FUDMA Journal of Sciences (FJS)
Maiden Edition
Vol. 1 No. 1, November, 2017, pp 84-87

SOME CHARACTERIZATIONS FOR THE DIMENSION OF ORDERED MULTISETS

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Abstract
The paper presents a study of *dimension* as an important combinatorial parameter of ordered multisets defined over partially ordered base sets. The relationship between the dimension of a partially ordered multiset and that of the underlying generic set is investigated and some results are presented.
Mathematics Subject Classification (2010): 06A07, 03E04, 06F25, 91B16

***Keywords**: Partially ordered multiset, multiset linear extension, multiset realizer, dimension*

Introduction

The notion of *dimension* is recognized as the most widely studied combinatorial parameter in the theory of ordered sets (Felsner *et al*., 2015; Joret *et al*., 2016; Streib and Trotter, 2014; Trotter, 2017). The dimension of a partially ordered set (poset) is the minimum number of linear extensions needed to characterize it. Dushnik and Miller (1941) defined the dimension of a poset P, denoted $\dim(P)$, as the least positive integer d for which P has a realizer of size d. An extension of this concept naturally arises if the linear orders which form a realizer of P are required to have certain additional properties (see Kelly (1981), Trotter (1992) and Trotter (1995) for extensive background information on the theory of posets and their dimension). Several characterizations have been established for the dimension of a poset. It can be deduced from Dilworth's decomposition theorem (Dilworth, 1950) that for any poset P, $\dim(P) \leq w$, where w is the width of P. Hiraguchi (1951) proved that $\dim(P) \leq \frac{|S|}{2}$, where $|S|$ is the cardinality of the ground set. In Kierstead and Trotter (1985), several inequalities bounding the greedy dimension of a poset P as a function of other parameters of P were developed. The notion of super greedy dimension was studied in Kierstead et al., (1987). There has been renewed interest in studying combinatorial properties of a poset determined by geometric properties of its order diagram and topological properties of its cover graph. An early result that has motivated research in this direction is the work of Trotter and Moore (1977). Recent results characterizing the dimension of a poset using these properties include, Felsner *et al*., (2015), Trotter and Wang (2015), Joret *et al*., (2016) and Trotter (2017). In Joret *et al*., (2016), it was shown that the dimension of a finite poset is bounded in terms of its height and the tree-width of its cover graph. Felsner *et al*., (2015) proved that the dimension of a poset is bounded if its cover graph is outerplanar. In Trotter and Wang (2015), it was proved that the dimension of a planar poset with t minimal element is at most $2t+1$ and that this bound is tight for $t=1$ and $t=2$, and for $t \geq 3$, there exist planar ordered sets with t minimal elements having dimension sets has come to an age.

In view of numerous applications of *multisets* (msets) found in both hard and soft sciences (Blizard (1989), Singh and Isah, (2016), Singh *et al*., (2007) and Wildberger (2003), are excellent expositions on msets), researchers have shown interest in extending concepts and results on ordered sets to msets (Kilbarda and Jovovi, 2004; Conder et al., 2007; Girish and Sunil 2009). This paper focuses on explicating the concept of dimension of a partially ordered multiset (pomset). Results on the dimension of a poset are generalized using an ordering that introduces hierarchies between the points of a finite mset.

Multisets

When repetition of elements of a set is significant, the mathematical structure obtained is called an mset. If $S = \{x_1, x_2, \ldots, x_n\}$ is a set, an mset M over S is a cardinal-valued function i.e., $M: S \rightarrow \mathbb{N}$ such that $x \in Dom\ (M)$ implies $M(x)$ is a cardinal number and $M(x) = m(x) > 0$, where $m(x)$ denotes the frequency (usually called *multiplicity*) of the object x in M. The set S represents the ground set of the mset M. In this work, for an element $x \in M$ and its multiplicity m, mx will represent a point in M. So that $M = [m_1x_1, m_2x_2, \ldots, m_nx_n]$, where $M(x_i) = m_i$ with $i \in [1, n]$. The set $\{x \in S | M(x) > 0\}$ is called the *root set* of M, and it is denoted M^*. Elements of the root set represent *objects* in an mset, and each individual occurrence of an object is called an *element* of the mset. The *cardinality* of an mset is the sum of the multiplicities of all its distinct elements.

The class of all finite msets containing objects from a set S will be denoted by $M(S)$. For two msets M, N in $M(S)$, M is a *submset* of N, denoted $M \subseteq N$, if $M(x) \leq N(x)$ for all $x \in S$ and M is strictly contained in N if and only if $M(x) < N(x)$ for at least one x. The *null* mset, denoted $\emptyset$, which is contained in every mset is given by, $\emptyset(x) = 0, \forall x \in S$. A submset is called *whole* if it contains all multiplicities of common elements from the parent set. A *full* submset contains all objects of the parent mset. Clearly, every mset contains a unique full submset, its root set. For any two msets M and N, if $M \subseteq N$ and $Dom(M) = Dom(N)$, then M is a full submset of N.

The union of two msets M and N is the mset given by, $(M \cup N)(x) = max\{m, n\}$ such that $mx \in M$ and $nx \in N$, for all $x \in S$. Their intersection is the mset given by $(M \cap N)(x) = min\{m, n\}$, for all $x \in S$. The union $M \uplus N$ is defined as the mset containing $m + n$ occurrences of any object occurring m times in M and n times in N. An mset M is *finite* if both the number of objects in M and their multiplicities are finite. The msets dealt with in this work are finite msets with nonnegative integral multiplicities.

Partially Ordered Multisets (Pomsets)

We present some basic definitions and notations on the ordered mset structure in order to make the article self-contained (see Balogun and Tella (2017) for details). A point in an mset M will be denoted by $m_i x_i$, where, m_i is the multiplicity of x_i in M. Let M be an mset defined over a partially ordered base set $P = (S, \preccurlyeq)$, with points $m_1x_1, m_2x_2, \dots, m_nx_n$. For any pair m_ix_i and m_jx_j in M, with $i, j \in \{1,2,\dots,n\}$, $m_ix_i \preccurlyeq\leq m_jx_j$ if and only if $x_i \preccurlyeq x_j$ in P. The points m_ix_i and m_jx_j coincide if and only if $x_i = x_j$ (principle of uniqueness of multiplicity of an object).

The points m_ix_i and m_jx_j are comparable if and only if, $m_ix_i \preccurlyeq\leq m_jx_j \vee m_jx_j \preccurlyeq\leq m_ix_i$ (denoted $m_ix_i \bowtie m_jx_j$), otherwise they are *incomparable* (denoted $m_ix_i || m_jx_j$).

The ordering $\preccurlyeq\leq$ is called a *partial mset order* or simply an mset order if it is reflexive, antisymmetric and transitive, and a *linear mset order* (or a total mset order) if it is a partial mset order, and all points m_ix_i, m_jx_j in M are comparable under $\preccurlyeq\leq$.

A pomset is a pair $\mathcal{M} = (M, \preccurlyeq\leq)$, where $\preccurlyeq\leq$ is a partial mset order on the mset M. The strict order associated with $\preccurlyeq\leq$ will be denoted by $<<$, where $m_ix_i << m_jx_j$ implies $m_ix_i \preccurlyeq\leq m_jx_j$ and $m_ix_i \neq\neq m_jx_j$.

The dual of the pomset $\mathcal{M}$ is the pomset, denoted $\mathcal{M}^d$, with $m_jx_j \preccurlyeq\leq_{\mathcal{M}^d} m_ix_i$ for all $m_ix_i \preccurlyeq\leq_{\mathcal{M}} m_jx_j$.

Given any mset $M \in M(S)$ defined over a partially ordered base set $P = (S, \preccurlyeq)$, it can be verified that $\mathcal{M} = (M, \preccurlyeq\leq)$ is a pomset. For a pomset $\mathcal{M}$, a point m_ix_i in M is maximal, respectively minimal, when there is no point m_jx_j with $m_ix_i << m_jx_j$, respectively $m_jx_j << m_ix_i$. If $\mathcal{M}$ has a unique maximal, respectively minimal, point, then it is called the maximum, respectively minimum, point of $\mathcal{M}$.

A suborder $\preccurlyeq\leq_{\mathcal{K}}$ is the restriction of $\preccurlyeq\leq$ to pairs of points in $N \subseteq M$, such that, $n_ix_i \preccurlyeq\leq_{\mathcal{K}} n_jx_j \Leftrightarrow m_ix_i \preccurlyeq\leq m_jx_j$ where $n_ix_i, n_jx_j \in N$ and $m_ix_i, m_jx_j \in M$ with $n_i \leq m_i$. The pair $(N, \preccurlyeq\leq_{\mathcal{K}})$ which we simply denote by $\mathcal{K}$, is called a subpomset of $\mathcal{M}$. A subpomset $\mathcal{C} = (N, \preccurlyeq\leq_{\mathcal{C}})$ of a pomset $\mathcal{M}$ is called an mset chain if $\mathcal{C}$ is linearly (or totally) ordered, i.e. $n_ix_i \bowtie n_jx_j$ for all pairs of points $n_ix_i, n_jx_j \in N$. Also, a subpomset $\mathcal{A} = (L, \preccurlyeq\leq_{\mathcal{A}})$ of $\mathcal{M}$ is called an mset antichain if, $l_ix_i || l_jx_j$ for all pairs $l_ix_i, l_jx_j \in L$. An mset chain $\mathcal{C}$ in a pomset $\mathcal{M}$ is *maximal* if it is not strictly contained in any other mset chain of $\mathcal{M}$, and *maximum*, if it contains maximum number of points. Maximal and maximum mset antichains are defined analogously.

The *height* of $\mathcal{M}$ is the number of points in a maximum mset chain and the *width* of $\mathcal{M}$ is the number of points in a maximum mset antichain.

Dimension of a Pomset

Definition 1
For an mset $M \in M(S)$, let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ and $\mathcal{N} = (M, \preccurlyeq\leq_{\mathcal{N}})$ be two pomsets defined over partially ordered base sets P and Q, respectively, where P and Q have the same ground set. Then $\mathcal{N}$ is an *mset extension* of $\mathcal{M}$ if $m_ix_i \preccurlyeq\leq_{\mathcal{M}} m_jx_j$ implies that $m_ix_i \preccurlyeq\leq_{\mathcal{N}} m_jx_j$ i.e. the relation $\preccurlyeq\leq_{\mathcal{M}}$ is contained in $\preccurlyeq\leq_{\mathcal{N}}$. An mset extension of $\mathcal{M}$ that is a linear mset order is called an *mset linear extension* of $\mathcal{M}$.

Example 1
Let $\mathcal{M} = (M, \preccurlyeq\leq_{\mathcal{M}})$ be a pomset where, $M = [3x_1, 5x_2, 2x_3, x_4, x_5, 4x_6, 3x_7]$. Suppose that the mset order on M is defined as follows:
$3x_1 \preccurlyeq\leq_{\mathcal{M}} 5x_2 \preccurlyeq\leq_{\mathcal{M}} x_4 \preccurlyeq\leq_{\mathcal{M}} 3x_7$
$3x_1 \preccurlyeq\leq_{\mathcal{M}} 2x_3 \preccurlyeq\leq_{\mathcal{M}} 4x_6$
$3x_1 \preccurlyeq\leq_{\mathcal{M}} 2x_3 \preccurlyeq\leq_{\mathcal{M}} x_5 \preccurlyeq\leq_{\mathcal{M}} 3x_7$
$3x_1 \preccurlyeq\leq_{\mathcal{M}} 5x_2 \preccurlyeq\leq_{\mathcal{M}} x_5 \preccurlyeq\leq_{\mathcal{M}} 3x_7$
Then, the sets
$\ell_1 = \{3x_1, 5x_2, 2x_3, x_4, x_5, 4x_6, 3x_7\}$ and
$\ell_2 = \{3x_1, 2x_3, 4x_6, 5x_2, x_5, x_4, 3x_7\}$
With,
$3x_1 \preccurlyeq\leq_{\ell_1} 5x_2 \preccurlyeq\leq_{\ell_1} 2x_3 \preccurlyeq\leq_{\ell_1} x_4 \preccurlyeq\leq_{\ell_1} x_5 \preccurlyeq\leq_{\ell_1} 4x_6 \preccurlyeq\leq_{\ell_1} 3x_7$
and
$3x_1 \preccurlyeq\leq_{\ell_2} 2x_3 \preccurlyeq\leq_{\ell_2} 4x_6 \preccurlyeq\leq_{\ell_2} 5x_2 \preccurlyeq\leq_{\ell_2} x_5 \preccurlyeq\leq_{\ell_2} x_4 \preccurlyeq\leq_{\ell_2} 3x_7$
, are mset linear extensions of $\mathcal{M}$ since, $m_ix_i \preccurlyeq\leq_{\ell_1} m_jx_j$ and $m_ix_i \preccurlyeq\leq_{\ell_2} m_jx_j$ whenever $m_ix_i \preccurlyeq\leq_{\mathcal{M}} m_jx_j$, i.e., $\preccurlyeq\leq_{\mathcal{M}}$ is contained in $\preccurlyeq\leq_{\ell_1}$ and $\preccurlyeq\leq_{\ell_2}$.

Mset Realizers
Let $\mathcal{J} = \{(m_ix_i, m_jx_j) \in M : m_ix_i || m_jx_j \text{ in } \mathcal{M}\}$. For any pair m_ix_i, m_jx_j in $\mathcal{J}$, there is a corresponding pair $x_i || x_j$ in P (by the definition of $\preccurlyeq\leq$). By Szpilrajn's extension theorem (Szpilrajn, 1930) for ordered sets, there exist two linear extensions l_1, l_2 with $x_i \preccurlyeq x_j$ in l_1 and $x_j \preccurlyeq x_i$ in l_2. Consequently, $m_ix_i \preccurlyeq\leq_{\ell_1} m_jx_j$ and $m_jx_j \preccurlyeq\leq_{\ell_2} m_ix_i$, for some linearly ordered mset extensions ℓ_1 and ℓ_2 of $\mathcal{M}$ induced by l_1, l_2. An *mset realizer* of a pomset is defined as follow:

Definition 2
A family $\{\ell_1, \ell_2, \dots, \ell_t\}$ of mset linear extensions of $\mathcal{M}$ is called an *mset realizer* of $\mathcal{M}$ if $\mathcal{M} = \ell_1 \cap \ell_2 \cap \dots \cap \ell_t$, i.e., for each pair $m_ix_i || m_jx_j$ in $\mathcal{M}$ there exist ℓ_i, ℓ_j in $\bigcap_{i=1}^{t} \ell_i$ such that $m_ix_i \preccurlyeq\leq_{\ell_i} m_jx_j$ and $m_jx_j \preccurlyeq\leq_{\ell_j} m_ix_i$.

Theorem 1
Let $\mathcal{M}$ be a pomset defined over a poset P. If $\mathcal{R} = \{\ell_1, \ell_2, \dots \ell_t\}$ and $R = \{l_1, l_2, \dots, l_s\}$ are realizers for $\mathcal{M}$ and P, respectively, then $t \leq s$.

Proof
Given a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$, where M has points $m_1x_1, m_2x_2, \dots, m_nx_n$. Let $\mathcal{J} = \{(m_ix_i, m_jx_j) \in M | m_ix_i || m_jx_j \text{ in } \mathcal{M}\}$. For any pair m_ix_i, m_jx_j in $\mathcal{J}$, there exist mset linear extensions ℓ_i, ℓ_j with $m_ix_i <<_{\ell_i} m_jx_j$ and $m_jx_j <<_{\ell_j} m_ix_i$. This implies $t \leq 2n$ for n incomparable pairs in $\mathcal{M}$. Now, for any pair m_ix_i, m_jx_j in M such that $m_ix_i || m_jx_j \text{ in } \mathcal{M}$, there is a corresponding pair $x_i || x_j$ in $(S, \preccurlyeq)$. Also, there exist linear extensions l_i, l_j say, such that $x_i \preccurlyeq_{l_i} x_j$ and $x_j \preccurlyeq_{l_j} x_i$. Similarly, $s \leq 2m$ for m incomparable pairs in P. Let $\mathcal{U} = \{(x_i, x_j) \in S | x_i || x_j \text{ in } P\}$. If for each pair $(x_i, x_j) \in \mathcal{U}$, there exists a corresponding pair $(m_ix_i, m_jx_j) \in \mathcal{J}$, then $m = n$. Since each mset linear order ℓ_i is induced by the linear order l_i, this implies $t = s$, otherwise $t < s$, as $t > s$ is a

contradiction. Hence the result. □

Definition 3
The *dimension* of $\mathcal{M}$, denoted by $\dim(\mathcal{M})$, is the least positive integer t for which $\mathcal{M} = \bigcap_{i=1}^{t} \ell_i$, with $\mathcal{M} \subset \ell_i$ for each i, where ℓ_i is an mset linear order on $\mathcal{M}$.

Corollary 2
Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset defined over a partially ordered base set $P = (S, \preccurlyeq)$. Then for any $M \in M(S)$, $\dim(\mathcal{M}) \leq \dim(P)$.

Proof
The result follows from the preceding theorem. If for each pair $(x_i, x_j) \in \mathcal{U}$, there exists a corresponding pair $(m_i x_i, m_j x_j) \in \mathcal{J}$, then $m = n$ (for m and n incomparable pairs in $\mathcal{J}$ and $\mathcal{U}$, respectively). Consequently $t = s$. Therefore $\dim(\mathcal{M}) = \dim(P)$.
Next, assume that $\dim(\mathcal{M}) > \dim(P)$, this implies that $m > n$ i.e., there exist some $(m_i x_i, m_j x_j) \in \mathcal{J}$ with no corresponding generating pair $(x_i, x_j) \in \mathcal{U}$, this is a contradiction. Hence, $\dim(\mathcal{M}) < \dim(P)$. Therefore $\dim(\mathcal{M}) \leq \dim(P)$ as required. □

Corollary 3
Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset defined over a poset $P = (S, \preccurlyeq_P)$, and Q a poset induced by a subset X of S. If the subset $X = M^*$, then the dimension of $\mathcal{M}$ is equal to the dimension of the subposet Q.

Proof
Let $\preccurlyeq_Q$ be the restriction of $\preccurlyeq_P$ to elements of X. Suppose $X = M^*$, then for any x_i, x_j in X with $x_i \preccurlyeq_Q x_j$ there exist points $m_i x_i, m_j x_j$ in M, with $m_i x_i \preccurlyeq \leq m_j x_j$ in $\mathcal{M}$. If the pomset $\mathcal{M}$ is an mset chain, the result is straightforward. Suppose $\mathcal{M}$ is not connected, then, for all $x_i || x_j$ in Q, there is a corresponding pair $m_i x_i || m_j x_j$ in $\mathcal{M}$. This implies $m = n$, for m and n incomparable pairs in $\mathcal{M}$ and Q, respectively. It follows from theorem 1 and corollary 2 that $\dim(\mathcal{M}) = \dim(Q)$. □

Theorem 4
The dimension of a pomset $\mathcal{M} = (M, \preccurlyeq \leq)$ is at most its width.

Proof
Let $\mathcal{M}$ be a pomset defined over a poset $P = (S, \preccurlyeq_P)$. Let m and ϖ be the dimension and width of $\mathcal{M}$ respectively. Let $Q = (M^*, \preccurlyeq_Q)$ be such that there exists an embedding of Q in P, this implies $x_i \preccurlyeq_Q x_j$ whenever $x_i \preccurlyeq_P x_j$. Let n and w be the dimension and width of Q, respectively. It follows from Dilworth's decomposition theorem that,
$n \leq w$
(1)by corollary 3 we have,
$m = n$ (2)
Since for each $x_i || x_j$ in Q there exists $m_i x_i || m_j x_j$ in $\mathcal{M}$, then
$\varpi = w$ (3)
Consequently, (1)-(3) imply $m \leq \varpi$. □

Concluding Remarks

The concept of dimension was defined on a partially ordered multiset structure. The relationship between the dimension of a pomset $\mathcal{M}$ and that of its underlying generic set was investigated and presented via some results. An mset is an extended notion of a set, hence, studying the concept of dimension on an ordered mset structure leads to generalizations. The problem of extending the concepts of greedy and super greedy dimensions studied in Kierstead and Trotter (1985), and Kierstead et al., (1987) via the ordered mset structure used in this work seems promising.

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