

**A COMPREHENSIVE ALTERNATIVE PROOF OF THE EQUIVALENCE OF DERSHOWITZ-MANNA AND HUET-OPPEN ORDERINGS**

\*<sup>1</sup>Peter, C. M. & <sup>2</sup>Singh, D.

<sup>1</sup>Department of Mathematical Science and Information Technology,  
Federal University, Dutsin-Ma, Nigeria  
macpee3@yahoo.com

<sup>2</sup>Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria  
(Former professor, Indian Institute of Technology, Bombay, India  
mathdss@yahoo.com

**Abstract**

The research note outlines an alternative proof of the equivalence of the Dershowitz-Manna ordering and the Huet-Oppen ordering which was originally presented by Jouannaud and Lescanne. First, some algebraic foundations of multiset orderings are laid which directly or indirectly leads to the result obtained by means of the Huet-Open ordering.

*Keywords:* Dershowitz-Manna, Huet-Oppen, Multiset, Partial ordering, Operations on multisets.

**Introduction**

The multiset ordering proposed by Dershowitz and Manna is a basis for many functions used for proving the termination of programs and term rewriting systems (Dershowitz and Manna, 1979). An alternative definition of the ordering was given in Huet and Oppen (1980). While the former proves to be more elegant, the latter is more precise.

There seems to exist a conflict of choice as to which definition of multiset ordering is more suitable. The first attempt in tackling this issue is the need to show that the orderings are equivalent. Jouannaud and Lescanne (1982) was the first to make an attempt to show the proof of equivalence of the orderings. In this paper, we present a comprehensive alternative to the proof of their equivalence. First, we present the results of some of the algebraic foundations of multiset orderings.

**Some Preliminary Concepts**

**Definition 2.1 Multiset**

A multiset is an unordered collection of *objects* in which, unlike the elements of the Cantorian set, the objects are allowed to repeat where each occurrence of an object is referred to as an *element* of the multiset. The total number of occurrences of all the objects of a multiset is known as its *cardinality*.

**Definition 2.2 Multiset equality**

Let  $E$  be a base set of the multisets  $M$  and  $N$ , then  $M = N$  if and only if  $M(x) = N(x) \forall x \in E$  where  $M(x)$  and  $N(x)$  denote the multiplicities of  $x$  in  $M$  and  $N$ , respectively.

**Definition 2.3 Submultiset**

Given a multiset  $M$  over a domain set  $E$ , a multiset  $A$  over  $E$  is called a *submultiset* of  $M$  written as  $A \subseteq M$  or  $M \supseteq A$  if  $A(x) \leq M(x)$  for all  $x \in E$ , where  $A(x)$  and  $M(x)$  are the multiplicities of  $x$  in the multisets  $A$  and  $M$ , respectively. Also, if  $A \subseteq M$  and  $A \neq M$ , then  $A$  is called a *proper submultiset* of  $M$ . A multiset is called the *supermultiset* in relation to its submultiset.

**Definition 2.4 Union**

Let  $A$  and  $B$  be two multisets of finite cardinality over a given domain set  $E$ . The *union* of  $A$  and  $B$  denoted  $A \cup B$  is the multiset defined by  $\max(A(x), B(x))$ . That is, an object  $x$  occurring  $m$  times in  $A$  and  $n$  times in  $B$ , occurs  $\max(m, n)$  times in  $A \cup B$ . For example, if  $A = [a, a, b, b, b, c, d]$ ,  $B = [a, b, c, c]$ , then  $A \cup B = [a, a, b, b, b, c, c, d]$ .

**Definition 2.5 Intersection**

The *intersection* of  $A$  and  $B$  denoted  $A \cap B$  is the multiset defined by  $\min(A(x), B(x))$ . That is, an object  $x$  with multiplicity  $m$  in  $A$  and multiplicity  $n$  in  $B$  has multiplicity of  $\min(m, n)$  in  $A \cap B$ . For example, if  $A = [a, a, b, b, d, d, d]$  and  $B = [a, b, b, c, c]$ , then  $A \cap B = [a, b, b]$ .

**Definition 2.6 Additive union**

The *additive union or sum or merge* of two multisets  $A$  and  $B$  denoted  $A \uplus B$  is the multiset defined by  $A(x) + B(x)$ , the direct sum of two numbers. That is, an object  $x$  occurring  $m$  times in  $A$  and  $n$  times in  $B$ , occurs  $m + n$  times in  $A \uplus B$ . For example, given that  $A = [a, a, b, b, d]$ ,  $B = [a, b, c, c, d]$ , then  $A \uplus B = [a, a, a, b, b, b, c, c, d, d]$ .

**Definition 2.7 Multiset Difference**

The *difference* of multisets  $A$  and  $B$  denoted  $A - B$  is the multiset defined by  $\max(A(x) - B(x), 0)$  for all  $x$  in  $D$ . For example, if  $A = [a, a, b, b, d]$ ,  $B = [a, b, c, c, d]$ , then  $A - B = [a, b]$ .

See Singh *et al.* (2007) and Tella *et al.* (2014) for further details on the above definitions.

**Definition 2.8 Partial ordering.**

A binary relation  $\preceq$  on a set  $X$  is called a *partial order* on  $X$  if  $\preceq$  satisfies the following properties on  $X$ :

1.  $x \preceq x$  for all  $x \in X$ . (*Reflexive property*)
2.  $x \preceq y$  and  $y \preceq x$  implies  $x = y$  for all  $x, y \in X$ . (*Anti symmetry property*)
3.  $x \preceq y$  and  $y \preceq z$  implies  $x \preceq z$  for all  $x, y, z \in X$ . (*Transitivity property*)

The set  $X$  is said to be *partially ordered* with respect to  $\preceq$ . We note that for some pair of elements  $x, y$  in  $X$ , neither  $x \preceq y$  nor  $y \preceq x$  may hold. If either  $x \preceq y$  or  $y \preceq x$  holds for all  $x, y$  in  $X$  then  $X$  is said to be *totally ordered* or *linearly ordered* or a *chain*. If on the other hand both  $x \preceq y$  and  $y \preceq x$  hold, then Property 2 would be a *symmetric property* and the ordering would be called an *equivalence relation* (Dornhoff and Hohn, 1978).

In most texts, the usual ordering is symbolized by " $\leq$ " and is often used between real numbers, whereas the symbol " $\preceq$ " is used for the generality of the items that could be considered as elements of a multiset or set (including real numbers). The strict cases of the orderings (that is a case of *irreflexive* property) is denoted by " $<$ " and " $\prec$ ", respectively. Moreover, some texts consider symbols such as " $\ll$ " or " $\prec\prec$ " for the strict case of the orderings that exist among sets or multisets.

**Definition 2.9 Dershowitz-Manna Multiset Ordering**

Let  $M$  and  $N$  be multisets defined over a base set  $E$  and let  $>$  be a partial ordering defined on  $E$ , then  $M \ll N$  if and only if there exist two multisets  $X$  and  $Y$  satisfying the following:

1.  $\emptyset \neq Y \subseteq N$ ,
2.  $M = (N \setminus Y) \uplus X$
3. For all  $x$  in  $X$  there exists  $y$  in  $Y$  such that  $y > x$ .

In other words,  $M \ll N$  if  $M$  is obtained from  $N$  by removing at least one element (those in  $Y$ ) from  $N$ , and replacing each such element  $y$  by zero or any finite number of elements (those in  $X$ ), each of which is strictly less than (in the ordering  $<$ ) one of the elements  $y$  that have been removed (Dershowitz and Manna, 1979).

**Definition 2.10 Huet-Oppen Multiset Ordering**

Let  $M$  and  $N$  be multisets over a base set  $E$  and let  $>$  be a partial ordering on  $E$ , then  $M \ll N$  if and only if  $M \neq N$  and  $M(x) > N(x)$  implies there exists  $y$  in  $E$  where  $y > x$  such that  $N(y) > M(y)$ .

It is easy to see that the last two multiset orderings defined above are *irreflexive*, *anti-symmetric* and *transitive*, (Huet and Oppen, 1980).

**Some Results Holding for Algebraic Operations on Multiset Ordering**

In the results that follow, the order symbol  $\succcurlyeq$  is the Huet-Oppen ordering. As usual, the symbol  $>$  is the ordering defined on an arbitrary multiset, while the symbol  $>$  is the usual ordering on the set of real numbers, integers in this case.

**Lemma 1**

Let  $E$  be a based set of multisets  $M, N$  and  $T$  such that  $N(x) > M(x)$  where  $x \in E$ , then  $[N \uplus T](x) > [M \uplus T](x)$ .

*Proof*

Since  $N(x)$  denotes the multiplicity of the object  $x$  in  $N$ , then  $N(x)$  is a real number. Similarly,  $M(x)$  is a real number. It follows that  $N(x) + T(x)$  and  $M(x) + T(x)$  are real numbers since  $T(x)$  is a real number for a multiset  $T$ . Since  $N(x) > M(x)$  holds by the hypothesis, then  $N(x) + T(x) > M(x) + T(x)$  holds by the properties of real numbers. By the definition of additive union, we have  $[N \uplus T](x) > [M \uplus T](x)$ .

**Lemma 2**

Let  $E$  be a based set of multisets  $M, N$  and  $T$  such that  $N(x) > M(x)$  where  $x \in E$  and  $T \subseteq M$ , then  $[N \setminus T](x) > [M \setminus T](x)$ .

*Proof*

Suppose  $T \subseteq M$ . By the definition of submultiset  $M(x) \geq T(x)$  for all  $x$  in  $E$ . By hypothesis,  $N(x) > M(x)$  holds and  $M(x), N(x)$  and  $T(x)$  are real numbers. By the transitivity property of real numbers,  $N(x) > M(x) \geq T(x)$ . It therefore follows from the properties of real numbers that  $N(x) - T(x) > M(x) - T(x)$ . Therefore,  $[N \setminus T](x) > [M \setminus T](x)$  holds by the definition of multiset difference.

**Lemma 3**

Let  $E$  be a based set of multisets  $M$  and  $N$  such that  $N(x) > M(x)$  where  $x \in E$ , then  $[N \setminus M](x) > [M \setminus N](x) = 0$ .

*Proof*

Since by hypothesis  $N(x) > M(x)$ , then  $N(x) + N(x) > M(x) + M(x)$ . This implies  $N(x) - M(x) > M(x) - N(x)$  by the properties of real numbers. In particular,  $M(x) - N(x) \leq 0$ . By the definition of multiset difference,  $[M \setminus N](x) = \max(M(x) - N(x), 0)$  which is 0 for the object  $x \in E$ .

Hence,  $[N \setminus M](x) > [M \setminus N](x) = 0$ .

**Lemma 4**

Let  $M, N$  and  $T$  be multisets such that  $N \gg M$ , where  $\gg$  is the Huet-Open multiset ordering, then  $(N \uplus T) \gg (M \uplus T)$ .

*Proof*

Let  $E$  be a base set of  $M, N$  and  $T$ . Suppose  $[M \uplus T](x) > [N \uplus T](x)$  for  $x \in E$ , then by definition of additive union  $M(x) + T(x) > N(x) + T(x)$ . That is  $[M(x) + T(x)] - T(x) > [N(x) + T(x)] - T(x)$

It follows that  $M(x) > N(x)$ . By the definition of Huet-Open ordering, there exists  $y$  in  $E$  where  $y \succ x$  such that  $N(y) > M(y)$ . This implies  $[N(y) + T(y)] > [M(y) + T(y)]$ . Again by the definition of additive union  $[N \uplus T](y) > [M \uplus T](y)$ . Hence,  $[N \uplus T] \gg [M \uplus T]$ .  $\square$

**Lemma 5**

Let  $M, N$  and  $T$  be multisets such that  $N \gg M$  and  $T \subseteq M$ , where  $\gg$  is the Huet-Open multiset ordering, then  $(N \setminus T) \gg (M \setminus T)$ .

*Proof*

Let  $E$  be a base set of  $M, N$  and  $T$ . Suppose  $[M \setminus T](x) > [N \setminus T](x)$  holds for an object  $x \in E$ , then by the definition of multiset difference,  $M(x) - T(x) > N(x) - T(x)$ . Thus,  $[M(x) - T(x)] + T(x) > [N(x) - T(x)] + T(x)$

This implies  $M(x) > N(x)$ . By the definition of Huet-Open ordering, there exists  $y \in E$  where  $y \succ x$  such that  $N(y) > M(y)$ . This implies  $[N(y) - T(y)] > [M(y) - T(y)]$ , and by the definition of multiset difference  $[N \setminus T](y) > [M \setminus T](y)$ . Hence,  $[N \setminus T] \gg [M \setminus T]$ .  $\square$

**Lemma 6**

Let  $M, N$  and  $T$  be multisets such that  $N \gg M$  and  $N \subseteq T$ , where  $\gg$  is the Huet-Open multiset ordering, then  $(T \setminus M) \gg (T \setminus N)$ .

*Proof*

Let  $E$  be a base set of  $M, N$  and  $T$ . Suppose  $[T \setminus N](x) > [T \setminus M](x)$  holds for an object  $x \in E$ , then  $T(x) - N(x) > T(x) - M(x)$  holds by the definition of multiset difference. This implies  $[T(x) - N(x)] + T(x) > [T(x) - M(x)] + T(x)$

Hence,  $M(x) > N(x)$ . By the definition of Huet-Open ordering, there exists  $y \in E$  where  $y \succ x$  such that  $N(y) > M(y)$ . This implies  $-M(y) > -N(y)$ , which implies  $[T(y) - M(y)] > [T(y) - N(y)]$  by the properties of real numbers. Subsequently, we get  $[T \setminus M](y) > [T \setminus N](y)$  by multiset difference. Therefore,  $[T \setminus M] \gg [T \setminus N]$ .  $\square$

**Lemma 7**

Let  $M, N$  and  $T$  be multisets such that  $N \gg M$ , where  $\gg$  is the Huet-Open multiset ordering, then  $[N \setminus M] \gg [M \setminus N]$ .

*Proof*

Let  $E$  be a base set of  $M, N$  and  $T$ . Suppose  $[M \setminus N](x) > [N \setminus M](x)$  for  $x \in E$ , then  $x$  is necessarily in  $[M \setminus N]$ . This implies  $M(x) > N(x)$ . Since  $N \gg M$  where  $\gg$  is the Huet-Open ordering then there exists  $y \in E$  where  $y \succ x$  such that  $N(y) > M(y)$  and  $M \neq N$ . By Lemma 3,  $[N \setminus M](y) > [M \setminus N](y)$ , hence  $[N \setminus M] \gg [M \setminus N]$ .  $\square$

**Lemma 8**

Let  $M, N, S$  and  $T$  be multisets such that  $N \gg M$  and  $T \gg S$ , then  $N \uplus T \gg M \uplus S$ .

*Proof*

By Lemma 4,  $N \uplus T \gg M \uplus T$  and  $M \uplus T \gg M \uplus S$ . It follows by transitivity of the ordering that  $N \uplus T \gg M \uplus S$ .  $\square$

**Lemma 9**

Let  $M, N, S$  and  $T$  be multisets such that  $N \gg M$  and  $T \gg S$ . If  $S \subseteq M$  and  $T \subseteq N$ , then  $N \setminus S \gg M \setminus T$ .

*Proof*

By Lemma 5,  $N \setminus S \gg M \setminus S$  and by Lemma 6,  $M \setminus S \gg M \setminus T$ . It follows by transitivity of the ordering that  $N \setminus S \gg M \setminus T$ .  $\square$

**The Equivalence of Dershowitz-Manna Ordering and Huet-Oppen Ordering**

We present an alternative proof of the equivalence of the Dershowitz-Manna ordering and the Huet-Oppen ordering presented in Lemma 2.6 of Jouannaud and Lescanne (1982) in the following theorem.

**Theorem 10**

*The Dershowitz-Manna multiset ordering and the Huet-Oppen multiset ordering are equivalent.*

*Proof*

Let  $E$  be a base set of the multisets  $M$  and  $N$  and let  $\ll_{DM}$  and  $\ll_{HO}$  denote Dershowitz-Manna and Huet-Oppen orderings, respectively. Assume  $M \ll_{HO} N$  holds and define  $X$  and  $Y$  as follows:  $X = M \setminus N$  and  $Y = N \setminus M$ . Thus,  $Y \subseteq N$  is clear. We claim  $Y \neq \emptyset$ . Suppose the contrary, that is  $N \setminus M = \emptyset$ . Either  $N \subset M$  or  $x \in M \setminus N$  for some  $x \in E$ . The former is a contradiction of  $M \ll_{HO} N$  since multiset inclusion satisfies the ordering as well. If the latter is the case then  $M(x) > N(x)$  and by  $\ll_{HO} \exists y \in E$  such that  $N(y) > M(y)$ . This implies  $y \in N \setminus M$  – a contradiction of the assumption  $Y = \emptyset$ . To prove 2, we consider that  $M \setminus X = M \setminus [M \setminus N]$  and  $N \setminus Y = N \setminus [N \setminus M]$ . But  $M \setminus [M \setminus N] = M \cap N = N \setminus [N \setminus M]$ . Thus, we have  $M \setminus X = N \setminus Y$ . By definition  $X \subseteq M$ , hence  $M = [N \setminus Y] \uplus X$ . To prove 3, let  $x \in X$ , then  $x \in [M \setminus N]$  implies  $M(x) > N(x)$ . By the definition of Huet-Oppen ordering, there exists  $y \in E$  where  $y > x$  such that  $N(y) > M(y)$ . Hence,  $y \in [N \setminus M]$  and this implies  $y \in Y$ .

Conversely, assume  $M \ll_{DM} N$ . From 2,  $M \setminus X = N \setminus Y$  and from 3,  $X \neq Y$ . This implies  $M \neq N$ . Assume  $M(x) > N(x)$ . From 2,  $M(x) = N(x) - Y(x) + X(x)$  and we have  $X(x) - Y(x) > 0$ . Hence,  $x \in X$ . From 3, there exists  $y$  where  $y > x$  such that  $y \in Y$ ; thus  $y \notin X$  and  $Y(y) > 0$  implies  $X(y) = 0$ . Moreover,  $M(y) = N(y) - Y(y) + X(y)$  holds by Property 2. This implies  $N(y) - M(y) = Y(y) > 0$ . Hence,  $N(y) > M(y)$ . Since  $x$  and  $y$  are arbitrarily chosen from  $E$ , then  $M \ll_{HO} N$ .  $\square$

**Conclusion**

Firstly, we examined both the Dershowitz-Manna definition and the Huet-Oppen definition of multiset ordering. Secondly, we exploited the Huet-Oppen ordering which is handier in the implementation of the ordering in laying some useful algebraic foundations. Thirdly, we provided a simple but comprehensive alternative to the original version of the proof of equivalence of the two orderings.

**Acknowledgement**

We are grateful to Jean-Pierre Jouannaud and Pierre Lescanne for their work which serves as the basis for this paper. We also thank the anonymous reviewers for their useful comments and corrections.

**References**

Dershowitz, N. and Manna, Z. (1979), Ordering for term rewriting systems, *Proc. 20th Symp. on Foundations of Computer Science*, 123-131.

Dershowitz, N. and Manna, Z. (1979), Proving Termination with Multiset Ordering, *Comm. ACM*, Vol. 22, 465-476.

Dornhoff, L. L. and Hohn, F. E. *Applied Modern Algebra*, Macmillan Publishing Co, 1978.

Huet, G. and Oppen, D. C. (1980). Equations and rewrite rules. *Formal language theory: perspectives and open problems*, 349-405.

Jouannaud, J. and Lescanne (1982), P., On multiset ordering, *Information processing letters*, Volume 15, Number 2, 57-63.

Singh, D., Ibrahim, A., Yohanna, T., and Singh, J. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, 37(3), 73- 92.

Tella, Y., Singh, D. and Singh, J. N. (2014). A Comparative Study of Multiset Orderings. *International Journal of Mathematics and Statistics Invention (IJMSI)*, 2(5), 59-70