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ON THE PROPERTIES OF TOPP-LEONE KUMARASWAMYWEIBUL DISTRIBUTION WITH APPLICATIONS TO BIOMEDICAL DATA

[1]Ismail Kolawole Adekunle, *[2]Ibrahim Sule, [2]Olalekan Akanji Bello

[1]Department of Mathematics and Statistics, School of Applied Sciences, Kaduna Polytechnic, Kaduna.
[2]Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria.

*Corresponding authors' email: ibrahimsule76@gmail.com

ABSTRACT

In this study, a new four-parameter lifetime distribution called the Topp Leone KumaraswamyWeibull distribution was derived using the Topp-Leone Kumaraswamy-G family of distributions. The model includes several important sub-models as special cases such as Topp-Leone exponentiatedWeibull,Topp Leone Weibull, exponentiatedWeibull and Weibull distributions. An expansion for the probability distribution function was carried out which was used to derive some of the mathematical properties. Some mathematical properties of the distribution were presented such as moments, moment generating function, quantile function, survival function, hazard function as well as mean, 1st quartile, median and 3rd quartile. The probability distribution function of order statistics of the Topp-Leone KumaraswamyWeibull distribution was obtained. Estimation of the parameters by maximum likelihood estimation method was discussed. Two real-life application of the distribution was presented and the analysis showed the fit and flexibility of the new distribution over some lifetime models considered. The analysis showed that the model is effective in fitting biomedical data.

Keywords: Statistical distributions, Topp-Leone Kumaraswamy-G family, Maximum order statistic, Minimum order statistic, Skewness, Kurtosis

INTRODUCTION

Numerous statistical distributions have been used widely to describe and forecast current occurrences in a variety of fields, including biology, engineering, economics, geography, and many more. However, the data in many of these areas typically exhibit complex behavior and a variety of forms that are linked to varying levels of skewness and kurtosis. As a result, a lot of the classical distributions that are currently in use have certain limits when fitting these data, hence using these classical distributions on the data sets may not result in an acceptable fit.

In statistical distribution theory, researchers are in the quest to generate new distributions by adding more parameters to the classical distributions in order to make them more robust, versatile and flexible in fitting different kinds of data. Some new ideas of generalizing probability distributions can be found in Beta-G family due to Eugene *et al.,* (2002), Transmuted-G family due to (Shaw & Buckley, 2007), gamma-G family due to (Zografos & Balakrishanan, 2009), Kumaraswamy-G family due to (Cordeiro & de Castro 2011), McDonald-G family due to (Alexander *et al.,* 2012), T-X family due to (Alzaatreh *et al.,* 2013), the exponentiated T-X family due to (Alzaghal *et al.,* 2013), the Weibull-G family due to (Bourguignon *et al.,* 2014), a quantile based T-XY approach due to (Aljarrah *et al.,* 2014), the logistic-G family due to (Tahir *et al.,* 2016), Topp-Leone-G due to (Al-Shomrani *et al.,*, 2016), Topp-Leone Expononentiated-G family due to (Ibrahim *et al.,*, 2020a), Type I Half Logistic Exponentiated-G family due to (Bello *et al.,* 2020), Type II Half Logistic Exponentiated-G family due to (Bello *et al.,* 2021), extended Topp-Leone exponentiated generalized-G family due to (Sule *et al.,* 2022).

The Weibull distribution is a very popular model and has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is generally adequate for modeling monotone hazard rates (Bourguignon *et al.,* 2014).

The cumulative density function (cdf) and probability distribution function (pdf) of Weibull distribution are given respectively as:

$$G(x;\lambda,\beta)=1-e^{-(\beta x)^{\lambda}} \tag{1}$$

$$g(x;\lambda,\beta)=\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}} \tag{2}$$

There are cases when the standard Weibull distribution fails to model data suitably and this is where it is necessary to apply generalized Weibull distribution because of its flexibility and better fit of the data than the classical Weibull distribution. The importance of such generalization has been proved in recent times on various problems and many classical distributions have been generalized. Some recent generalizations on Weibull distribution are; Mudholkar and Srivastava (1993) proposed the Exponentiated Weibull distribution, (Lai *et al.* 2003) introduced modified Weibull distribution, (Lee *et al.* 2007) studied Beta-Weibull distribution, (Bebbington *et al.,* 2007) suggested a flexible Weibull distribution, (Carrasco *et al.,* 2008) proposed generalized modified Weibull distribution, Silva *et al.* (2010) and Nadarajah *et al.* (2011) suggested Beta modified Weibull distribution and Singla *et al.* (2012) studied the mathematical properties of the Beta generalized Weibull distribution, Elbatal and Aryal (2013) studied transmuted additive Weibull distribution, Nofal *et al.* (2016) proposed .Kumaraswamy transmuted exponentiated additive Weibull distribution, Al-Sulami (2020) studied Exponentiated Exponential Weibull Distribution, Ibrahim (2021) derived Topp-Leone exponentiated Weibull distribution.

MATERIALS AND METHODS

Ibrahim *et al.* (2020b) proposed Topp-Leone Kumarawamy-G family of distributions with cdf and pdf given respectively as

$$F(x;\alpha,\sigma,\theta)=\left[1-\left[1-\left[G(x;\varphi)\right]^{\alpha}\right]^{2\sigma}\right]^{\theta} \tag{3}$$

$$f(x;\alpha,\sigma,\theta)=2\alpha\sigma\theta\left[G(x;\varphi)\right]^{\alpha-1}\left[1-\left[G(x;\varphi)\right]^{\alpha}\right]^{2\sigma-1}\left[1-\left[1-\left[G(x;\varphi)\right]^{\alpha}\right]^{2\sigma}\right]^{\theta-1} \tag{4}$$

The Topp-Leone KumarawamyWeibull(TLKW) Distribution

The Topp-Leone KumaraswamyWeibull distribution is obtained by inserting equation (1) into equation (2) as

$$F(x;\alpha,\lambda,\sigma,\beta,\theta)=\left[1-\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]^{\theta} \tag{5}$$

On differentiating equation (5), we have the pdf of TLKW distribution given as

$$f(x;\alpha,\lambda,\sigma,\beta,\theta)=2\alpha\lambda\sigma\theta\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}$$
$$\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma-1}\left[1-\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]^{\theta-1} \tag{6}$$

$$x\geq 0,\alpha,\lambda,\sigma,\beta,\theta>0$$

Where β is the scale parameter and $\alpha,\lambda,\sigma,\theta$ are the shape parameters.

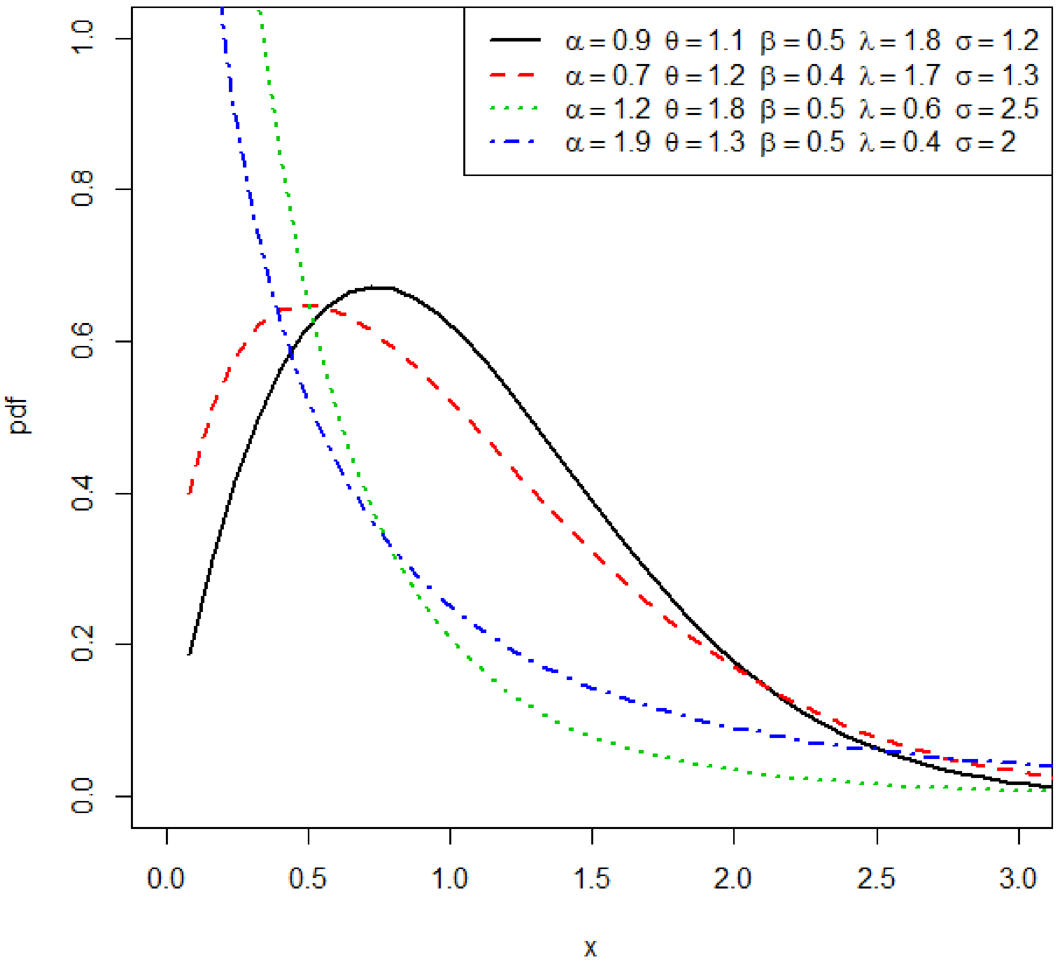

Figure 1: Plots of pdf of TLKW distribution with different parameter values.

Figure 1 shows the shape of the TLKW distribution with different values for each of the parameters which indicates that the TLKW distribution can be used to model highly skewed data.

Expansion of Density

In this sub-session, the pdf of the TLKW distribution is expanded using the binomial expansion given as:

$$(1-y)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(b)}{i!\Gamma(b-i)}y^{i} \qquad (7)$$

Using equation (7) on equation (6), we have the expanded form of the TLKW distribution given as

$$f(x)=2\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))}{i!j!k!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)}x^{\lambda-1}\left[e^{-(\beta x)^{\lambda}}\right]^{k+1} \qquad (8)$$

From equation (8), some properties of the TLKW distribution can be derived.

Properties of TLKW distribution

Some properties of TLKW distribution comprising moments, moment generating function, quantile function, median, survival function, hazard function odd function and distributions of Order Statistics are provided under this sub-section.

Moments

The moments of TLKW distribution is given as:

$$E(x^{r})=2\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(r+\lambda+1)}{i!j!k!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\left[\lambda(k+1)\beta\right]^{r+\lambda}} \qquad (9)$$

Mean

The mean of the TLKW distribution is obtained by setting $r=1$ in equation (9) and it is given as

$$E(x)=2\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda+2)}{i!j!k!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\left[\lambda(k+1)\beta\right]^{\lambda+1}} \qquad (10)$$

Moment generating function (MGF)

The MGF of TLKW distribution is given as

$$E(e^{tx})=2\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))t^{m}\Gamma(m+\lambda+1)}{i!j!k!m!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\left[\lambda(k+1)\beta\right]^{m+\lambda}}$$

(11)

Quantile function (QF)

The QF of TLKW is obtained by inverting the cdf in equation (5) and it is given as

$$x=Q(u)=\frac{1}{-\beta}\left\{\log\left[1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\sigma}}\right]^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\lambda}} \qquad (12)$$

By setting $u=\frac{1}{4}$, $u=\frac{1}{2}$ and $u=\frac{3}{4}$ we have the 1st quartile, median and the 3rd quartile of the TLKW distribution respectively given as:

$$x=Q(0.25)=\frac{1}{-\beta}\left\{\log\left[1-\left[1-\left[1-\left(0.25\right)^{\frac{1}{\theta}}\right]^{\frac{1}{2\sigma}}\right]^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\lambda}} \qquad (13)$$

$$x=Q(0.5)=\frac{1}{-\beta}\left\{\log\left[1-\left[1-\left[1-\left(0.5\right)^{\frac{1}{\theta}}\right]^{\frac{1}{2\sigma}}\right]^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\lambda}} \qquad (14)$$

$$x = Q(0.75) = \frac{1}{-\beta}\left\{\log\left[1-\left[1-\left[1-(0.75)^{\frac{1}{\theta}}\right]^{\frac{1}{2\sigma}}\right]^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\lambda}} \quad (15)$$

Survival function

The survival function of TLKW distribution is given as:

$$S(x;\alpha,\lambda,\sigma,\beta,\theta) = 1-\left[1-\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]^{\theta} \quad (16)$$

Hazard function

The hazard function of TLKW distribution is given as:

$$H(x) = \frac{2\alpha\lambda\sigma\theta\beta^{\lambda}x^{\lambda-1}e^{-(\beta x)^{\lambda}}\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha-1}\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma-1}\left[1-\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]^{\theta-1}}{1-\left[1-\left[1-\left[1-e^{-(\beta x)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]^{\theta}} \quad (17)$$

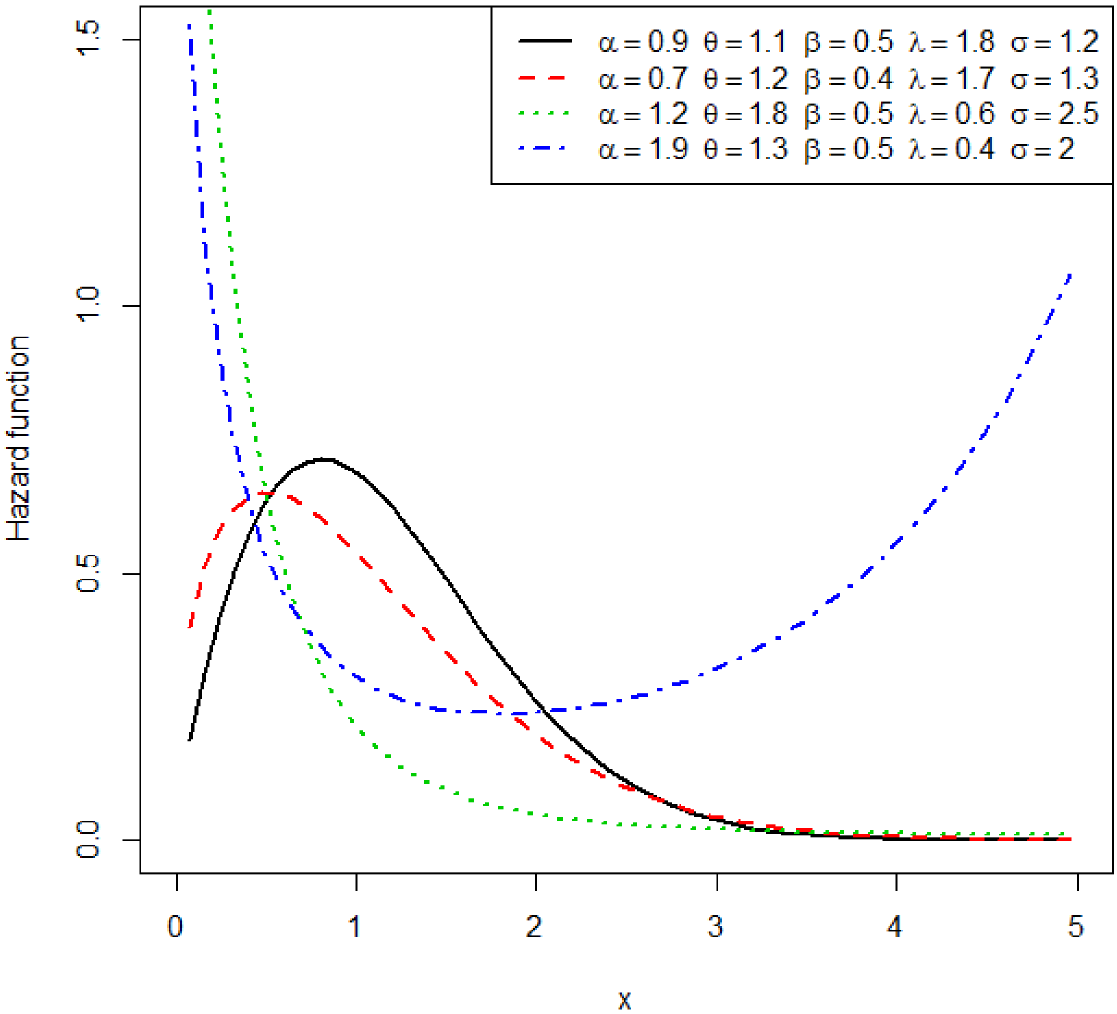

Figure 2: Plots of hazard function of TLKW distribution with different parameter value.

Figure 2 shows different shapes of hazard function of the TLKW distribution. It can be deduced from the plot that the distribution has an increasing, decreasing and bathtub shape which makes it a good distribution for modeling biomedical data.

Distribution of order Statistics

The r^{th}order statistic of TLKW distribution is given as:

$$f_{r:n}(x)=\frac{2\alpha\lambda\sigma\theta\beta^{\lambda}}{B(r,n-r+1)}\sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma(\theta(r+i))\Gamma(2\sigma(j+1))\Gamma(\alpha(k+1))}{j!k!l!\Gamma(\theta(r+i)-j)\Gamma(2\sigma(j+1)-k)\Gamma(\alpha(k+1)-l)}x^{\lambda-1}\left[e^{-(\beta x)^{\lambda}}\right]^{k+1} \quad (18)$$

The pdf of the minimum order statistic of TLKW distribution is obtained by setting $r=1$ in equation (18) and it is given as:

$$f_{1:n}(x)=2n\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma(\theta(i+1))\Gamma(2\sigma(j+1))\Gamma(\alpha(k+1))}{j!k!l!\Gamma(\theta(i+1)-j)\Gamma(2\sigma(j+1)-k)\Gamma(\alpha(k+1)-l)}x^{\lambda-1}\left[e^{-(\beta x)^{\lambda}}\right]^{k+1} \quad 19)$$

The pdf of the maximum order statistic of TLKW distribution is obtained by setting $r=n$ in equation (18) and it is given as:

$$f_{n:n}(x)=2n\alpha\lambda\sigma\theta\beta^{\lambda}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\theta(n+i))\Gamma(2\sigma(j+1))\Gamma(\alpha(k+1))}{j!k!l!\Gamma(\theta(n+i)-j)\Gamma(2\sigma(j+1)-k)\Gamma(\alpha(k+1)-l)}x^{\lambda-1}\left[e^{-(\beta x)^{\lambda}}\right]^{k+1} \quad (20)$$

Parameter Estimation

In this sub-section, we estimate the parameters of the TLKW distribution using maximum likelihood estimator (MLE). For a random sample $X_1, X_2, ..., X_n$ of size n from the $TLKW(\alpha,\beta,\theta,\sigma,\lambda)$, the log-likelihood function $L(\alpha,\beta,\theta,\sigma,\lambda)$ of equation (6) is given as

$$\begin{aligned}\log(L)&=n\log 2+n\log\alpha+n\log\lambda+n\log\theta+n\log\sigma+n\lambda\log\beta\\&+(\lambda-1)\sum_{i=1}^{n}\log(x_i)+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-(\beta x_i)^{\lambda}}\right]\\&-\sum_{i=1}^{n}(\beta x_i)^{\lambda}+(2\sigma-1)\sum_{i=1}^{n}\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]+(\theta-1)\sum_{i=1}^{n}\left[1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right]\end{aligned} \quad (21)$$

Differentiating equation (21) with respect to each parameter and equating to zero, we have

$$\frac{\delta\log(L)}{\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-(\beta x_i)^{\lambda}}\right]-(2\sigma-1)\sum_{i=1}^{n}\left[\frac{\lambda\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\log\left[1-e^{-(\beta x_i)^{\lambda}}\right]}{1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}}\right]+(\theta-1)\sum_{i=1}^{n}\left[\frac{2\sigma\lambda\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma-1}\log\left[1-e^{-(\beta x_i)^{\lambda}}\right]}{1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}}\right] \quad (22)$$

$$\frac{\delta\log(L)}{\beta}=\frac{n}{\beta}+(\alpha-1)\sum_{i=1}^{n}\left[\frac{\lambda(\beta x_i)^{\lambda-1}\left[e^{-(\beta x_i)^{\lambda}}\right]}{\left[1-e^{-(\beta x_i)^{\lambda}}\right]}\right]-(2\sigma-1)\sum_{i=1}^{n}\left[\frac{\alpha\lambda(\beta x_i)^{\lambda-1}\left[e^{-(\beta x_i)^{\lambda}}\right]\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha-1}}{1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}}\right]+(\theta-1)\sum_{i=1}^{n}\left[\frac{2\sigma\alpha\lambda(\beta x_i)^{\lambda-1}\left[e^{-(\beta x_i)^{\lambda}}\right]\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha-1}\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}}{1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}}\right] \quad (23)$$

$$\begin{aligned}\frac{\delta\log(L)}{\lambda}&=\frac{n}{\lambda}+nlo\beta+\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}(\beta x_i)^{\lambda}\log(\beta x_i)+(\alpha-1)\sum_{i=1}^{n}\left[\frac{(\beta x_i)^{\lambda}\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\log(\beta x_i)}{1-e^{-(\beta x_i)^{\lambda}}}\right]\\&-\sum_{i=1}^{n}\left[\frac{\alpha(\beta x_i)^{\lambda}\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]\log(\beta x_i)}{1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}}\right]+(\theta-1)\sum_{i=1}^{n}\left[\frac{2\sigma\alpha(\beta x_i)^{\lambda}\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\log(\beta x_i)}{1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}}\right]\end{aligned} \quad (24)$$

$$\frac{\delta\log(L)}{\sigma}=\frac{n}{\sigma}+\sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]-(\theta-1)\sum_{i=1}^{n}\left[\frac{\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]}{1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}}\right] \quad (25)$$

$$\frac{\delta\log(L)}{\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left[1-\left[1-\left[1-e^{-(\beta x_i)^{\lambda}}\right]^{\alpha}\right]^{2\sigma}\right] \quad (26)$$

Now, equation (22), equation (23), equation (24), equation (25) and equation (26) do not have a simple form and are therefore intractable. As a result, we have to resort to non-linear estimation of the parameters using iterative procedures.

Monte Carlo Simulation Processes

In this sub-section, simulation study to see the performance of MLEs of TLKW distribution is performed. The random number generation is obtained using the quantile function of the TLKW distribution. We note that the u^{th}qf of the TLKW distribution is given in (12). Hence, if U has uniform random variable on (0, 1), then $x$has the TLKW random variable.

We generated $N=10000$ samples of sizes $n=50$, *100, 200 and 500* from TLKW distribution with its qf. Then we computed the empirical means, biases and mean squared errors (MSE) of the MLEs with

$$Bias_{\hat{\psi}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_i\right) \qquad (27)$$

and

$$MSE_{\hat{\psi}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_i\right)^2, \qquad (28)$$

for $\psi = (\beta, \theta, \alpha, \lambda, \sigma)$

To examine the performance of the MLEs for the TLKW distribution, we perform a simulation study as follows:

i. Generate *N* samples of size *n* from the TLKW distribution with its qf.
ii. Compute the MLEs for the *N* samples, say $(\hat{\beta}, \hat{\theta}, \hat{\alpha}, \hat{\lambda}, \hat{\sigma})$, for $i = 1, 2, ..., N$
iii. Compute the MLEs for *N* samples
iv. Compute the biases and MSEs given in equation (27) and equation (28).

We repeat these steps for *N= 10000* and *n =20, 50, 100, 25, 5000 and 1000* with different values of $\psi = (\beta, \theta, \alpha, \lambda, \sigma)$. Table 1 shows how the biases and MSE vary with respect to *n*. As expected the MSEs of the estimated parameters decreases as *n* increases it proves the consistency of the estimators.

RESULTS AND DISCUSSION

In this section, a performance comparison with other distributions is performed to assess the flexibility and capability of TLKW distribution. We fit the TLKW distribution to two real data sets and conduct a comparative analysis with fits to the Type II Exponentiated Half Logistic Weibull (TIIEHLW) distribution by (Al-Mofleh *et al.*, 2020), Half-Logistic Generalized Weibull (HLGW) Distribution by (Masood & Amna 2018), ExponentiatedWeibull (EW) by (Pal *et al.*, 2006), Weibull Distribution by (Xie & Lai 1996) and Topp-Leone Generated Weibull (TLGW) Distribution by (Aryal *et al.*, 2017).

Table 1: Monte Carlo simulation results for some values of parameters

N	Parameters	(1,1,2.5,1,1)		
		Estimated Values	**Bias**	**MSE**
20	α	0.9785	-0.0215	0.2340
	θ	1.0646	0.0646	0.4410
	β	1.2108	0.2108	0.5204
	λ	1.2100	0.2100	0.5201
	σ	1.1172	0.1172	0.4237
50	α	1.0069	0.0069	0.1789
	θ	1.0330	0.0330	0.3138
	β	1.0813	0.0813	0.3087
	λ	1.0798	0.0798	0.3057
	σ	1.0460	0.0460	0.2582
100	α	1.0195	0.0195	0.1239
	θ	1.0340	0.0340	0.2275
	β	1.0197	0.0197	0.2022
	λ	1.0183	0.0183	0.1962
	σ	1.0365	0.0365	0.1779
250	α	1.0136	0.0136	0.0889
	θ	1.0192	0.0192	0.1481
	β	1.0044	0.0044	0.1397
	λ	1.0029	0.0029	0.1309
	σ	1.0124	0.0124	0.1247
500	α	1.0188	0.0188	0.0651
	θ	1.0173	0.0173	0.1104
	β	0.9900	-0.0100	0.1059
	λ	0.9883	-0.0117	0.0941
	σ	1.0054	0.0054	0.0878
1000	α	1.0144	0.0144	0.0505
	θ	1.0103	0.0103	0.0796

β	0.9914	-0.0086	0.0854
λ	0.9899	-0.0101	0.0715
σ	1.0000	0.0000	0.0592

Table 1 presents the mean values of the estimates based on the corresponding actual parameter values chosen. The table also presents the MSE of the estimates and as expected, the MSE decreases as the sample size increases which make the TLKW distribution flexible in fitting datasets from diverse field of human endeavor.
The pdf of the competing models used are:
The TIIEHLW distribution developed by (Al-Mofleh *et al.*, 2020) has pdf defined as:

$$f(x;\alpha,\lambda,\beta,\theta)=2\alpha\lambda\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha-1}}{\left[1+\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha+1}} \quad (29)$$

The HLGW distribution developed by Masood and Amna (2018) has pdf defined as:

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\alpha-1}\left[1+\theta x^{\alpha}\right]^{\lambda-1}exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]}{\left[1+exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]\right]^{2}} \quad (30)$$

The EW distribution proposed by (Pal *et al.*, 2006) has pdf given as:

$$f(x;\alpha,\lambda,\beta)=\alpha\lambda^{\beta}\beta x^{\beta-1}\left[1-exp(-\lambda x)^{\beta}\right]^{\alpha-1}exp(-\lambda x)^{\beta} \quad (31)$$

The Weibull Distribution proposed by Xie and Lai (1996) has pdf given as:

$$f(x;\theta,\beta)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}} \quad (32)$$

The TLGW distribution developed by (Aryal *et al.*, 2017) has pdf defined as:

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta\alpha-1}\left[1-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]\left[2-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]^{\alpha-1} \quad (33)$$

When compared to the aforementioned comparator distributions, the two datasets utilized as examples in the application show how the new proposed distribution is more adaptable, applicable, and "best fit" when modeling the datasets experimentally. R programming is used to carry out each and every calculation.

Data set 1
The first data set represents the remissions times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients, previously used by (Lee & Wang, 2003):

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

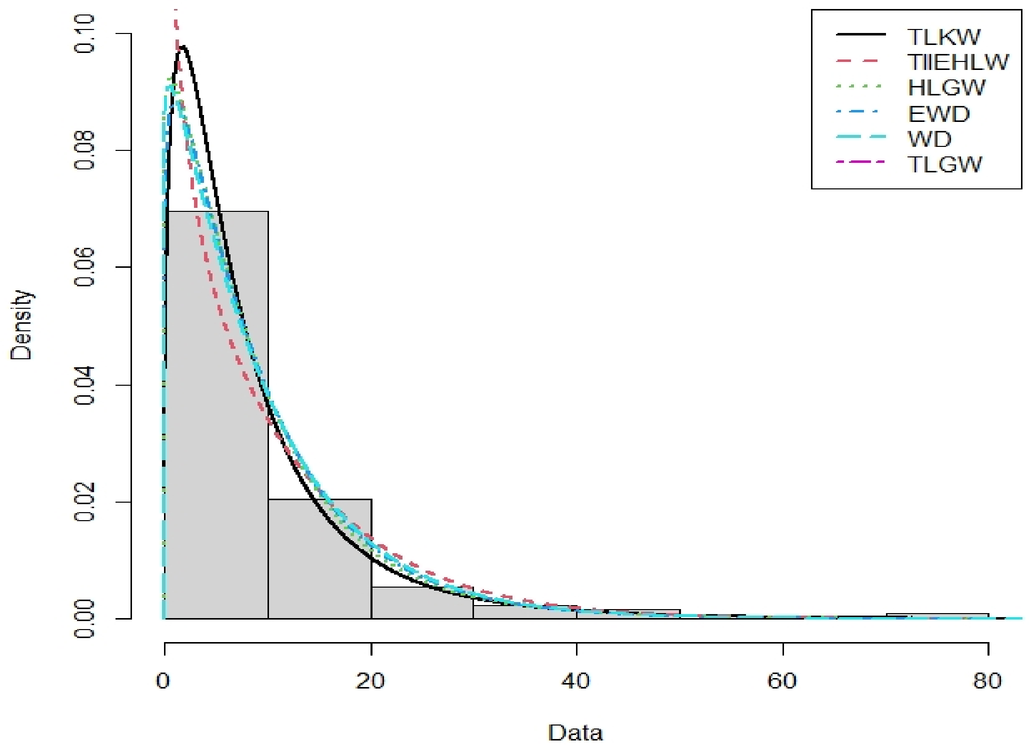

Figure 3: Fitted pdfs for the TLKW, TIIEHLW, HLGW, EWD, WD, and TLGW distributions to the data set 1

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	θ	β	λ	σ	**LL**	**AIC**
TLKW	0.4984	3.5456	3.3930	0.6384	0.1140	-410.2248	830.4496
TIIEHLW	0.2368	0.2634	1.1245	0.8929	-	-418.4258	844.8516
HLGW	1.0581	0.2868	-	0.6613	-	-412.4861	830.9721
EWD	1.1545	-	0.9861	0.1188		-413.1202	832.2403
WD	-	0.0939	1.0478	-		-414.0869	832.1738
TLGW	6.6269	4.1785	0.2522	0.0219	-	-442.2653	880.5306

The parameters of the new proposed distribution and the five comparator distributions were estimated using maximum likelihood, and the results are shown in Table 2. The new proposed distribution reported the minimum AIC value according to the goodness of fit measure, though the HLGW was closely behind it. The proposed distribution's dominance over its competitors is also confirmed by a visual evaluation of the fit shown in Figure 3.

Thus, among the variety of distributions taken into consideration, the new proposed distribution best fit the data set of bladder cancer patients.

Data set 2

The second data set represents the life time data relating to times (in months from 1st January, 2013 to 31st July, 2018) of 105 patients who were diagnosed with hypertension and received at least one treatment related to hypertension in the hospital where death is the event of interest, previously used by (Umeh & Ibenegbu, 2019):

45, 37, 14, 64, 67, 58, 67, 55, 64, 62, 9, 65, 65, 43, 13, 8, 31, 30, 66, 9, 10, 31, 31, 31, 46, 37, 46, 44, 45, 30, 26, 28, 45, 40, 47, 53, 47, 41, 39, 33, 38, 26, 22, 31, 46, 47, 66, 61, 54, 28, 9, 63, 56, 9, 49, 52, 58, 49, 53, 63, 16, 67, 61, 67, 28, 17, 31, 46, 52, 50, 30, 33, 13, 63, 54, 63, 56, 32, 33, 37, 7, 56, 1, 67, 38, 33, 22, 25, 30, 34, 53, 53, 41, 45, 59, 59, 60, 62, 14, 57, 56, 57, 40, 44, 63.

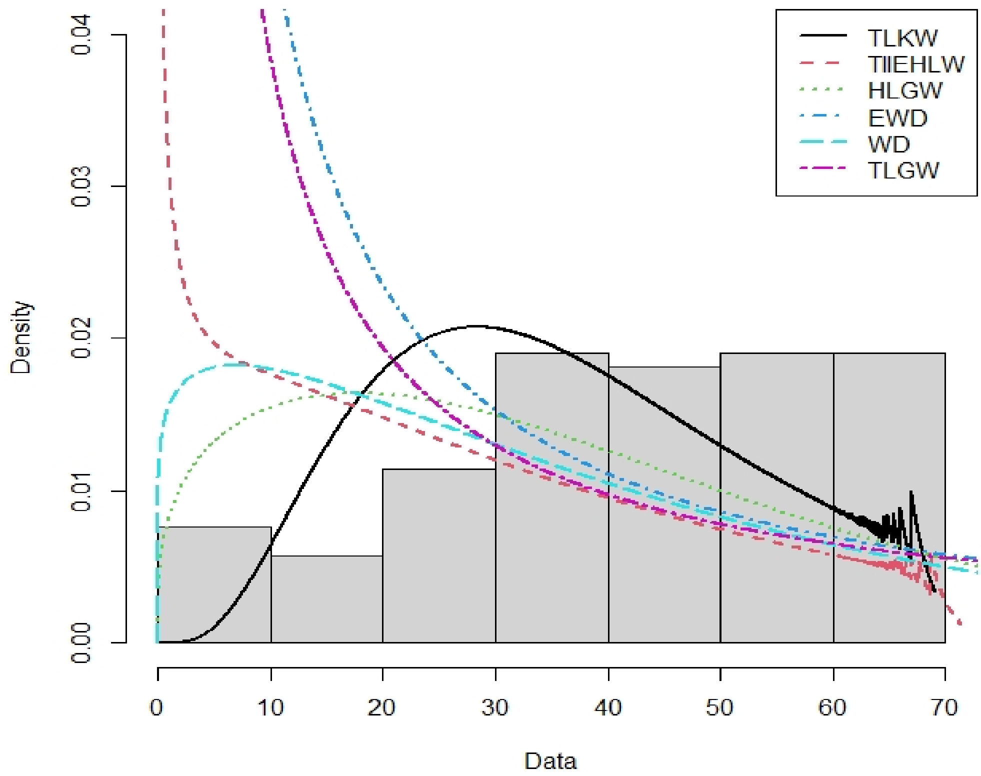

Figure 4: Fitted pdfs for the TLKW, TIIEHLW, HLGW, EWD, WD, and TLGW distributions to the data set 2

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	θ	β	λ	σ	LL	AIC
TLKW	0.9227	0.2429	0.0115	8.0233	5.3340	-441.3385	892.6769
TIIEHLW	0.0487	0.3378	1.1028	0.5271	-	-495.9288	999.8576
HLGW	1.3181	0.0175	-	0.7136		-475.625	957.2501
EWD	5.0005	-	0.1826	4.4779		-464.4788	934.9575
WD	-	0.0138	1.1406	-		-487.8239	979.6479
TLGW	10.8957	0.0624	8.2172	0.0120		-471.7036	951.4072

The parameters of the TLKW distribution and the five comparator distributions were estimated using maximum likelihood, as shown in Table 3. The new distribution returned the smallest AIC value, indicating that it is the distribution that best fits the hypertension patients based on the goodness of fit measure AIC. The new distribution's dominance over its competitors is also confirmed by a visual examination of the fit shown in Figure 4.

CONCLUSION

This paper has derived a new distribution called the Topp-Leone KumaraswamyWeibulldistribution that generalized the Weibull distribution. Some properties of the new distribution were derived such as the survival function, hazard rate function, quantile function, mean, 1st quartile, median, 3rd quartile and the distribution of order statistics. The shapes of the proposed distribution were shown by plotting the graphs of the pdf and hazard rate function. The estimation of the model parameters by the method of the maximum likelihood was carried out using a package in *R* known as *AdequacyModel*. Monte Carlo simulation was carried out to see the performance of MLEs of the TLKW distribution and as expected, the MSEs of the estimated parameters decreaseas the sample size*n* increases which proves the consistency of the estimators. Applications of the new distribution to two real data sets were carried out and the results are presented in Table 2and Table 3 respectively. The results showed that the Topp LeoneKumaraswamyWeibull distribution is quite effective and superior in fitting the two real data sets considered. Also, the flexibility of the new model can be seen from the histogram and fitted pdf plots for the two data sets, it can be deduced that the new model fits the two data sets better than the competing models considered.

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REFERENCES

Alexander, C., Cordeiro, G. M., Ortega, E. M. M. and Sarabia, J. M. (2012). Generalized Beta-generated Distributions.

Computational Statistics and Data Analysis, 56(6), 1880-1896.

Aljarrah, M. A., Lee, C. and Famoye, F. (2014). On generating T-X family of distributions using quantile functions. *Journal of Statistical Distributions and Applications*, 1(2), 1-24.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.

Alzaghal, A., Famoye, F. and Lee, C. (2013). Exponentiated T-X family of distributions with some applications. *International Journal of Statistics and Probability*, 2(3), 31-49.

Al-Mofleh, H., Elgarhy, M., Afify, A. Z. and Zannon, M. (2020). Type II Exponentiated Half Logistic generated family of distributions with applications. *Electronic Journal of Applied Statistical Analysis*, 13(2), 536-561 DOI: 10.1285/i20705948v13n2p536

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M. Q., (2016). Topp-Leone family of distributions: Some properties and application, *Pakistan Journal of Statistics and Operation Research.* XII, 3, 443-451.

Al-Sulami, D. (2020). Exponentiated Exponential Weibull Distribution: Mathematical Properties and Application, *American Journal of Applied Sciences*, 17, 188-195

Aryal, G.R., Ortega, E.M.M., Hamedani, G.G., and Yousof, H.M. (2017). The Topp-Leone generated Weibull distribution: regression model, characterizations and applications. *International Journal of Statistics and Probability*; 6(1), 126-141.

Bebbington, M., Lai, C. D., and Zitikis, R. (2007). A flexible Weibull extension. *Reliability Engineering and System Safety*, 92(6), 719-726.

Bello, O. A., Doguwa, S. I., Yahaya, A. and Jibril, H. M. (2020). A type I Half-Logistic Exponentiated-G Family of Distributions: Properties and Applications, *Communication in Physical Sciences*, 7(3), 147-163.

Bello, O. A., Doguwa, S. I., Yahaya, A. and Jibril, H. M. (2021). A type II Half-Logistic Exponentiated-G Family of Distributions with Applications to Survival Analysis, *FUDMA Journal of Sciences*, 5(3), 177-190.

Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G family of probability distribution. *Journal of Data Science*, 12, 53-68.

Carrasco, J. M., Ortega, E. M., and Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, 53(2), 450-462.

Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81, 883-893.

Elbatal, I. and Aryal, G. (2013).On the transmuted additive Weibull distribution. *Austrian Journal of Statistics*, 42, 117-132.

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications.
Communications in Statistics - Theory and Methods, 31, 497-512.

Ibrahim, S., Doguwa, S.I., Audu, I. and Jibril, H.M., (2020a). On the Topp Leone exponentiated-G Family of Distributions: Properties and Applications, *Asian Journal of Probability and Statistics*; 7(1), 1-15.

Ibrahim, S., Doguwa S. I., Audu, I. and Jibril, H. M., (2020b). The Topp Leone Kumaraswamy-G Family of Distributions with Applications to Cancer Disease Data, *Journal of Biostatistics and Epidemiology*, 6(1), 37-48.

Ibrahim, S. (2021). The properties of Topp-Leone exponentiatedWeibull distribution with application to survival data, *Research Journal of Mathematical and Statistical Sciences*, 9(1), 1-9.

Lai, C. D., Xie, M. and Murphy, D. N. P. (2003). A Modified Weibull Distribution. *IEEE Transactions on Reliability*, 52(1), 33 – 37.

Lee, E. T. and Wang, J. W. (2003). Statistical methods for survival data analysis (3rdEdition), John Wiley and Sons, New York, USA, 535 Pages, ISBN 0-471-36997-7.

Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull Distribution: Some Properties and Applications to Censored Data. *Journal of Modern Applied Statistical Methods,* 6(1), 173 – 186.

Masood, A. and Amna, B. (2018). The Half-Logistic Generalized Weibull Distribution. *Journal of Probability and Statistics*, 8767826, 12.

Mudholkar, G. S. and Srivastava, D. K. (1993). ExponentiatedWeibull Family for Analyzing Bathtub Failure-rate Data. *IEEE Transactions on Reliability*, 42, 299-302.

Nofal, Z. M., Afify, A. Z., Yousof, H. M., Granzotto, D. C. T. andLouzada, F. (2016). Kumaraswamy transmuted exponentiated additiveWeibull distribution. *International Journal of Statistics and Probability*, 5, 78-99.

Nadarajah, S., Cordeiro, G. M., and Ortega, E. M. (2011). General results for the beta-modified Weibull distribution. *Journal of Statistical Computation and Simulation*, 81(10), 1211-1232.

Pal, M., Ali, M.M., and Woo, J. (2006). ExponentiatedWeibull distribution. *STATISTICA, anno* LXVI, 2.

Shaw, W. T. and Buckley, I. R. (2007). The Alchemy of Probability Distributions: Beyond Gram-Charlier Expansions and a Skew-kurtotic-normal Distribution from a Rank Transmutation Map. (http://library.wolfram.com/infocenter/Articles/6670/alchemy.pdf).

Sule, I.,Lawal, H.O. and Bello, O.A. (2022). Properties of a new generalized family of distributions with applications to relief times of patients data, *Journal of Statistical Modeling and Analytics*, 4(1), 39 - 55.

Silva, G. O., Ortega, E. M., andCordeiro, G. M. (2010). The beta modified Weibull distribution.*Lifetime data analysis*, 16(3), 09-430.

Singla, N., Jain, K., and Sharma, S. K. (2012). The beta generalized Weibull distribution: Properties and applications,*Reliability Engineering and System Safety*, 102, 5-15.

Tahir, M. H., Zubair, M., Cordeiro, G. M., Alzaatreh, A. and Mansoor, M. (2016). The Poisson-X Family of Distributions. *Journal of Statistical Computation and Simulation*, 86, 2901-2921.

Umeh, E. and Ibenegbu, A. (2019). A Two-Parameter Pranav Distribution with Properties and Its Application,*Journal of Biostatistics and Epidemiology*, 5(1), 74-90.

Xie, M., Lai, C. D. (1996). On the increase of the expected lifetime by parallel redundancy. *Asia Pacific Journal of Operation Research*. 13, 171-179.

Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodolog*y, 6(4), 344-362.