 = [5x_3, 8x_5]$$

Also, the following are mset antichains in $\mathcal{M}$:

$$\mathcal{A}_1 = [x_1, 3x_2, 2x_6]$$
$$\mathcal{A}_2 = [3x_2, 5x_3]$$
$$\mathcal{A}_3 = [5x_3, 2x_6]$$

Semimset Chains

With the ordering $\preccurlyeq\leq_{\mathcal{M}}$, cases where the incomparable pairs of points in a given event satisfy condition i or ii below abound.

i. $((x_i \preccurlyeq x_j) \vee (x_j \preccurlyeq x_i)) \wedge ((m_i \nleq m_j) \vee (m_j \nleq m_i))$

ii. $((x_i \npreccurlyeq x_j) \vee (x_j \npreccurlyeq x_i)) \wedge ((m_i \leq m_j) \vee (m_j \leq m_i))$

That is, we have incomparable pairs m_ix_i, m_jx_j such that $m_ix_i \preccurlyeq\nleq_{\mathcal{M}} m_jx_j$ or $m_ix_i \npreccurlyeq\leq_{\mathcal{M}} m_jx_j$.

We define subpomsets whose incomparable points are of the form i or/and ii only as follows:

Definition 7

A subpomset ζ of a pomset $\mathcal{M}$ is called a semimset chain if $m_ix_i \preccurlyeq\nleq_{\mathcal{M}} m_jx_j \vee m_ix_i \npreccurlyeq\leq_{\mathcal{M}} m_jx_j$ holds for all incomparable points m_ix_i and m_jx_j in M.

Example 3

Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset and $M = [6x_1, 4x_2, 6x_3, 5x_4, 4x_5]$. Assume that the root set and multiplicities are ordered as follows respectively:

$x_1 \preccurlyeq x_2 \preccurlyeq x_5, x_3 \preccurlyeq x_4$ and $1 < 3 < 5$, $2 < 4 < 6$, i.e., $m_i < m_j$ if and only if m_i and m_j are both odd (even) and m_i is smaller than m_j.

Then, the following are subpomsets of $\mathcal{M}$:

$$H_1 = [2x_1, 4x_2, 3x_5]$$
$$H_2 = [x_3, 3x_2, 5x_4]$$
$$H_3 = [3x_3, 2x_4, 4x_5]$$

H_1 and H_2 are semimset chains in $\mathcal{M}$, while H_3 is an mset antichain.

Observe that,

For the subpomset H_1, we have,

$$2x_1 \preccurlyeq\leq_{\mathcal{M}} 4x_2$$
$$2x_1 \preccurlyeq\nleq_{\mathcal{M}} 3x_5$$
$$4x_2 \preccurlyeq\nleq_{\mathcal{M}} 3x_5$$

For the subpomset H_2 we have,

$$x_3 \npreccurlyeq\leq_{\mathcal{M}} 3x_2$$
$$x_3 \preccurlyeq\leq_{\mathcal{M}} 5x_4$$
$$3x_2 \preccurlyeq\leq_{\mathcal{M}} 5x_4$$

For the subpomset H_3 we have,

$$3x_3 \preccurlyeq\nleq_{\mathcal{M}} 2x_4$$
$$3x_3 \npreccurlyeq\nleq_{\mathcal{M}} 4x_5$$
$$2x_4 \npreccurlyeq\leq_{\mathcal{M}} 4x_5$$

Remark 3

The notion of semimset chains could be quite useful when characterizing an embedding of the pomset $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ into a linear order. This has application in the scheduling or jump number problem.

Definition 8

A semimset chain ζ in a pomset $\mathcal{M}$ is maximal if it is not strictly contained in any other semimset chain of $\mathcal{M}$.

Proposition 4

An mset antichain $\mathcal{A}$ in a pomset $\mathcal{M}$ is a semimset chain if its root set is well ordered.

Proof

Let $\mathcal{A}^*$ denote the root set of $\mathcal{A}$. Now, $\mathcal{A}$ is an mset antichain implies that $m_ix_i || m_jx_j$ for all m_ix_i, m_jx_j, hence neither $m_ix_i \preccurlyeq\leq_{\mathcal{M}} m_jx_j$ nor $m_jx_j \preccurlyeq\leq_{\mathcal{M}} m_ix_i$ holds in $\mathcal{M}$ for all i, j. But for all $x_i, x_j \in \mathcal{A}^*$, we have $(x_i \preccurlyeq x_j) \vee (x_j \preccurlyeq x_i)$, suppose $\mathcal{A}^* = x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n$.

Since $\mathcal{A}$ is an mset antichain, and $x_i \preccurlyeq x_j$ for all x_i, x_j with $i \leq j$, then it must be the case that $m_i \nleq m_j$ holds for all points m_ix_i, m_jx_j. Thus $m_ix_i \preccurlyeq\nleq_{\mathcal{M}} m_jx_j$ for all $m_ix_i, m_jx_j \in \mathcal{A}$. Therefore $\mathcal{A}$ is a semimset chain. □

Proposition 5

A semimset chain ζ in a pomset $\mathcal{M}$ is an mset antichain if for all points $m_ix_i, m_jx_j \in \zeta$, we have $m_ix_i \npreccurlyeq\leq_{\mathcal{M}} m_jx_j \vee m_ix_i \preccurlyeq\nleq_{\mathcal{M}} m_jx_j$.

The result is straightforward since $m_ix_i || m_jx_j$ for all $m_ix_i, m_jx_j \in \zeta$.

CONCLUSIONS

A partial ordering $\preccurlyeq\leq_{\mathcal{M}}$ was defined on a finite mset M using a partially ordered base set and an ordered set of multiplicities. This partial mset order is found suitable for extending existing notions on partially ordered sets to ordered msets where repetition is significant. The concept of semimset chains which is peculiar to the defined ordered mset structure was outlined. In a semimset chain (a substructure which is a consequence of definition 1) the ordering between the incomparable points are of the form $\preccurlyeq\not\leq_{\mathcal{M}}$ or $\not\preccurlyeq\leq_{\mathcal{M}}$, characterizing an embedding of structures with such incomparable points promises to be useful in modelling application problems like the *scheduling problem* or *jump number* problem (Faigle and Schrader, 1984).

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