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ON ISOMORPHIC SOFT LATTICES AND SOFT SUBLATTICES

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Abstract
This paper crisply presents the fundamentals of soft set theory to emphasize that soft set has enough developed basic supporting tools through which various algebraic structures in theoretical point of view could be developed. We defined the term soft lattice and present the concept of upper bound and least upper bound, lower bound and greatest lower bound in terms of soft set context. Soft lattice is redefined in terms of supremum and infimum and some related results are established. A perception named isomorphic soft lattice and soft sublattice are introduced where some related results are established.

Keywords: *Soft set, Parameters, soft Lattice, isomorphic soft lattice and soft sublattice.*

Introduction

Primarily the aim of soft set theory is to provide a tool with enough parameters to deal with uncertainty associated with the data, whereas on the other hand it has ability to represent the data in a useful manner. Since, the introduction of soft set by Molodtsov, (1999) as a general Mathematical tool for handling uncertainties about vague concept, many researchers have been working on this new area of mathematics. Composition of soft set relations and construction of transitive closure are presented in Ibrahim and Dauda (2012). Feng *et al*. (2008), initiate the study of soft semirings by using the soft set theory. Jun *et al*. (2011) apply the notion of soft sets by Molodtsov to the theory of BCK-algebras and their basic properties are derived. Ibrahim and Yusuf , (2015) focus their discussion on the development of soft set on lattice theory structure and some concepts of lattices are also discussed in details. The hybrid soft sets and some of their applications and operations are presented in [Kharal and Ahmad (2010), Atagun and Sezgin(2011), Manemaran, (2011), Sut , (2012), Aktas and Cagman, (2007), Gong et al., (2010)].
In this paper, we defined the term soft lattices and present the concept of upper bound and least upper bound, lower bound and greatest lower bound in terms of soft set context. We also introduced a perception named isomorphic soft lattice and soft sublattice where some related results are established.

Preliminaries

Let U be a universal set and E be the set of all possible parameters under consideration with respect to U. Let the power set of U (i.e., the set of all subsets of U) be denoted by $P(U)$ and A is a subset of the parameters, E ($A \subseteq E$). The parameters are attributes, characteristics or properties associated with the objects in U. Then we have the followings which can be found in [Qin and Hong (2010), Babitha and Sunil, (2010), Irfan et al., (2009), Cagman et al., (2011), Maji et al., (2002), Maji and Roy (2003)].

Definition 1.1
A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U. That is, a soft set is a parametrized family of subsets of the set U. For all $e \in E, F(e)$ is considered as the set of e −approximate elements of the soft set (F, E).

Definition 1.2
A soft set (F, E) over a universe U is said to be *null* or *empty* soft set denoted by $\widetilde{\emptyset}$, if $\forall e \in E, F(e) = \emptyset$.

Definition 1.3
A soft set (F, A) over a universe U is called *absolute* or *universal soft set* denoted by $\widetilde{(F, A)}$ or $\widetilde{U}$, if $\forall e \in E$, $F(e) = U$.

Definition 1.4
Let $E = \{e_1, e_2, e_3, \ . \ . \ ., e_n\}$ be a set of parameters. The *not-set* of *E* denoted by $\neg E$ is defined as $\neg E$ = $\{\neg e_1, \neg e_2, \neg e_3, \ . \ . \ ., \neg e_n\}$.

Definition 1.5
The *complement* of a soft set (F, E), denoted by $(F, E)^c$, is defined as $(F, E)^c = (F^c, \neg E)$.
Where $F^c : \neg E \rightarrow P(U)$ is a mapping given by $F^c(\alpha) = U - F(\neg\alpha), \forall \alpha \in \neg E$. F^c is called the soft complement function of F. Consequently, $(F^c)^c = F$ and $((F, E)^c)^c = (F, E)$

Definition 1.6
Let (F, A) and (G, B) be any two soft sets over a common universe U, then (F, A) is called a soft subset of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$ if ;

(i) $A \subset B$, and

(ii) $\forall e \in A, F(e) = G(e)$

(F, A) is said to be a soft super set of (G, B), if (G, B) is a subset of (F, A) and it is denoted by $(F, A) \tilde{\supseteq} (G, B)$.

Definition 1.7
Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal, denoted by $(F, A) = (G, B)$, if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 1.8
If (F, A) and (G, B) are two soft sets then "(F, A) AND (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined as $(F, A) \wedge (G, B) = (H, A \times B)$, Where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall (\alpha, \beta) \in A \times B$.

Definition 1.9
If (F, A) and (G, B) are two soft sets then "(F, A) OR (G, B)" denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (P, A \times B)$, where,
$P(\alpha, \beta) = F(\alpha) \cup G(\beta), \forall (\alpha, \beta) \in A \times B$.

The redefined concept of definition 1.8 and 1.9 are as follows:

Definition 1.10 (Ibrahim and Yusuf, 2015)
If (F, A) and (G, B) are two soft sets then "(F, A) AND (G, B)" denoted by $(F, A) \wedge (G, B)$ is defined as $(F, A) \wedge (G, B) = (H, A \cap B)$, where $H(\alpha) = F(\alpha) \cap G(\alpha), \forall \alpha \in A \cap B$.

Definition 1.11 (Ibrahim and Yusuf, 2015)
If (F, A) and (G, B) are two soft sets then "(F, A) OR (G, B)" denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (P, A \cup B)$, where, $P(\alpha) = F(\alpha) \cup G(\alpha), \forall \alpha \in A \cup B$.

Definition 1.12
Let (F, A) and (G, B) be two soft sets over a common universe *U*. The union or extended union of (F, A) and (G, B), denoted by $(F, A) \cup (G, B)$ or $(F, A) \cup_E (G, B)$, is the soft set (H, C) satisfying the following conditions:
(i) $C = A \cup B$, (ii) $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B \\ G(e) & \text{if } e \in B \backslash A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

Definition 1.13
The intersection of two soft sets (F, A) and (G, B) over a common universe set U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C, H(e) = F(e)$ or $G(e)$, we write $(F, A) \cap (G, B) = (H, C)$

Definition 1.14
The extended intersection of soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \cap_E (G, B)$, is the soft set (H, C), where $C = A \cup B$ $\forall e \in C$ and

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B \\ G(e) & \text{if } e \in B \backslash A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

Definition 1.15
The restricted intersection of soft sets (F, A) and (G, B) over a common universe U, denoted by $(F, A) \cap_R (G, B)$, is the soft set (H, C), where $C = A \cap B \neq \emptyset$ such that $H(e) = F(e) \cap G(e)$, $\forall e \in C$.

Definition 1.16
Let (F, A) and (G, B) be two soft sets over a common universe U such that $\cap B \neq \emptyset$. The restricted union of (F, A) and (G, B), denoted by $(F, A) \cup_R (G, B)$, is defined as $(F, A) \cup_R (G, B) = (H, C)$, where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \cup G(e)$.

Definition 1.17
Let (F, A) and (G, B) be two soft sets over a common universe U such that $\cap B \neq \emptyset$. The restricted difference of (F, A) and (G, B) denoted by $(F, A) \sim_R (G, B)$, is defined as $(F, A) \sim_R (G, B) = (H, C)$, Where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \setminus G(e)$.

Definition 1.18
The restricted symmetric difference of two soft sets (F,A) and (G,B) over a common universe U is defined as
$(F,A)\,\Delta(G,B) = (F,A)\ \cup_R\ (G,B) \sim_R \big((F,A)\ \cap_R\ (G,B)\big)$

Various properties of these operations and algebraic structures defined on soft sets could be found in [Burris and Sankappanavar, (1980), Roitman, (1990), Bilkhoff, (1935), Nation et al., (1988)].

Isomorphic soft lattices, and soft sublattices
The word isomorphism is used to signify that two structures are the same except for the nature of their elements. So many researchers have studied isomorphic lattices and sublattice in standard or classical setting. In this section we present isomorphic soft lattice and soft sublattice where some related results are derived.

Definition 2.1
Let (Γ,E) be a soft set. Let $A,B,C\subseteq E$ such that $(F,A),(G,B)$ and (H,C) are all defined. Then (Γ,E) together with the binary operations $\vee$ (disjunction) and $\wedge$ (conjunction) is called soft lattice if the following axioms are satisfied:
L1: (a) $(F,A)\vee(G,B)=(G,B)\vee(F,A)$
(b) $(F,A)\wedge(G,B)=(G,B)\wedge(F,A)$
(Commutative laws)
L2: (a)
$(F,A)\vee\big((G,B)\vee(H,C)\big)=\big((F,A)\vee(G,B)\big)\vee(H,C)$
(b)
$(F,A)\wedge\big((G,B)\wedge(H,C)\big)=\big((F,A)\wedge(G,B)\big)\wedge(H,C)$
(Associative laws)
L3: (a) $(F,A)\vee(F,A)=(F,A)$
(b) $(F,A)\wedge(F,A)=(F,A)$
(Idempotent laws)
L4: (a) $(F,A)=(F,A)\vee\big((F,A)\wedge(G,B)\big)$
(b) $(F,A)=(F,A)\wedge\big((F,A)\vee(G,B)\big)$,
(Absorption laws)
We denote the soft lattice (Γ,E) by $L(\Gamma,E)$. For convenience we simply write L. Where $\vee$ and $\wedge$ are as defined in Definition 1.10 and *Definition 1.11*

Definition 2.2
A soft set (Γ,E) is called an ordered soft set if the parameter set E is ordered.
Remark 2.1
If (Γ,E) is order soft set then for $A,B,C\subseteq E$, $(F,A),(G,B),(H,C)$ are all order soft sets.

Definition 2.4
A binary relation $\subseteq$ defined on the set of parameters E is a partial order on E if for every $A,B,C\subseteq E,(F,A),(G,B),(H,C)$ are defined such that the following axioms are satisfied
(i) $(F,A)\,\widetilde{\subseteq}\,(F,A)$
(Reflexivity)
(ii) $(F,A)\,\widetilde{\subseteq}\,(G,B)$
and
$(G,B)\,\widetilde{\subseteq}\,(F,A)\Rightarrow(F,A)=(G,B)$
(Antisymmetry)
(iii) $(F,A)\,\widetilde{\subseteq}\,(G,B)$
and
$(G,B)\,\widetilde{\subseteq}\,(H,C)\Rightarrow(F,A)\,\widetilde{\subseteq}\,(H,C)$
(Transitivity)

If, in addition, for every $A,B\subseteq E$
(iv) $(F,A)\,\widetilde{\subseteq}\,(G,B)$ or $(G,B)\,\widetilde{\subseteq}\,(F,A)$, then we say $\widetilde{\subseteq}$ is a total order on E.
A non- empty soft set (Γ,E) with a partial order on it is called *partially ordered soft set* denoted as $\big((\Gamma,E),\widetilde{\subseteq}\big)$. If the relation is total order then we say that $\big((\Gamma,E),\widetilde{\subseteq}\big)$ is called *totally ordered soft set.*

Definition 2.4 upper bound
Let (Γ,E) be a partially ordered soft set. Let $A\subseteq E$ such that (F,A) is also a partial ordered soft set. Then a set $\Gamma(e)$ in (Γ,E) is called an *upper bound* for (F,A) if
$\mathrm{F}(a)\subseteq\Gamma(e),e\in E\ \forall\mathrm{F}(a)\in(\mathrm{F},A)$,
and $a\in A$

Definition 2.5 Least upper bound
Let (Γ,E) be a partially ordered soft set. Let $A\subseteq E$ such that (F,A) is also a partial ordered soft set. Then a set $\Gamma(e)$ in $(\Gamma,E),e\in E$ is called the *least upper bound* (lub), or *supremum* of $(\mathrm{F},A)\big(sup(\mathrm{F},A)\big)$ if $\Gamma(e)$ is an *upper bound* of (F,A), and if
$\forall\mathrm{F}(a)\in(\mathrm{F},A),\forall a\in A,\exists e_1\in E$ such that
$\mathrm{F}(a)\subseteq\Gamma(e_1),\forall\mathrm{F}(a)\in(\mathrm{F},A)\Rightarrow\Gamma(e)\subseteq\Gamma(e_1)$
{i.e., $\Gamma(e)$ is the smallest among the *upper bound* of (F,A)}.

Definition 2.6 Lower bound
Let (Γ,E) be a partially ordered soft set. Let $A\subseteq E$ such that (F,A) is also a partially ordered soft set. Then a set $\Gamma(e)$ in (Γ,E) is called a *lower bound* for (F,A) if
$\Gamma(e)\subseteq\mathrm{F}(a),e\in E,\forall\mathrm{F}(a)\in(\mathrm{F},A),\forall a\in A$

Definition 2.7 Greatest lower bound

Let (Γ, E) be a partially ordered soft set. Let $A \subseteq E$ such that (F, A) is also a partially ordered soft set. Then a set $\Gamma(e)$ in $(\Gamma, E), e \in E$ is called the *greatest lower bound* (g. l. b), or *infimum* of $(\mathrm{F}, A)\left(imf\,(\mathrm{F}, A)\right)$ if $\Gamma(e)$ is a *lower bound* of (F, A), and if $\forall F(a) \in (F, A), \forall a \in A, \exists e_1 \in E$ such that $\Gamma(e_1) \subseteq \mathrm{F}(a), \forall \mathrm{F}(a) \in (\mathrm{F}, A), \forall a \in A \Rightarrow \Gamma(e_1) \subseteq \Gamma(e)$ {i.e., $\Gamma(e)$ is the greatest among the lower bound of (F, A)}.

Definition 2.8 Soft lattice in terms of supremum and infimum.

A partially ordered soft set (Γ, E) is a soft lattice denoted by $L(\Gamma, E)$ if and only if for every partially ordered soft subset (F, A) of (Γ, E), the supremum of (F, A) and the infimum of (F, A) exists in (Γ, E).

Definition 2.9

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that $(\mathrm{F}, \mathrm{A}), (\mathrm{G}, \mathrm{B})$, and (H, C) are all defined. Then two soft lattices (F, A) and (G, B) are isomorphic if there is a bijection $\alpha: (\mathrm{F}, \mathrm{A}) \to (\mathrm{G}, \mathrm{B})$ and for $\forall \mathrm{F}(e_1)$, $\mathrm{F}(e_2) \in (\mathrm{F}, \mathrm{A})$,

(i) $\alpha\left(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)\right) = \alpha\left(\mathrm{F}(e_1)\right) \vee \alpha\left(\mathrm{F}(e_2)\right)$

or

(ii) $\alpha\left(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2)\right) = \alpha\left(\mathrm{F}(e_1)\right) \wedge \alpha\left(\mathrm{F}(e_2)\right)$.

Such an α is called an isomorphism.

Remark 2.1

It is useful to note if $\alpha: (F, A) \to (G, B)$ is an isomorphism then $\alpha^{-1}: (G, B) \to (F, A)$ is an isomorphism, and if $\beta: (G, B) \to (H, C)$ is an isomorphism then $\beta o \alpha: (F, A) \to (H, C)$ is an isomorphism.

Definition 2.10

If (F, A) and (G, B) are two partial ordered soft sets (possets) and $\alpha: (F, A) \to (G, B)$ is a map, then we say α is order-preserving if $\alpha\left(\mathrm{F}(a_1)\right) \subseteq \alpha\left(\mathrm{F}(a_2)\right)$ holds in (G, B) whenever $\mathrm{F}(a_1) \subseteq \mathrm{F}(a_2)$ holds in $(F, A), \forall a_1, a_2 \in A$.

Definition 2.11

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that $(\mathrm{F}, \mathrm{A}), (\mathrm{G}, \mathrm{B})$, and (H, C) are all defined. Then two soft lattices (F, A) and (G, B) are isomorphic if there is a bijection $\alpha: (\mathrm{F}, \mathrm{A}) \to (\mathrm{G}, \mathrm{B})$ and for $\forall \mathrm{F}(e_1)$, $\mathrm{F}(e_2) \in (\mathrm{F}, \mathrm{A})$,

(i) $\alpha\left(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)\right) = \alpha\left(\mathrm{F}(e_1)\right) \vee \alpha\left(\mathrm{F}(e_2)\right)$

or

(ii) $\alpha\left(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2)\right) = \alpha\left(\mathrm{F}(e_1)\right) \wedge \alpha\left(\mathrm{F}(e_2)\right)$.

Such an α is called an isomorphism.

Remark 2.1

It is useful to note if $\alpha: (F, A) \to (G, B)$ is an isomorphism then $\alpha^{-1}: (G, B) \to (F, A)$ is an isomorphism, and if $\beta: (G, B) \to (H, C)$ is an isomorphism then $\beta o \alpha: (F, A) \to (H, C)$ is an isomorphism.

Definition 2.12

If (F, A) and (G, B) are two partially ordered soft sets (possets) and $\alpha: (F, A) \to (G, B)$ is a map, then we say α is order-preserving if $\alpha\left(\mathrm{F}(a_1)\right) \subseteq \alpha\left(\mathrm{F}(a_2)\right)$ holds in (G, B) whenever $\mathrm{F}(a_1) \subseteq \mathrm{F}(a_2)$ holds in $(F, A), \forall a_1, a_2 \in A$.

Proof

If $\alpha: (F, A) \to (G, B)$ is an isomorphism, and $\mathrm{F}(e_1) \subseteq \mathrm{F}(e_2)$ holds in (F, A), then

$$\mathrm{F}(e_1) = \mathrm{F}(e_1) \wedge \mathrm{F}(e_2) \text{ so,}$$

$$\alpha\left(\mathrm{F}(e_1)\right) = \alpha\left(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2)\right)$$
$$= \alpha\left(\mathrm{F}(e_1)\right) \wedge \alpha\left(\mathrm{F}(e_2)\right).$$

Hence, $\alpha\left(\mathrm{F}(e_1)\right) \subseteq \alpha\left(\mathrm{F}(e_2)\right)$ and thus, α is order-preserving. As α^{-1} is an isomorphism, it is also order-preserving.

Conversely, let $\alpha: (F, A) \to (G, B)$ be a bijection such that both α and α^{-1} are order-preserving. For $\mathrm{F}(e_1), \mathrm{F}(e_2) \in (F, A)$, we have

$$\mathrm{F}(e_1) \subseteq \mathrm{F}(e_1) \vee \mathrm{F}(e_2) \text{ and}$$

$$\mathrm{F}(e_2) \subseteq \mathrm{F}(e_1) \vee \mathrm{F}(e_2) \text{ so,}$$

$$\alpha\left(\mathrm{F}(e_1)\right) \subseteq \alpha\left(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)\right)$$

and $\alpha\left(\mathrm{F}(e_2)\right) \subseteq \alpha\left(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)\right)$,

hence,

$$\alpha\left(\mathrm{F}(e_1)\right) \vee \alpha\left(\mathrm{F}(e_2)\right) \subseteq \alpha\left(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)\right)$$

Furthermore,

if $\alpha\left(\mathrm{F}(e_1)\right) \vee \alpha\left(\mathrm{F}(e_2)\right) \subseteq F(e_3)$,

where $F(e_3)$ is arbitrary set in (F, A),

then

$\alpha(\mathrm{F}(e_1)) \subseteq F(e_3)$
and $\alpha(\mathrm{F}(e_2)) \subseteq F(e_3)$

hence,

$$\mathrm{F}(e_1) \subseteq \alpha^{-1}(F(e_3))$$

and $\mathrm{F}(e_2) \subseteq \alpha^{-1}(F(e_3))$ so,

$$\mathrm{F}(e_1) \vee \mathrm{F}(e_2) \subseteq \alpha^{-1}(F(e_3)),$$

and thus

$$\alpha(\mathrm{F}(e_1) \vee \mathrm{F}(e_2)) \subseteq F(e_3)$$

$$\Rightarrow \alpha(\mathrm{F}(e_1)) \vee \alpha(\mathrm{F}(e_2)) = \alpha(\mathrm{F}(e_1) \vee \mathrm{F}(e_2))$$

.

Similarly, let $\alpha: (F, A) \to (G, B)$ be a bijection, such that both α and α^{-1} are order-preserving. For $\mathrm{F}(e_1), \mathrm{F}(e_2) \in (F, A)$, we have

$$\mathrm{F}(e_1) \subseteq \mathrm{F}(e_1) \wedge \mathrm{F}(e_2)$$

and $\mathrm{F}(e_2) \subseteq \mathrm{F}(e_1) \wedge \mathrm{F}(e_2)$

so,

$$\alpha(\mathrm{F}(e_1)) \subseteq \alpha(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2))$$

and

$$\alpha(\mathrm{F}(e_2)) \subseteq \alpha(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2))$$

hence,

$$\alpha(\mathrm{F}(e_1)) \wedge \alpha(\mathrm{F}(e_2)) \subseteq \alpha(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2))$$

Furthermore,
if $\alpha(\mathrm{F}(e_1)) \wedge \alpha(\mathrm{F}(e_2)) \subseteq F(e_3)$,
where $F(e_3)$ is arbitrary set in (F, A),
then

$$\alpha(\mathrm{F}(e_1)) \subseteq F(e_3)$$

and $\alpha(\mathrm{F}(e_2)) \subseteq F(e_3)$

hence,

$$\mathrm{F}(e_1) \subseteq \alpha^{-1}(F(e_3))$$

and $\mathrm{F}(e_2) \subseteq \alpha^{-1}(F(e_3))$ so,

$$\mathrm{F}(e_1) \wedge \mathrm{F}(e_2) \subseteq \alpha^{-1}(F(e_3)),$$

and thus

$$\alpha(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2)) \subseteq F(e_3)$$

$$\Rightarrow \alpha(\mathrm{F}(e_1)) \wedge \alpha(\mathrm{F}(e_1)) = \alpha(\mathrm{F}(e_1) \wedge \mathrm{F}(e_2)).$$

Theorem 2.2
Let (Γ, E) be an ordered soft set, and $A, B \subseteq E$, such that (F, A) and (G, B) are defined. Let $\theta: (F, A) \to (G, B)$ be define as
$\theta(F(e_1)) = \{F(e_2) \in (F, A): F(e_2) \subseteq F(e_1), \forall F(e_1) \in (F, A)\}$. Then (F, A) is isomorphic to the range of θ ordered by $\subseteq$.

Proof
If $F(e_1) \subseteq F(e_2)$,
then, $\theta(F(e_1)) \subseteq \theta(F(e_2))$,
Since $F(e_1) \subseteq \theta(F(e_1))$
(By reflexivity)

$$\theta(F(e_1)) \subseteq \theta(F(e_2))$$
$$\Rightarrow F(e_1) \subseteq F(e_2)$$

Thus, $F(e_1) \subseteq F(e_2)$ if and only if

$$\theta(F(e_1)) \subseteq \theta(F(e_2))$$

That θ is one - to – one then follows by antisymmetry.

Theorem 2.3
Let (Γ, E) be a soft lattice. Let $A, B \subseteq E$, such that (F, A) and (G, B) are all defined. If $\alpha: (F, A) \to (G, B)$ is a mapping then the following holds:

(i) $(F, A) \subseteq (G, B) \Rightarrow \alpha((F, A)) \subseteq \alpha((G, B))$

(ii) $\alpha((F, A) \wedge (G, B)) \subseteq \alpha((F, A)) \wedge \alpha((G, B)$,
equality hold if α is one-one.

Proof
(i) $(F, A) \subseteq (G, B) \Rightarrow \alpha((F, A)) \subseteq \alpha((G, B))$
Suppose $(F, A) \subseteq (G, B)$

$$\Rightarrow (F, A) = (F, A) \wedge (G, B)$$
$$\Rightarrow \alpha((F, A)) = \alpha((F, A) \wedge (G, B))$$
$$\Rightarrow \alpha((F, A)) = \alpha((F, A)) \wedge \alpha((G, B))$$

$\Rightarrow \alpha((F, A)) \subseteq \alpha((G, B))$. Hence (1) holds'

(ii) $\alpha((F, A) \wedge (G, B)) \subseteq \alpha((F, A)) \wedge \alpha((G, B)$
Let $G(b) \in \alpha((F, A) \wedge (G, B))$

$$\Rightarrow G(b) \in \alpha((F, A))$$

and $G(b) \in \alpha((G, B)$

$$\Rightarrow G(b) \in \alpha((F, A)) \wedge \alpha((G, B)$$

Hence,
$\alpha((F, A) \wedge (G, B)) \subseteq \alpha((F, A)) \wedge \alpha((G, B)$

Theorem 2.4
If $\alpha: (F, A) \to (G, B)$ is a bijective mapping, then $\alpha^{-1}: (G, B) \to (F, A)$ is also a bijective mapping.

Proof
Let $G(b_1) \neq G(b_2)$
for $G(b_1), G(b_1) \in (G, B)$

$\Rightarrow \alpha^{-1}(G(b_1)) = F(a_1)$ for $F(a_1) \in (F, A)$ and $\alpha^{-1}(G(b_2)) = F(a_2)$ for $F(a_2) \in (F, A)$

$\Rightarrow G(b_1) = \alpha(F(a_1))$

and $G(b_2) = \alpha(F(a_2))$

$\Rightarrow \alpha(F(a_1)) \neq \alpha(F(a_2))$ since α is one-one

$\Rightarrow \alpha^{-1}(G(b_1)) \neq \alpha^{-1}(G(b_2))$. Hence, α^{-1} is one - one.

Let $F(a) \in (F, A)$. Since α is one-one, there exists a unique element $G(b)$ in (G, B) such that

$\alpha(F(a)) = G(b)$

$\Rightarrow F(a) = \alpha^{-1}(G(b))$

$\Rightarrow \alpha^{-1}$ is one-one

$\Rightarrow \alpha^{-1}$ is bijective.

Theorem 2.5

Let $\alpha: (F, A) \rightarrow (G, B)$ and $\beta: (G, B) \rightarrow (H, C)$ be two bijective mapping of soft lattices. Then $\beta o \alpha: (F, A) \rightarrow (H, C)$ is also a bijective mapping.

Proof

Suppose $F(a_1), F(a_2) \in (F, A)$.

Then $F(a_1) \neq F(a_2)$

$\Rightarrow \alpha(F(a_1)) \neq \alpha(F(a_2))$, since α is injective

$\Rightarrow \beta(F(a_1)) \neq \beta(F(a_2))$, since β is injective

$\Rightarrow \beta o \alpha(F(a_1)) \neq \beta o \alpha(F(a_2))$

Hence, $\beta o \alpha$ is one-one.

Let $H(c) \in (H, C)$ there exists $G(b)$ in (G, B) such that $\beta(G(b)) = H(c)$ as β is onto.

Since, α is onto there exists $F(a)$ in (F, A) such that $\alpha(F(a)) = G(b)$.

Then $\beta\left(\alpha(F(a))\right) = H(c)$ for all $H(c)$ in (H, C)

$\Rightarrow \beta o \alpha(F(a)) = H(c)$

$\Rightarrow \beta o \alpha$ is onto. Hence $\beta o \alpha$ is bijective.

Definition 2.13

If (Γ, E) is a soft lattice and $(F, A) \neq \emptyset$ is a soft subset of (Γ, E) such that $\forall$ $F(e_1), F(e_2) \in (F, A)$ both $F(e_1) \vee F(e_2)$ and $F(e_1) \wedge F(e_2)$ are in (F, A), where $\vee$ and $\wedge$ are the soft lattice operations of (Γ, E).

Then we say that (F, A) with the same operations (restricted to (F, A)) is a soft sublattice of (Γ, E).

Remark 2.2

If (F, A) is a soft sublattice of (Γ, E) then for $F(e_1), F(e_2) \in (F, A)$ we will of course have $F(e_1) \subseteq F(e_2) \in (F, A)$ if and only if $F(e_1) \subseteq F(e_2) \in (\Gamma, E)$. It is interesting to note that given a soft lattice (Γ, E), one can often find soft subsets (F, A) which as partial ordered soft sets (using the same order relation) are soft lattices, but which do not qualify as soft sublattice as the operation $\vee$ and $\wedge$ do not agree with those of the original soft lattice (Γ, E).

Example 2.1

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ be the universal set, let $P(U)$ be the power set of U.

Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the sets of parameters, and $e_1 \subseteq e_2 \subseteq e_3 \subseteq e_4 \subseteq e_5$. Let $A = \{e_1, e_2, e_3, e_5\} \subseteq E$. Let (Γ, E) be a soft lattice and (F, A) be soft subset of (Γ, E). Let

$(\Gamma, E) = \{\Gamma(e_1) = \{h_1, h_2, h_4\},$
$\Gamma(e_2) = \{h_1, h_2, h_4\}, \Gamma(e_3) = \{h_1, h_2, h_3, h_4\},$

$\Gamma(e_4) = \{h_1, h_2, h_3, h_4\}, \Gamma(e_5) = \{h_1, h_2, h_3, h_4, h_5\}$
$\}$

$(F, A) = \{F(e_1) = \{h_1, h_2, h_4\},$
$F(e_2) = \{h_1, h_2, h_4\}, F(e_3) = \{h_1, h_2, h_3, h_4\}, F(e_5) = \{h_1, h_2, h_3, h_4, h_5\}$
$\}$

From figure 1, we note that (F, A) as partial ordered soft set is a soft lattice, but (F, A) is not a soft sublattice of soft lattice (Γ, E).

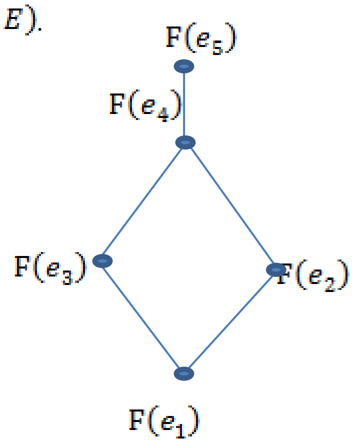

Figure1: soft lattice structure

Definition 2.14
A soft lattice (F,A) can be embedded into a soft lattice (Γ,E) if there is a soft sublattice of (Γ,E) isomorphic to (F,A); in this case we also say (Γ,E) contains a copy of (F,A) as a sublattice.

Example 2.2
Let (F,A) and (Γ,E) be two soft lattices, and (H,C) a soft sublattice of (Γ,E). If there exists an isomorphism $\alpha:(\mathrm{H},C)\to(\mathrm{F},A)$, such that for $\forall\mathrm{H}(e)\in(\mathrm{H},C)$, we have $\alpha\big(\mathrm{H}(e)\big)\in(\mathrm{F},A)$, then we say that (F,A) is embedded in (Γ,E) or (Γ,E) contain a copy of (F,A) as a soft sublattice.

Conclusion

In this research work, we have introduced the concept of soft lattices and studied some of their algebraic properties. Isomorphic soft lattice and soft sublattice were also introduced and their properties have been studied.

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