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SELECTED SINGLE-STEP HYBRID BLOCK FORMULA FOR SOLVING THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH APPLICATION IN THIN FILM FLOW

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ABSTRACT

This research paper examines the derivation of selected hybrid single-step block method for the numerical integration of third order ordinary differential equations. The method has the advantage of selecting only odd off-grid points within the interval of interest. The basis function for the formula is interpolated at three selected non-grid points within a single-step interval and collocated at all points. Further analysis of the basic numerical properties were established. The method was found to be A-stable, zero-stable and consistent. The small scale errors observed from numerical experiment indicate that the derived formula has better numerical approximations than some comparable methods in literature, while its application on practical thin film flow problem also showed improved solutions.

Keywords: Selected single-step, basis function, Hybrid block formula, interpolation and collocation and thin film flow problem

INTRODUCTION

Interestingly, numerical integration remained an alternative approximation approach to analytical solutions for several modeled problems, either as initial value problems (IVPs) or boundary value problems (BVPs) in ordinary differential equations (ODEs). Such problems exist in biology, neural networks, electric circuits, electromagnetic waves and thin film flow on solid surfaces, deflection of a curve beam and a three layer beam in engineering. However, numerical methods have also been used by (see, Arqub & Maayah, 2019a, 2019b and 2018) for the approximation of other differential equations as Bernoulli equations, fractional volterra-integro-differential equation, among others. Over the years, block methods have been introduced as one of the efficient numerical methods for the numerical integration of ODEs, Henrici, (1962). Subsequently, hybrid block methods were introduced to evaluate off-grid and grid points within any interval of discretization, (see Gragg & Stetter, 1964). Hybrid block methods have the advantages of varying step-sizes, utilizing data at off-step points and most importantly, their ability to overcome Dahlquist barrier of zero-stability, (see Henrici, 1962) and Dahlquist (1956)).

However, recently, several research scholars (see, Althemai et al., (2022), Kuboye et al., (2020), Duromola (2022), Duromola (2019), Abdelrahim & Omar (2016) and Adeyeye & Omar (2017)) among others have redirected their research quest for robust numerical methods to formulating single-step hybrid block methods. This is because most of the derivations of single-step hybrid block methods require less human efforts and reduced complex computer codes thereby minimize numerical errors. Because of sufficient stability properties in hybrid block methods as a result of fixed step discretization, ability to use smaller step-sizes for approximations without error growth resulting from perturbation, they have been widely used for solving third order ordinary differential equations (ODEs) (see, Duromola (2019), modebei et al., (2021), Haweel et al., (2021), Lawal et al., (2018), Aigbiremhon et al., (2021) and Obarhua & Kayode (2016), among others. Also, hybrid numerical methods have been used for the approximations of higher order ODEs (see, Abolarin et al., (2020) and Areo & Omole (2015)) and results from their applications showed improved absolute errors. Similarly, adaptive polynomial method was formulated and implemented on linear, non-linear and a thin film flow problem and compared with some other hybrid block methods (see, Momoniat and Mahomed (2010), Mecheel et al., (2013) and Yap et al., (2014)).

In this research paper, selected single-step hybrid block method is proposed. The method has the advantage of selecting only odd off-grid points in the single-step considered thereby reducing human efforts and complex computer codes for the method.

he research paper is structured as follows. The formulation of the method is considered in section two while the basic numerical properties are analyzed in section three. Section four considered numerical experiments of the method and results. In section five, the application of the method to thin film flow problem is considered. Section six presents results and discussions and finally, in section seven, we draw the conclusion and future research.

Proposed Selected Single-step Hybrid Block Method

Let the basis function for our method be given in a single variable x as:

$$y(x) = \sum_{j=0}^{(p+l)-1} a_j x^j \tag{1}$$

Which is the approximate solution to the third order ordinary differential equation of the form:

$$y'''(x) = f(x, y, y', y''), \quad y(x_0) = y_0, \quad y'(0) = y_0', \quad y''(0) = y_0'' \tag{2}$$

Where p and l denote points of collocation and interpolation respectively. So that the third derivative is:

$$y'''(x) = \sum_{j=0}^{(p+l)-1} j(j-1)(j-2)a_j x^{j-3} \tag{3}$$

However, in this research work, four off-grid points have been introduced to formulate the method. We have selected the odd number off-step points only to ensure reduced human efforts at derivations, reduced complex computer program during implementation in order to minimize round-off errors and finally the zero-stability and consistency of the method in order to overcome the first and second Dahlquist barriers of zero-stability and consistency properties, which are critical stability properties for all numerical methods.

Interpolating (1) at x_{n+i}, $i = \frac{1}{8}\left(\frac{2}{8}\right)\frac{5}{8}$ and collocating at x_{n+v}, $v = 0, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, 1$ we get a system of nonlinear equations of the form:

$$AX = B \tag{4}$$

Where,

$$A = \begin{bmatrix}
1 & x_{n+\frac{1}{8}} & x^2_{n+\frac{1}{8}} & x^3_{n+\frac{1}{8}} & x^4_{n+\frac{1}{8}} & x^5_{n+\frac{1}{8}} & x^6_{n+\frac{1}{8}} & x^7_{n+\frac{1}{8}} & x^8_{n+\frac{1}{8}} \\
1 & x_{n+\frac{3}{8}} & x^2_{n+\frac{3}{8}} & x^3_{n+\frac{3}{8}} & x^4_{n+\frac{3}{8}} & x^5_{n+\frac{3}{8}} & x^6_{n+\frac{3}{8}} & x^7_{n+\frac{3}{8}} & x^8_{n+\frac{3}{8}} \\
1 & x_{n+\frac{5}{8}} & x^2_{n+\frac{5}{8}} & x^3_{n+\frac{5}{8}} & x^4_{n+\frac{5}{8}} & x^5_{n+\frac{5}{8}} & x^6_{n+\frac{5}{8}} & x^7_{n+\frac{5}{8}} & x^8_{n+\frac{5}{8}} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{8}} & 60x^2_{n+\frac{1}{8}} & 120x^3_{n+\frac{1}{8}} & 210x^4_{n+\frac{1}{8}} & 336x^5_{n+\frac{1}{8}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{8}} & 60x^2_{n+\frac{3}{8}} & 120x^3_{n+\frac{3}{8}} & 210x^4_{n+\frac{3}{8}} & 336x^5_{n+\frac{3}{3}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{5}{8}} & 60x^2_{n+\frac{5}{8}} & 120x^3_{n+\frac{5}{8}} & 210x^4_{n+\frac{5}{8}} & 336x^5_{n+\frac{5}{8}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{7}{8}} & 60x^2_{n+\frac{7}{8}} & 120x^3_{n+\frac{7}{8}} & 210x^4_{n+\frac{7}{8}} & 336x^5_{n+\frac{7}{8}} \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x^2_{n+1} & 120x^3_{n+1} & 210x^4_{n+1} & 336x^5_{n+1}
\end{bmatrix},$$

$$X = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^\top, \quad B = \begin{bmatrix} y_{n+\frac{1}{8}} & y_{n+\frac{3}{8}} & y_{n+\frac{5}{8}} & f_n & f_{n+\frac{1}{8}} & f_{n+\frac{3}{8}} & f_{n+\frac{5}{8}} & f_{n+\frac{7}{8}} & f_{n+1} \end{bmatrix}^\top$$

whose unknowns ay_s are solved for using Gaussian elimination technique and results are substituted into Equation 1 to give a continuous linear multistep method of the form:

$$y(x) = \sum_{j=\sigma_i} \alpha_j(x) y_{n+j} + h^3\left(\sum_{j=0} \beta_j(x) f_{n+j} + \sum_{j=\sigma_i} \beta_{\sigma_i(x) f_{n+\sigma_i}}\right) \tag{5}$$

where, i = 0(1)5 with $\sigma_i = 0, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ *and* 1, from which we have:

$$\left.\begin{aligned}
\alpha_{\frac{1}{8}}(\xi) &= \frac{15}{8} - \frac{8\xi}{h} + \frac{8\xi^2}{h^2} \\
\alpha_{\frac{3}{8}}(\xi) &= -\frac{5}{4} + \frac{12\xi}{h} - \frac{16\xi^2}{h^2} \\
\alpha_{\frac{5}{8}}(\xi) &= \frac{3}{8} - \frac{4\xi}{h} + \frac{8\xi^2}{h^2}
\end{aligned}\right\} \tag{6}$$

$$\left.\begin{array}{l}
\beta_0(\xi)=\frac{457\,h^3}{2293760}-\frac{71\,\xi\,h^2}{58800}-\frac{46393\,\xi^2 h}{2822400}+\frac{\xi^3}{6}-\frac{1513\,\xi^4}{2520\,h}+\frac{192\,\xi^5}{175\,h^2}-\frac{1712\,\xi^6}{1575\,h^3}+\frac{2048\,\xi^7}{3675\,h^4}\\
-\frac{256\,\xi^8}{2205\,h^5}\\
\beta_{\frac{1}{8}}(\xi)=-\frac{5557\,h^3}{2752512}+\frac{252053\,\xi\,h^2}{9031680}-\frac{77863\,\xi^2 h}{752640}+\frac{5\xi^4}{6h}-\frac{673\,\xi^5}{315\,h^2}+\frac{764\,\xi^6}{315\,h^3}-\frac{2944\,\xi^7}{2205\,h^4}+\frac{128\,\xi^8}{441\,h^5}\\
\beta_{\frac{3}{8}}(\xi)=-\frac{2137\,h^3}{655360}+\frac{3623\,\xi\,h^2}{102400}-\frac{114899\,\xi^2 h}{1612800}-\frac{7\,\xi^4}{18\,h}+\frac{137\,\xi^5}{75\,h^2}-\frac{604\,\xi^6}{225\,h^3}+\frac{128\,\xi^7}{75\,h^4}-\frac{128\,\xi^8}{315\,h^5}\\
\beta_{\frac{5}{8}}(\xi)=\frac{619\,h^3}{1966080}-\frac{22271\,\xi\,h^2}{6451200}+\frac{1019\,\xi^2 h}{179200}+\frac{7\,\xi^4}{30\,h}-\frac{269\,\xi^5}{225\,h^2}+\frac{476\,\xi^6}{225\,h^3}-\frac{2432\,\xi^7}{1575\,h^4}+\frac{128\,\xi^8}{315\,h^5}\\
\beta_{\frac{7}{8}}(\xi)=-\frac{491\,h^3}{2752512}+\frac{17389\,\xi\,h^2}{9031680}-\frac{2329\,\xi^2 h}{752640}-\frac{5\,\xi^4}{42\,h}+\frac{199\,\xi^5}{315\,h^2}-\frac{76\,\xi^6}{63\,h^3}+\frac{2176\,\xi^7}{2205\,h^4}-\frac{128\,\xi^8}{441\,h^5}\\
\beta_1(\xi)=\frac{421\,h^3}{6881280}-\frac{929\,\xi\,h^2}{1411200}+\frac{979\,\xi^2 h}{940800}+\frac{\xi^4}{24h}-\frac{352\,\xi^5}{1575\,h^2}+\frac{688\,\xi^6}{1575\,h^3}-\frac{4096\,\xi^7}{11025\,h^4}+\frac{256\,\xi^8}{2205\,h^5}
\end{array}\right\}\quad(7)$$

Evaluating Equations 6 and 7 at non-interpolating points, $x_{n+\sigma_i}$, where $\sigma_i = 0, \frac{7}{8}$ and 1, and set $\varepsilon = x - x_n$, so that $\varepsilon = 0h, \frac{7h}{8}$ and $1h$ in Equation 5 gives:

$$13762560\,y_n - 25804800\,y_{n+\frac{1}{8}} + 17203200\,y_{n+\frac{3}{8}} - 5160960\,y_{n+\frac{5}{8}} = h^3\Big(2742\,f_n -27785\,f_{n+\frac{1}{8}} - 44877\,f_{n+\frac{3}{8}} + 4333\,f_{n+\frac{5}{8}} - 2455\,f_{n+\frac{7}{8}} + 842\,f_{n+1}\Big) \quad (8)$$

$$67200\,y_{n+\frac{7}{8}} - 67200\,y_{n+\frac{1}{8}} + 201600\,y_{n+\frac{3}{8}} - 201600\,y_{n+\frac{5}{8}} = h^3\Big(8\,f_n - 15\,f_{n+\frac{1}{8}} +532\,f_{n+\frac{3}{8}} + 532\,f_{n+\frac{5}{8}} - 15\,f_{n+\frac{7}{8}} + 8\,f_{n+1}\Big) \quad (9)$$

$$2752512\,y_{n+1} - 5160960\,y_{n+\frac{1}{8}} + 14450688\,y_{n+\frac{3}{8}} - 12042240\,y_{n+\frac{5}{8}} = h^3\Big(446\,f_n -661\,f_{n+\frac{1}{8}} + 39991\,f_{n+\frac{3}{8}} + 49833\,f_{n+\frac{5}{8}} + 4405\,f_{n+\frac{7}{8}} + 66\,f_{n+1}\Big) \quad (10)$$

Again, we take the first and second derivatives of Equation 5 and 7, evaluating at all points, that is, $x_{n+\sigma_i}$ and $\sigma_i = 0, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ and 1 and that $\varepsilon = 0h, \frac{h}{8}, \frac{3h}{8}, \frac{5h}{8}, \frac{7h}{8}$ and h in Equation 7 gives:

$$45158400\,hy'_n + 361267200\,y_{n+\frac{1}{8}} - 541900800\,y_{n+\frac{3}{8}} + 180633600\,y_{n+\frac{5}{8}} = -h^3\Big(54528\,f_n -1260265\,f_{n+\frac{1}{8}} - 1597743\,f_{n+\frac{3}{8}} + 155897\,f_{n+\frac{5}{8}} - 86945\,f_{n+\frac{7}{8}} + 29728\,f_{n+1}\Big) \quad (11)$$

$$1411200\,hy'_{n+\frac{1}{8}} + 8467200\,y_{n+\frac{1}{8}} - 11289600\,y_{n+\frac{3}{8}} + 2822400\,y_{n+\frac{5}{8}} = -h^3\Big(1468\,f_n -8970\,f_{n+\frac{1}{8}} - 23023\,f_{n+\frac{3}{8}} + 1862\,f_{n+\frac{5}{8}} - 1125\,f_{n+\frac{7}{8}} + 388\,f_{n+1}\Big) \quad (12)$$

$$282240\,hy'_{n+\frac{3}{8}} + 564480\,y_{n+\frac{1}{8}} - 564480\,y_{n+\frac{5}{8}} = h^3\Big(36\,f_n - 227\,f_{n+\frac{1}{8}} - 2562\,f_{n+\frac{3}{8}} -203\,f_{n+\frac{5}{8}} + 20\,f_{n+\frac{7}{8}} - 4\,f_{n+1}\Big) \quad (13)$$

$$1411200\, hy'_{n+\frac{5}{8}} - 2822400\, y_{n+\frac{1}{8}} + 11289600\, y_{n+\frac{3}{8}} - 8467200\, y_{n+\frac{5}{8}} = h^3\Big(316\, f_n - 530\, f_{n+\frac{1}{8}} + 21329\, f_{n+\frac{3}{8}} + 9534\, f_{n+\frac{5}{8}} - 1765\, f_{n+\frac{7}{8}} + 516\, f_{n+1}\Big) \quad (14)$$

$$282240\, hy'_{n+\frac{7}{8}} - 1693440\, y_{n+\frac{1}{8}} + 4515840\, y_{n+\frac{3}{8}} - 2822400\, y_{n+\frac{5}{8}} = h^3\Big(124\, f_n - 153\, f_{n+\frac{1}{8}} + 13034\, f_{n+\frac{3}{8}} + 18011\, f_{n+\frac{5}{8}} + 1416\, f_{n+\frac{7}{8}} - 92\, f_{n+1}\Big) \quad (15)$$

$$9031680\, hy'_{n+1} - 72253440\, y_{n+\frac{1}{8}} + 180633600\, y_{n+\frac{3}{8}} - 108380160\, y_{n+\frac{5}{8}} = h^3\Big(2656\, f_n + 1261\, f_{n+\frac{1}{8}} + 540827\, f_{n+\frac{3}{8}} + 891555\, f_{n+\frac{5}{8}} + 235925\, f_{n+\frac{7}{8}} - 2304\, f_{n+1}\Big) \quad (16)$$

$$5644800\, h^2y''_{n} - 90316800\, y_{n+\frac{1}{8}} + 180633600\, y_{n+\frac{3}{8}} - 90316800\, y_{n+\frac{5}{8}} = -h^3\Big(185572\, f_n + 1167945\, f_{n+\frac{1}{8}} + 804293\, f_{n+\frac{3}{8}} - 64197\, f_{n+\frac{5}{8}} + 34935\, f_{n+\frac{7}{8}} - 11748\, f_{n+1}\Big) \quad (17)$$

$$70560\, h^2y''_{n+\frac{1}{8}} - 1128960\, y_{n+\frac{1}{8}} + 2257920\, y_{n+\frac{3}{8}} - 1128960\, y_{n+\frac{5}{8}} = h^3\Big(1068\, f_n - 8326\, f_{n+\frac{1}{8}} - 11403\, f_{n+\frac{3}{8}} + 1554\, f_{n+\frac{5}{8}} - 809\, f_{n+\frac{7}{8}} + 276\, f_{n+1}\Big) \quad (18)$$

$$352800\, h^2y''_{n+\frac{3}{8}} - 5644800\, y_{n+\frac{1}{8}} + 11289600\, y_{n+\frac{3}{8}} - 5644800\, y_{n+\frac{3}{8}} = -h^3\Big(1156\, f_n - 5025\, f_{n+\frac{1}{8}} - 1246\, f_{n+\frac{3}{8}} + 6489\, f_{n+\frac{5}{8}} - 2010\, f_{n+\frac{7}{8}} + 636\, f_{n+1}\Big) \quad (19)$$

$$352800\, h^2y''_{n+\frac{5}{8}} - 5644800\, y_{n+\frac{1}{8}} + 11289600\, y_{n+\frac{3}{8}} - 5644800\, y_{n+\frac{5}{8}} = h^3\Big(1308\, f_n - 3270\, f_{n+\frac{1}{8}} + 51177\, f_{n+\frac{3}{8}} + 43442\, f_{n+\frac{5}{8}} - 6285\, f_{n+\frac{7}{8}} + 1828\, f_{n+1}\Big) \quad (20)$$

$$352800\, h^2y''_{n+\frac{7}{8}} - 5644800\, y_{n+\frac{1}{8}} + 11289600\, y_{n+\frac{3}{8}} - 5644800\, y_{n+\frac{5}{8}} = -h^3\Big(708\, f_n - 2785\, f_{n+\frac{1}{8}} - 36918\, f_{n+\frac{3}{8}} - 101703\, f_{n+\frac{5}{8}} - 40370\, f_{n+\frac{7}{8}} + 4668\, f_{n+1}\Big) \quad (21)$$

$$5644800\,h^2y''_{n+1} - 90316800\,y_{n+\frac{1}{8}} + 180633600\,y_{n+\frac{3}{8}} - 90316800\,y_{n+\frac{5}{8}} = -h^3\Big(996\,f_n - 14775\,f_{n+\frac{1}{8}} - 650811\,f_{n+\frac{3}{8}} - 1519301\,f_{n+\frac{5}{8}} - 1147785\,f_{n+\frac{7}{8}} - 196324\,f_{n+1}\Big) \quad (22)$$

Equations 8 – 22 are then put in matrix form to produce:

$$RY_m = SY_{m-1} + TF_{m-1} + UF_m \quad (23)$$

Where,

$$Y_m = \left[y_{n+\frac{1}{8}}\ y_{n+\frac{3}{8}}\ y_{n+\frac{5}{8}}\ y_{n+\frac{7}{8}}\ y_{n+1}\ y'_{n+\frac{1}{8}}\ y'_{n+\frac{3}{8}}\ y'_{n+\frac{5}{8}}\ y'_{n+\frac{7}{8}}\ y'_{n+1}\ y''_{n+\frac{1}{8}}\ y''_{n+\frac{3}{8}}\ y''_{n+\frac{5}{8}}\ y''_{n+\frac{7}{8}}\ y''_{n+1} \right]^\top$$

$$Y_{m-1} = \left[y_n\ y'_n\ y''_n \right]^\top, \quad F_{m-1} = \left[f_n \right]^\top, \quad F_m = \left[f_{n+\frac{1}{8}}\ f_{n+\frac{3}{8}}\ f_{n+\frac{5}{8}}\ f_{n+\frac{7}{8}}\ f_{n+1} \right]^\top$$

The block matrices in Equation 23 is then resolved by multiplying throughout by R^{-1} to give the following discrete schemes:

$$y_{n+\frac{1}{8}} = y_n + \frac{1}{8}hy'_n + \frac{1}{128}h^2y''_n + \frac{150673}{722534400}h^3f_n + \frac{42451}{289013760}h^3f_{n+\frac{1}{8}} - \frac{10043}{206438400}h^3f_{n+\frac{3}{8}} + \frac{821}{29491200}h^3f_{n+\frac{5}{8}} - \frac{4027}{289013760}h^3f_{n+\frac{7}{8}} + \frac{3503}{722534400}f_{n+1}h^3 \quad (24)$$

$$y_{n+\frac{3}{8}} = y_n + \frac{3}{8}hy'_n + \frac{9}{128}h^2y''_n + \frac{205929}{80281600}h^3f_n + \frac{195939}{32112640}h^3f_{n+\frac{1}{8}} + \frac{261}{22937600}h^3f_{n+\frac{3}{8}} + \frac{4131}{22937600}h^3f_{n+\frac{5}{8}} - \frac{3483}{32112640}h^3f_{n+\frac{7}{8}} + \frac{3159}{80281600}f_{n+1}h^3 \quad (25)$$

$$y_{n+\frac{5}{8}} = y_n + \frac{5}{8}hy'_n + \frac{25}{128}h^2y''_n + \frac{201625}{28901376}h^3f_n + \frac{1444375}{57802752}h^3f_{n+\frac{1}{8}} + \frac{74125}{8257536}h^3f_{n+\frac{3}{8}} - \frac{3125}{8257536}h^3f_{n+\frac{5}{8}} + \frac{10625}{57802752}h^3f_{n+\frac{7}{8}} - \frac{1625}{28901376}f_{n+1}h^3 \quad (26)$$

$$y_{n+\frac{7}{8}} = y_n + \frac{7}{8}hy'_n + \frac{49}{128}h^2y''_n + \frac{199969}{14745600}h^3f_n + \frac{333739}{5898240}h^3f_{n+\frac{1}{8}} + \frac{1025227}{29491200}h^3f_{n+\frac{3}{8}} + \frac{184877}{29491200}h^3f_{n+\frac{5}{8}} + \frac{3773}{5898240}h^3f_{n+\frac{7}{8}} - \frac{2401}{14745600}f_{n+1}h^3 \quad (27)$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1553}{88200}h^3f_n + \frac{341}{4410}h^3f_{n+\frac{1}{8}} + \frac{169}{3150}h^3f_{n+\frac{3}{8}} + \frac{7}{450}h^3f_{n+\frac{5}{8}} + \frac{13}{4410}h^3f_{n+\frac{7}{8}} - \frac{37}{88200}f_{n+1}h^3 \quad (28)$$

Therefore, Equations 24 - 28 represent the formulated selected single-step hybrid block (SSHB) method for the solution of (2) and their associated first and second derivative discrete schemes are given below:

$$y'_{n+\frac{1}{8}} = y'_n + \frac{1}{8}hy''_n + \frac{48281}{11289600}h^2f_n + \frac{1217}{282240}h^2f_{n+\frac{1}{8}} - \frac{4051}{3225600}h^2f_{n+\frac{3}{8}} + \frac{1147}{1612800}h^2f_{n+\frac{5}{8}} - \frac{1601}{4515840}h^2f_{n+\frac{7}{8}} + \frac{1391}{11289600}h^2f_{n+1} \quad (29)$$

$$y'_{n+\frac{3}{8}} = y'_n + \frac{3}{8}hy''_n + \frac{17139}{1254400}h^2f_n + \frac{4905}{100352}h^2f_{n+\frac{1}{8}} + \frac{201}{22400}h^2f_{n+\frac{3}{8}} - \frac{549}{358400}h^2f_{n+\frac{5}{8}} + \frac{117}{250880}h^2f_{n+\frac{7}{8}} - \frac{171}{1254400}h^2f_{n+1} \quad (30)$$

$$y'_{n+\frac{5}{8}} = y'_n + \frac{5}{8}hy''_n + \frac{9925}{451584}h^2 f_n + \frac{45625}{451584}h^2 f_{n+\frac{1}{8}} + \frac{8875}{129024}h^2 f_{n+\frac{3}{8}} + \frac{25}{8064}h^2 f_{n+\frac{5}{8}} + \frac{625}{903168}h^2 f_{n+\frac{7}{8}} - \frac{125}{451584}h^2 f_{n+1} \quad (31)$$

$$y'_{n+\frac{7}{8}} = y'_n + \frac{7}{8}hy''_n + \frac{7007}{230400}h^2 f_n + \frac{14063}{92160}h^2 f_{n+\frac{1}{8}} + \frac{31213}{230400}h^2 f_{n+\frac{3}{8}} + \frac{26411}{460800}h^2 f_{n+\frac{5}{8}} + \frac{49}{5760}h^2 f_{n+\frac{7}{8}} - \frac{343}{230400}h^2 f_{n+1} \quad (32)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{379}{11025}h^2 f_n + \frac{79}{441}h^2 f_{n+\frac{1}{8}} + \frac{263}{1575}h^2 f_{n+\frac{3}{8}} + \frac{143}{1575}h^2 f_{n+\frac{5}{8}} + \frac{67}{2205}h^2 f_{n+\frac{7}{8}} - \frac{37}{22050}h^2 f_{n+1} \quad (33)$$

$$y''_{n+\frac{1}{8}} = y''_n + \frac{9679}{201600}hf_n + \frac{14339}{161280}hf_{n+\frac{1}{8}} - \frac{2203}{115200}hf_{n+\frac{3}{8}} + \frac{409}{38400}hf_{n+\frac{5}{8}} - \frac{851}{161280}hf_{n+\frac{7}{8}} + \frac{41}{22400}hf_{n+1} \quad (34)$$

$$y''_{n+\frac{3}{8}} = y''_n + \frac{663}{22400}hf_n + \frac{3963}{17920}hf_{n+\frac{1}{8}} + \frac{1869}{12800}hf_{n+\frac{3}{8}} - \frac{381}{12800}hf_{n+\frac{5}{8}} + \frac{213}{17920}hf_{n+\frac{7}{8}} - \frac{87}{22400}hf_{n+1} \quad (35)$$

$$y''_{n+\frac{5}{8}} = y''_n + \frac{295}{8064}hf_n + \frac{2125}{10752}hf_{n+\frac{1}{8}} + \frac{1325}{4608}hf_{n+\frac{3}{8}} + \frac{515}{4608}hf_{n+\frac{5}{8}} - \frac{125}{10752}hf_{n+\frac{7}{8}} + \frac{25}{8064}hf_{n+1} \quad (36)$$

$$y''_{n+\frac{7}{8}} = y''_n + \frac{889}{28800}hf_n + \frac{4949}{23040}hf_{n+\frac{1}{8}} + \frac{28469}{115200}hf_{n+\frac{3}{8}} + \frac{10633}{38400}hf_{n+\frac{5}{8}} + \frac{2779}{23040}hf_{n+\frac{7}{8}} - \frac{49}{3200}hf_{n+1} \quad (37)$$

$$y''_{n+1} = y''_n + \frac{103}{3150}hf_n + \frac{22}{105}hf_{n+\frac{1}{8}} + \frac{58}{225}hf_{n+\frac{3}{8}} + \frac{58}{225}hf_{n+\frac{5}{8}} + \frac{22}{105}hf_{n+\frac{7}{8}} + \frac{103}{3150}hf_{n+1} \quad (38)$$

ANALYSIS OF THE METHOD

In this section, we shall consider the order, error constants, zero-stability and consistency of SSHB method.

Order and Local Truncation Errors (LTEs) of the method

Theorem 1 (Dahlquist first barrier of zero-stability).

No zero-stable linear multistep method of step number k can have order exceeding $k + 1$ when k is odd, or $k + 2$ when k is even. If the method is also explicit, then it cannot attain an order greater than k.

Remark: (See Henrici (1962) and Lambert (1973)).

Therefore, the new method in Equation 5 and their associated linear differential operator $\gamma[y(x); h]$ is defined by:

$$\Upsilon[y(x);h] = \sum_{j=\sigma_i} \alpha_{\sigma_i}(x_n + \sigma_i h) - h^3\left(\sum_{j=0} \beta_j y'''(x_n + jh) + \sum_{j=\sigma_i} \beta_{\sigma_i} y'''(x_n + \sigma_i h)\right) \quad (39)$$

Where, $i = 0(1)4, \sigma_i = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$, 1 and y($x$) is an arbitrary function which is continuously differentiable in the interval [a, b]. Taylor series expansion of $y(x_n + jh), y'''(x_n + jh)$ and $y'''(x_n + \sigma_i h)$ gives:

$$\Upsilon[y(x);h] = C_0 y(x) + C_1 h y^{(1)}(x) + C_2 h^2 y^{(2)} + C_3 h^3 y^{(3)} + \ldots + C_p h^p y^{(p)} \tag{40}$$

Definition 1. The linear differential operator Y and its associated selected single-step hybrid block (SSHB) method is said to be of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = C_{p+2} = 0, C_{p+3} \neq 0$.

Definition 2. The term C_{p+3} in Definition 1 is called the error constant and it indicates the local truncation error, which is given by:

$$t_{n\_k} = C_{p+3} h^{p+3} y^{p+3}(X_n) + 0(h^{p+3})$$

Expanding (39) in Taylor's series gives:

$$\sum_{j=0}^{\infty} \frac{\left(\frac{1}{8}\right)^j h^j}{j!} y_n^j - y_n - \frac{1}{8} h y_n' - \frac{h^2}{128} y_n'' - \frac{150673}{722534400} h^3 y_n''' - h^3 \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+3} \left(\frac{42451}{289013760} \left(\frac{1}{8}\right)^j - \frac{10043}{206438400} \left(\frac{3}{8}\right)^j + \frac{821}{29491200} \left(\frac{5}{8}\right)^j - \frac{4027}{289013760} \left(\frac{7}{8}\right)^j + \frac{3503}{722534400} (1)^j \right) = 0 \tag{41}$$

$$\sum_{j=0}^{\infty} \frac{\left(\frac{3}{8}\right)^j h^j}{j!} y_n^j - y_n - \frac{3}{8} h y_n' - \frac{9}{128} h^2 y_n'' - \frac{205929}{80281600} h^3 y_n''' - h^3 \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+3} \left(\frac{195939}{32112640} \left(\frac{1}{8}\right)^j + \frac{261}{22937600} \left(\frac{3}{8}\right)^j + \frac{4131}{22937600} \left(\frac{5}{8}\right)^j - \frac{3483}{32112640} \left(\frac{7^j}{8}\right) + \frac{3159}{80281600} \right) = 0 \tag{42}$$

$$\sum_{j=0}^{\infty} \frac{\left(\frac{5}{8}\right)^j h^j}{j!} y_n^j - y_n - \frac{5}{8} h y_n' - \frac{25}{128} h^2 y_n'' - \frac{201625}{28901376} h^3 y_n''' - h^3 \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+3} \left(\frac{1444375 \left(\frac{1}{8}\right)^j}{57802752} + \frac{74125}{8257536} \left(\frac{3}{8}\right)^j - \frac{3125}{8257536} \left(\frac{5}{8}\right)^j + \frac{10625}{57802752} \left(\frac{7}{8}\right)^j - \frac{1625}{28901376} \right) = 0 \tag{43}$$

$$\sum_{j=0}^{\infty} \frac{\left(\frac{7}{8}\right)^j h^j}{j!} y_n^j - y_n - \frac{7}{8} h y_n' - \frac{49}{128} h^2 y_n'' - \frac{199969}{14745600} h^3 y_n''' - h^3 \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+3} \left(\frac{333739}{5898240} \left(\frac{1}{8}\right)^j + \frac{1025227}{29491200} \left(\frac{3}{8}\right)^j + \frac{184877}{29491200} \left(\frac{5}{8}\right)^j - \frac{3773}{5898240} \left(\frac{7}{8}\right)^j + \frac{2401}{14745600} \right) = 0 \tag{44}$$

$$\sum_{j=0}^{\infty} \frac{h^j y_n}{j!} - y_n - h y_n' - \frac{1}{2} h^2 y_n'' - \frac{1553\, h^3 y_n'''}{88200} - h^3 \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j+3} \left(\frac{341}{4410} \left(\frac{1}{8}\right)^j + \frac{169}{3150} \left(\frac{3}{8}\right)^j + \frac{7}{450} \left(\frac{5}{8}\right)^j + \frac{13}{4410} \left(\frac{7}{8}\right)^j - \frac{37}{88200} \right) = 0 \tag{45}$$

Comparing Equations 41 - 45 in terms of powers h^j and y^j, we get the order of our method as:

$$[\,6\;6\;6\;6\;6\,]^T$$

With the error constants:

$$C_{p+3} = \left[-\frac{8081}{48704929136640} \quad -\frac{4887}{3006477107200} \quad \frac{1375}{9740985827328} \quad \frac{16807}{6957847019520} \quad \frac{41}{7431782400} \right]^T$$

Theorem 2. Henrici (1962) A linear multistep method is said to be convergent if it is consistent (that is, $p \geq 1$) and it is zero-stable.

Consistency
From Equation 10, observe that:

$$p(t) = 2752512t - 12042240\,t^{\frac{5}{8}} + 14450688\,t^{\frac{3}{8}} - 5160960t^{\frac{1}{8}} \text{ and}$$
$$q(t) = 446 - 661\,t^{\frac{1}{8}} + 39991\,t^{\frac{3}{8}} + 49833\,t^{\frac{5}{8}} + 4405\,t^{\frac{7}{8}} + 66\,t$$

Therefore, our method in Equations 24 - 28 is said to be consistent if:

i. The order, $p \geq 1$. Thus, Our method whose order is 6 satisfied this condition.

ii. $\sum_{i=1}^{5} \alpha_i = 0$, so that from above:
$\alpha_{\frac{1}{8}} = -5160960, \quad \alpha_{\frac{3}{8}} = 14450688, \quad \alpha_{\frac{5}{8}} = -12042240, \quad \text{and} \quad \alpha_1 = 2752512$

iii. Since,
$$p(t) = 2752512t - 12042240\,t^{\frac{5}{8}} + 14450688\,t^{\frac{3}{8}} - 5160960t^{\frac{1}{8}} \text{ and}$$
$$p'(t) = 2752512 - 7526400\,t^{-\frac{3}{8}} + 5419008\,t^{-\frac{5}{8}} - 645120\,t^{-\frac{7}{8}},$$
it follows that, $p(1) = 0 = p'(1)$

iv. Recal that: $p'''(t) = -3880800\,t^{-\frac{19}{8}} + 5503680\,t^{-\frac{21}{8}} - 1058400\,t^{-\frac{23}{8}}$
So that $p'''(1) = 3!\,q(1) = 564480.$

Therefore, since the conditions above are met, it follows that our new method is consistent.

Zero stability
Definition 3. The continuous implicit linear multi-step method in Equation 5 is said to be zero stable if no root of the first characteristic polynomial, $\rho(r)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than one, (see, Abolarin et al., (2020)).
Therefore,

$$\rho(r) = det[rR^1 - S^1] \tag{46}$$

Where, R^1 and S^1 are coefficients of

$$\left[y_{n+\frac{1}{8}}\ y_{n+\frac{3}{8}}\ y_{n+\frac{5}{8}}\ y_{n+\frac{7}{8}}\ y_{n+1}\right]^\top \text{ and } \left[y_{n-\frac{7}{8}}\ y_{n-\frac{5}{8}}\ y_{n-\frac{3}{8}}\ y_{n-\frac{1}{8}}\ y_n\right]^\top$$

in Equations 24 - 28 respectively and are given by:

$$R^1 = \begin{bmatrix} 1\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 1 \end{bmatrix}, \quad S^1 = \begin{bmatrix} 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1 \end{bmatrix}$$

Evaluating Equation 46 and solving for r gives the roots of the fist characteristic polynomial as: $r_i = 0, 0, 0, 0$ and 1, $i = 1(1)5$. Thus, our new (SSHB) method is zero stable.
Hence, the selected single-step hybrid block (SSHB) method converges in line with Theorem 2 and Definitions 2 and 3.

Absolute Stability Region of the SSHB
Definition 4 (Source: (Lambert, 1991)).
The linear multistep method (SSHB) is said to be A-Stable if its region of absolute stability contains the whole of the left hand half plane (that is, $Re(h\lambda) < 0$).
In line with Butcher (2008), we determine the stability region by a single stability function. Therefore, for the differential equation $y''' = qy$, Equation 10 becomes:

$$22752512y_{n+1} - 5160960y_{n+\frac{1}{8}} + 14450688y_{n+\frac{3}{8}} - 12042240y_{n+\frac{5}{8}} = h^3\Big(446qy_n$$
$$+ 66qy_{n+1} - 661qy_{n+\frac{1}{8}} + 39991qy_{n+\frac{3}{8}} + 49833qy_{n+\frac{5}{8}} + 4405qy_{n+\frac{7}{8}}\Big) \tag{47}$$

After some necessary substitutions, manipulations and collection of like terms using Maple software environment, we arrived at a polynomial of the form:

$$(2752512 - 66z)w^5 - 4405zw^4 + (12042240 + 49833z)w^3 - (14450688 - 39991z)w^2 + (5160960 - 661z)w + 446z = 0 \quad (48)$$

Where w is nth order polynomial, $h^3q = z$ and it is the complex plane for which Equation 47 has only bounded solution as $n \to \infty$. Therefore, all solutions to (47) converge to zero as $n \to \infty$ if the interior stability region is considered for z in this set. Equation 48 is then solved for z and setting

$e^{\theta I} = w$ gives:

$$z(w) = \frac{344064e^{i\theta}\left(8\left(e^{i\theta}\right)^4 - 35\left(e^{i\theta}\right)^2 + 42\,e^{i\theta} - 15\right)}{66\left(e^{i\theta}\right)^5 + 4405\left(e^{i\theta}\right)^4 + 49833\left(e^{i\theta}\right)^3 + 39991\left(e^{i\theta}\right)^2 - 661\,e^{i\theta} + 446} \quad (49)$$

The above equation is in the form:

$z(w) = \frac{\rho(w)}{\sigma(w)}$.

Therefore, the above procedures for finding the stability region of our method by using a single stability function is called "Boundary Locus Method". Equation 49 is then coded in MATLAB software environment and the stability of our method is as shown below:

From Figure 1 below and in accordance with Definition 4, it follows that our new formulated (SSHB) method is A-stable.

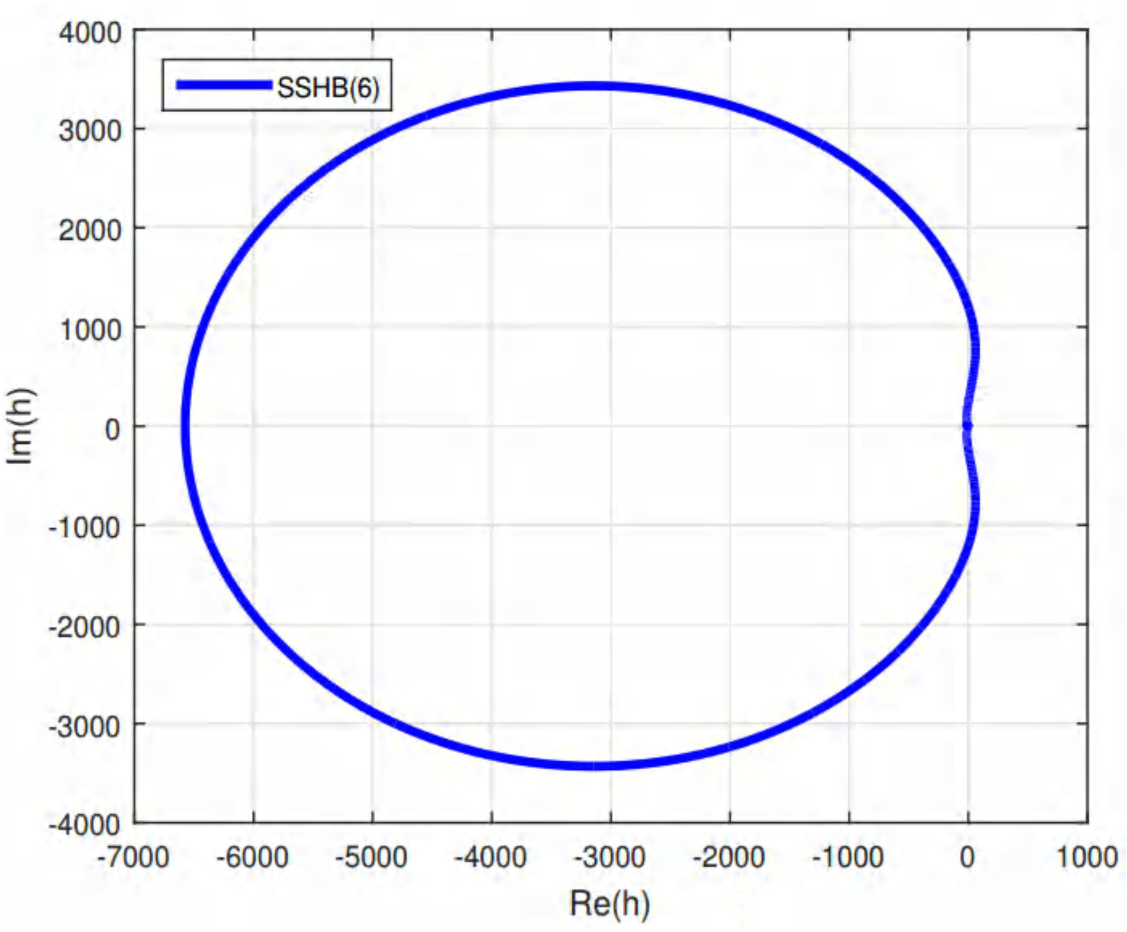

Figure 1: Absolute Stability Region of SSHB(6)

Numerical examples and implementation

Therefore, we present the numerical examples to justify the accuracy and performance of our method via Taylor series expansion and their corresponding derivatives which represent our explicit method that will incorporate all initial values for our derived formula and for the approximation of (2) and are implemented on MATLAB software environment. Thus, we take the Taylor series expansion that are of the same order as the order of our method: y_{n+1}, where $i = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ and 1. So that,

$$y_{n+i} = y(x_n + ih) = y_n + ihy'_n + \frac{(ih)^2}{2!}y''_n + \frac{(ih)^3}{3!}f_n + \frac{(ih)^4}{4!}f'_n + \frac{(ih)^5}{5!}f''_n + \frac{(ih)^6}{6!}f'''_n$$

$$y'_{n+i} = y'(x_n + ih) = y'_n + ihy''_n + \frac{(ih)^2}{2!}f_n + \frac{(ih)^3}{3!}f'_n + \frac{(ih)^4}{4!}f''_n + \frac{(ih)^5}{5!}f'''_n$$

$$y''_{n+i} = y''(x_n + ih) = y''_n + ihf_n + \frac{(ih)^2}{2!}f'_n + \frac{(ih)^3}{3!}f''_n + \frac{(ih)^4}{4!}f'''_n$$

Example 1. Consider the Initial Value Problem below:

$$y''' - y'' + y' - y = 0 \quad y(0) = 1 \quad y'(0) = 0, \quad y''(0) = -1, \quad h = 0.01$$

Theoretical solution: $y(x) = \cos x$

The absolute errors for our method are compared with Kuboye et al. (2020) in Table 1.

Example 2. $y''' = 3\sin x \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad h = 0.1$

Theoretical solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

The absolute errors for our method are compared with Kuboye et al., (2020), Kashkari et al. (2019) and Adeniran et al., (2016) and are shown in Tables 2, 3 and 4 respectively.

Example 3. Consider the initial value problem of the form:

$$y''' = e^x, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5, \quad h = 0.1$$

Theoretical solution: $y(x) = 2 + 2x^2 + e^x$

Our new method is compared in terms of absolute errors with Kuboye et al., (2020) whose method is of order 6 and the results are as shown in Table 5.

Example 4. Consider the third order linear problem:

$$y''' + e^x = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 3, \quad h = 0.1$$

Theoretical solution: $y(x) = 2(1 + x^2) - e^x$

Source: Kayode & Obarhua (2017) and Ogunware et al., (2020)
The results for Test Example 4 are presented in Table 6.

Example 5. Consider the non-linear initial value problem:

$$y''' = y'(2xy'' + y'), \quad y(0) = 1, \quad y'(0) = \frac{1}{2}, \quad y''(0) = 0, \quad h = 0.1$$

Theoretical solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Table 7 provides the results for Test Example 5 and results are compared with Kayode & Obarhua (2017).
The following abbreviations are used in the tables:

ES → Exact solution
CS → Computed solution
AbsErr → Absolute errors
EIFBM(6) → Error in fist block method with s = 52 of order 6 in Kuboye et al., (2020)
EISBM(6) → Error in second block method with s = 94 of order 6 in Kuboye et al., (2020)
BHCM → Block hybrid collocation method of order 6 in Yap et al., (2014)
SSHB(6) → The new formulated selected single-step hybrid block of uniform order 6.

Table 1: Absolute errors comparing our method with Kuboye et al., (2020) for Test Example 1 when h=0.01

X	ES	CS	EIFBM(6)	EISBM(6)	SSHB(6)
0.01	0.999950000416665260	0.999950000416665260	1.1102230E-16	0.000000E+00	0.000000E+00
0.02	0.999800006666577760	0.999800006666577870	5.5511151E-16	5.5511151E-16	1.110223E-16
0.03	0.999550033748987540	0.999550033748988880	8.6597396E-15	8.7707619E-15	1.332268E-15
0.04	0.999200106660977920	0.999200106660985020	6.4837025E-14	6.4614980E-14	7.105427E-15
0.05	0.998750260394966280	0.998750260394991150	2.6301183E-14	2.6290081E-14	2.486900E-14

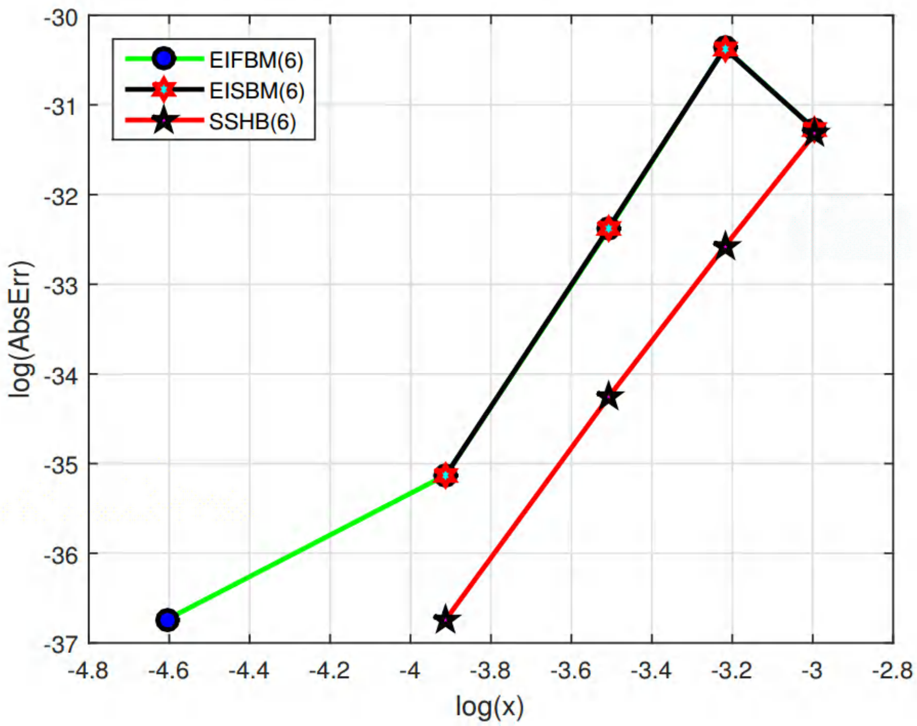

Figure 2: Efficiency curves for Table 1

Table 2: Absolute errors comparing our method with Kuboye et al., (2020) for Test Example 2 when h=0.1

x	ES	CS	EIFBM(6)	EISBM(6)	SSHB(6)
0.1	0.990012495834077020	0.990012495834077240	6.8911543e-13	5.9885430e-13	2.220446e-16
0.2	0.960199733523725120	0.960199733523724900	4.4015902e-12	3.8212766e-12	2.220446e-16
0.3	0.911009467376818090	0.911009467376818200	1.0999868e-11	9.5831121e-12	1.110223e-16
0.4	0.843182982008655380	0.843182982008655270	2.0601632e-11	1.7947976e-11	1.110223e-16
0.5	0.757747685671118280	0.757747685671118500	2.6853520e-11	3.2626124e-11	2.220446e-16
0.6	0.656006844729035250	0.656006844729034810	6.7268413e-11	6.0369598e-11	4.440892e-16
0.7	0.539526561853465480	0.539526561853466920	1.1150603e-10	1.0098744e-10	1.443290e-15
0.8	0.410120128041496110	0.410120128041499110	1.6985002e-10	1.5461399e-10	2.997602e-15
0.9	0.269829904811993430	0.269829904811998030	2.4948449e-10	2.2891933e-10	4.607426e-15
1.0	0.120906917604419300	0.120906917604426250	3.6226498e-10	3.3474887e-10	6.952772e-15
1.1	-0.034211635723267797	-0.034211635723257514	5.0769700e-10	4.7182869e-10	1.028344e-14
1.2	-0.192926736569979160	-0.192926736569964700	6.8618927e-10	6.4034714e-10	1.446065e-14

Table 3: Absolute errors comparing our method with Kashkari et al., (2019) for Test Example 2 when h=0.1 and h=0.01

x	**Error in Kashkari et al., (2019)** p = 6, h = *0.1*	**Error in our method** p = 6, h = *0.1*	**Error in Kashkari et al., (2019)** p = 6, h = *0.01*	**Error in our method** p = 6, h = *0.01*
0.1	4.1078E-15	2.2205E-16	4.4409E-16	2.2205E-16
0.2	1.6875E-14	2.2205E-16	1.2212E-15	4.4409E-16
0.3	5.0848E-14	1.1102E-16	2.4425E-15	2.2205E-16
0.4	1.1779E-13	1.1102E-16	3.7748E-15	2.2205E-16
0.5	2.4081E-13	2.2205E-16	5.5511E-15	2.2205E-16
0.6	4.3709E-13	4.4409E-16	8.4377E-15	3.3307E-16
0.7	7.3708E-13	1.4433E-15	1.1324E-14	2.2205E-16
0.8	1.1662E-12	2.9976E-15	1.4544E-14	5.5511E-17
0.9	1.7587E-12	4.6074E-15	1.8985E-14	5.5511E-16
1.0	2.5466E-12	6.953E-15	2.3870E-14	1.0131E-15

Table 4: Absolute errors in our method in comparison with Adeniran et al., (2016) for Test Example 2 when h = 0.1

x	ES	CS	Error in Our Method SSHB(6)	Error in Adeniran et al., (2016) p=6
0.1	0.990012495834077020	0.990012495834077240	2.2205e-16	1.0000e-14
0.2	0.960199733523725120	0.960199733523724900	2.2205e-16	2.7100e-13
0.3	0.911009467376818090	0.911009467376817980	1.1102e-16	1.1450e-12
0.4	0.843182982008655380	0.843182982008655270	1.1102e-16	2.9620e-12
0.5	0.757747685671118280	0.757747685671118500	2.2205e-16	6.0710e-12
0.6	0.656006844729035250	0.656006844729035700	4.4409e-16	1.0800e-11
0.7	0.539526561853465480	0.539526561853466920	1.4433e-15	1.7439e-11
0.8	0.410120128041496110	0.410120128041499110	2.9976e-15	2.6274e-11
0.9	0.269829904811993430	0.269829904811998030	4.6074e-15	3.7598e-11
1.0	0.120906917604419300	0.120906917604426250	6.9528e-15	5.1625e-11

Table 5: Absolute errors comparing our method with Kuboye et al., (2020) for Test Example 3 when h=0.1

x	ES	CS	EIFBM(6)	EISBM(6)	SSHB(6)
0.1	3.125170918075647700	3.125170918075647700	1.5227819E-12	1.3447021E-12	0.000000E+00
0.2	3.301402758160169700	3.301402758160169700	9.6922470E-12	8.5487173E-12	0.000000E+00
0.3	3.529858807576003300	3.529858807576002900	2.4267699E-11	2.1475266E-11	4.440892E-16
0.4	3.811824697641270600	3.811824697641270600	4.5451198E-11	4.0219827E-11	0.000000E+00
0.5	4.148721270700128200	4.148721270700126400	7.8387963E-11	7.0274453E-11	1.776357E-15
0.6	4.542118800390508900	4.542118800390507100	1.3159340E-10	1.1942358E-10	1.776357E-15
0.7	4.993752707470476600	4.993752707470473100	2.0471091E-10	1.8746782E-10	3.552714E-15
0.8	5.505540928492466800	5.505540928492462300	2.9804159E-10	2.7454572E-10	4.440892E-15
0.9	6.079603111156949100	6.079603111156942000	4.1925841E-10	3.8885162E-10	7.105427E-15
1.0	6.718281828459044600	6.718281828459034000	5.8107297E-10	5.4199667E-10	1.065814E-14

Table 6: Absolute errors comparing our method with Kayode et al., (2017) and Ogunware et al., (2020) for Test Example 4 when h=0.1

x	ES	CS	Error in our Method SSHB(6)	Error in Kayode et al., (2017), p=6	Error in Ogunware et al., (2020)
0.1	0.914829081924352310	0.914829081924352420	1.110223E-16	1.82410E-13	3.473600E-14
0.2	0.858597241839830220	0.858597241839830330	1.110223E-16	1.67078E-12	3.326900E-13
0.3	0.830141192423996980	0.830141192423997420	4.440892E-16	6.00142E-12	3.709100E-14
0.4	0.828175302358729940	0.828175302358730710	7.771561E-16	1.48598E-11	5.791840E-13
0.5	0.851278729299871810	0.851278729299873690	1.887379E-15	3.01205E-11	3.581010E-13
0.6	0.897881199609490870	0.897881199609493860	2.997602E-15	5.38418E-11	1.209298E-12
0.7	0.966247292529523350	0.966247292529527900	4.551914E-15	8.83157E-11	1.179995E-12
0.8	1.054459071507532400	1.054459071507539000	6.661338E-15	1.36060E-10	2.514500E-12
0.9	1.160396888843050300	1.160396888843059800	9.547918E-15	1.99870E-10	2.409110E-12
1.0	1.281718171540954500	1.281718171540968000	1.354472E-14	2.82814E-10	4.870670E-12

Table 7: Absolute errors comparing our method with Kayode et al., (2017) for Test Example 5 when h=0.1

x	Exact. sol.	Approx. sol.	Error in Our Method SSHB(6)	Error in Kayode et al., (2017) , p=6
0.1	1.050041729278491400	1.050041729278490900	4.44E-16	9.49E-08
0.2	1.100335347731075600	1.100335347723219400	7.86E-12	1.32E-06
0.3	1.151140435936466800	1.151140417012499700	1.89E-08	5.65E-06
0.4	1.202732554054082100	1.202732428419503100	1.26E-07	1.58E-05
0.5	1.255412811882995200	1.255412330374608100	4.82E-07	3.55E-05
0.6	1.309519604203111900	1.309518197933271100	1.41E-06	6.97E-05

0.7	1.365443754271396200	1.365440267755764900	3.49E-06	1.25E-04
0.8	1.423648930193601700	1.423641168185924900	7.76E-06	2.11E-04
0.9	1.484700278594051700	1.484684221401156200	1.61E-05	3.41E-04
1.0	1.549306144334054800	1.549274550494045900	3.16E-05	5.32E-04

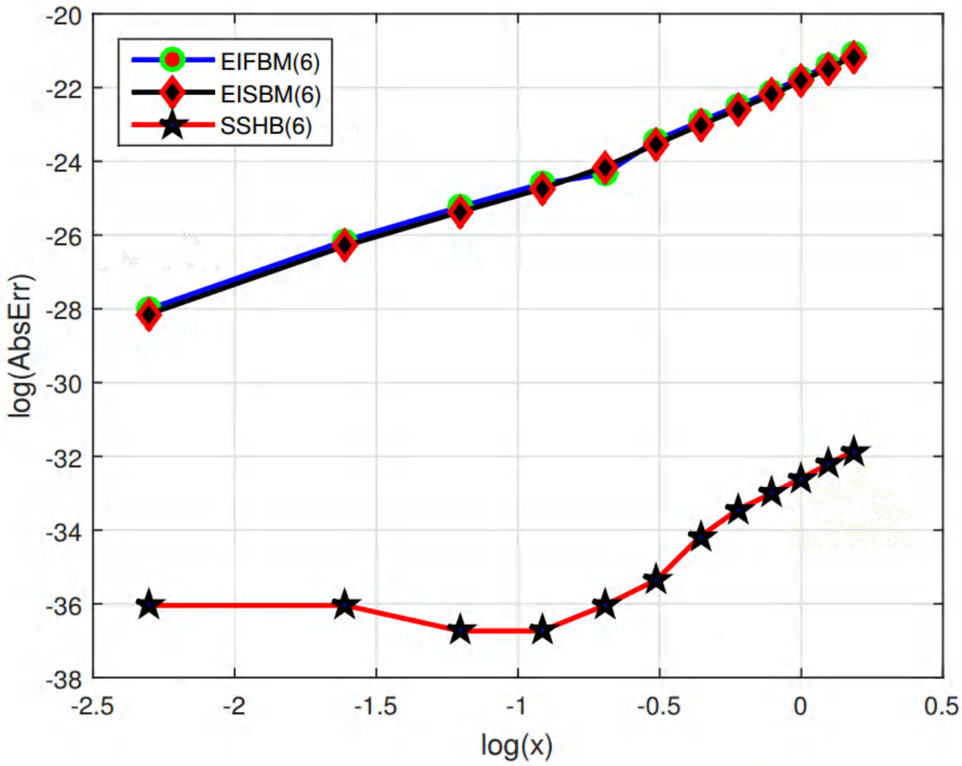

Figure 3: Efficiency curves for Table 2

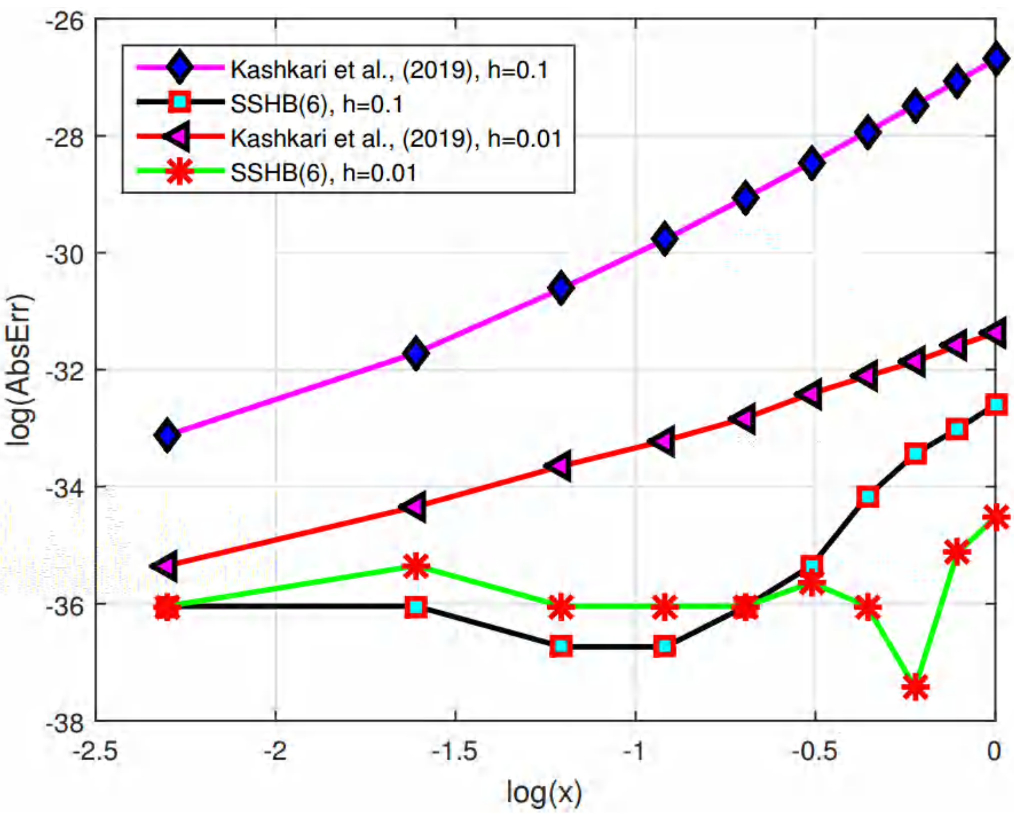

Figure 4: Efficiency curves for Table 3

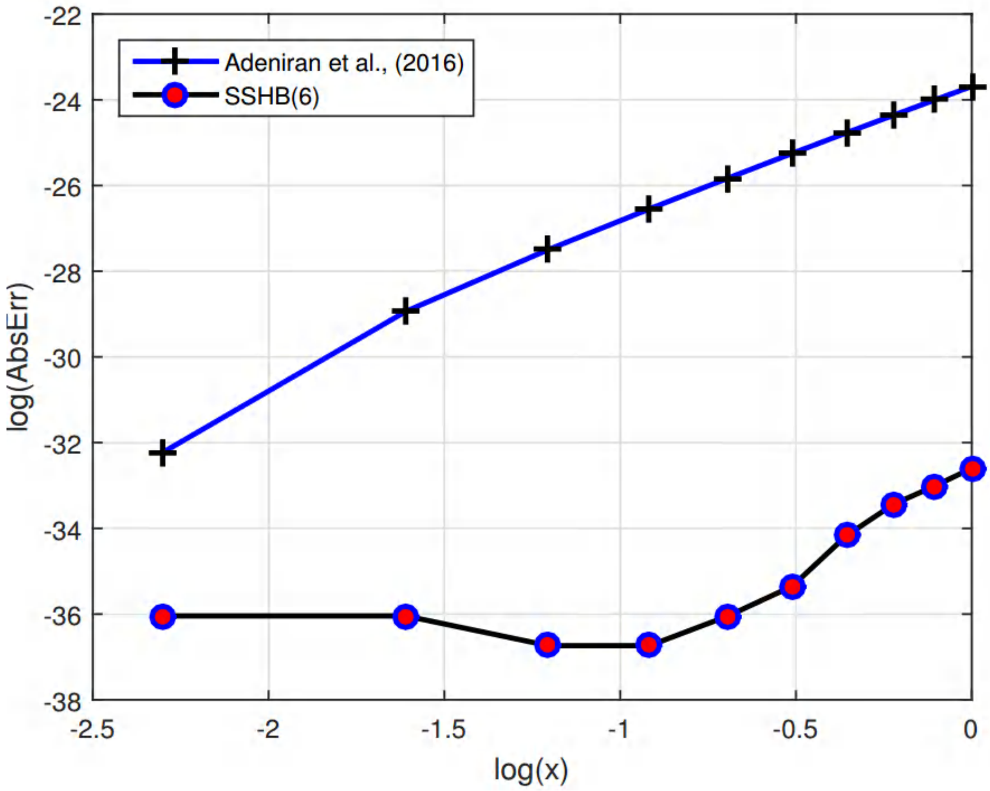

Figure 5: Efficiency curves for Table 4

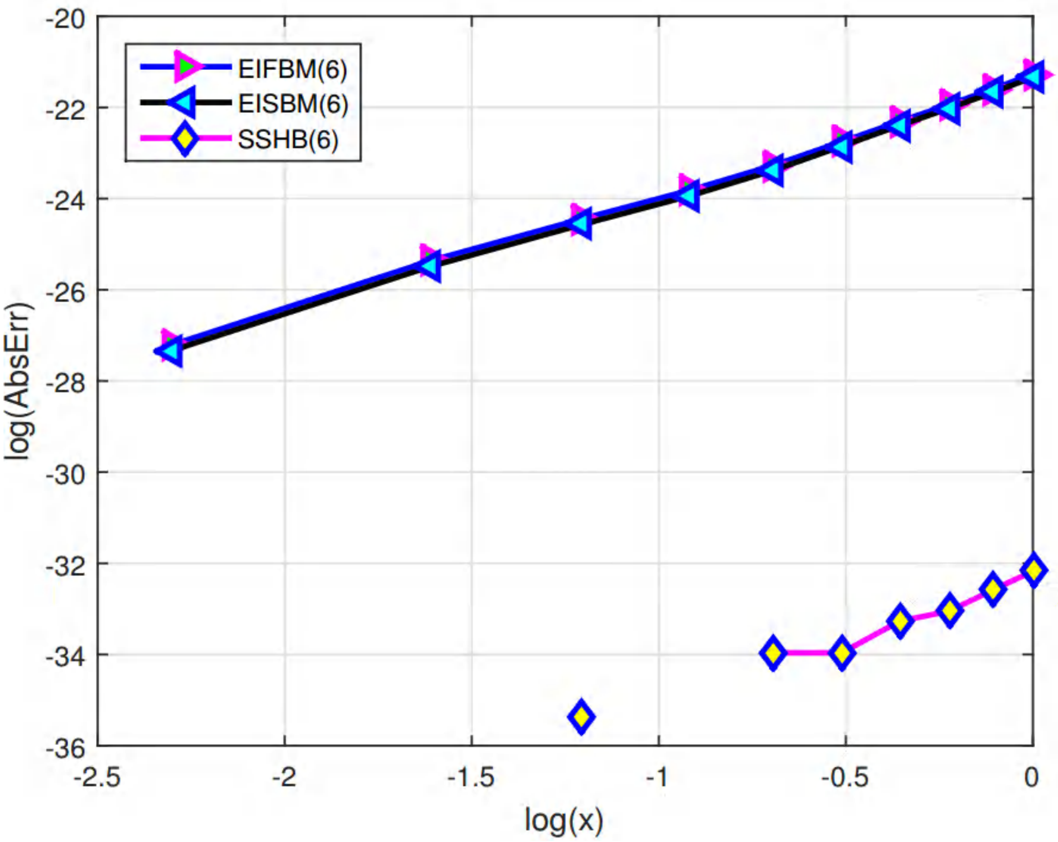

Figure 6: Efficiency curves for Table 5

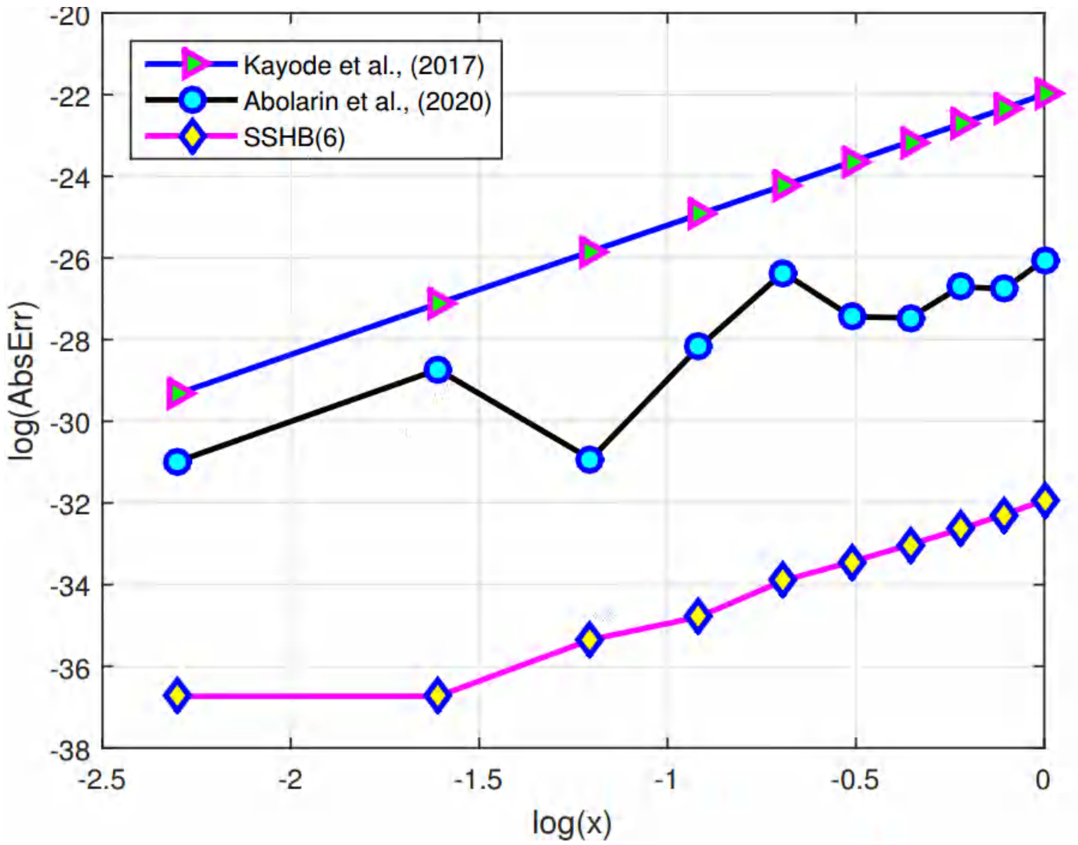

Figure 7: Efficiency curves for Table 6

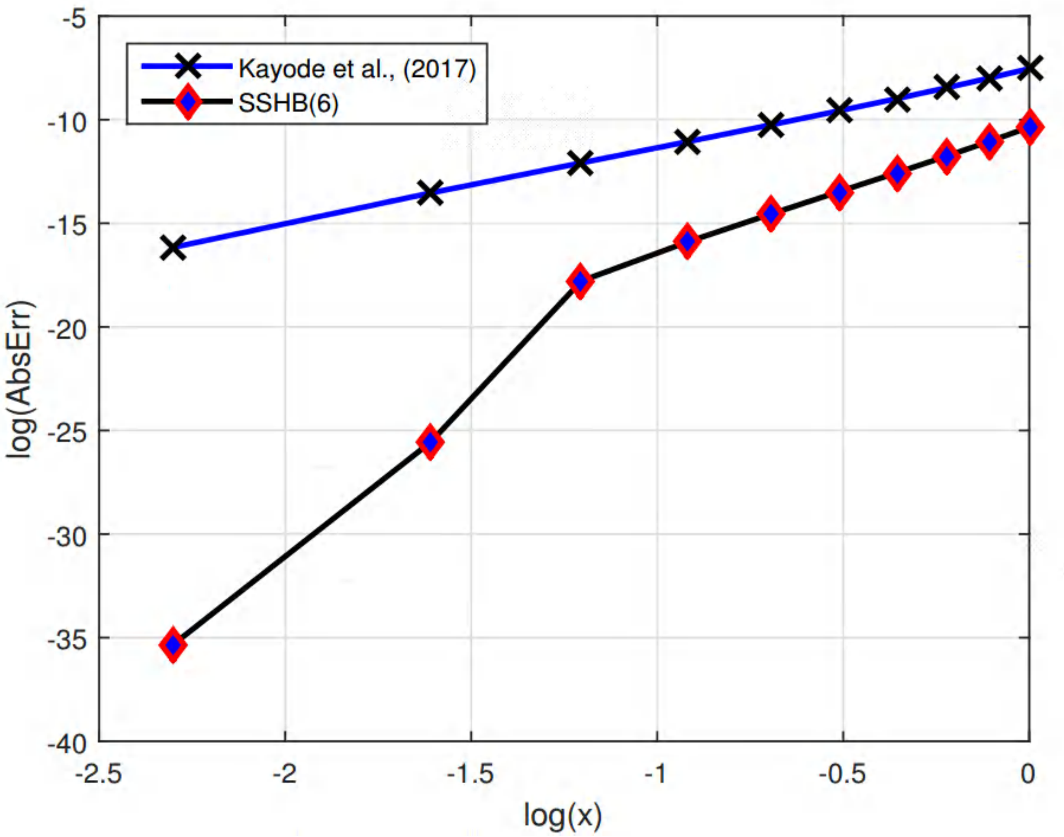

Figure 8: Efficiency curves for Table 7

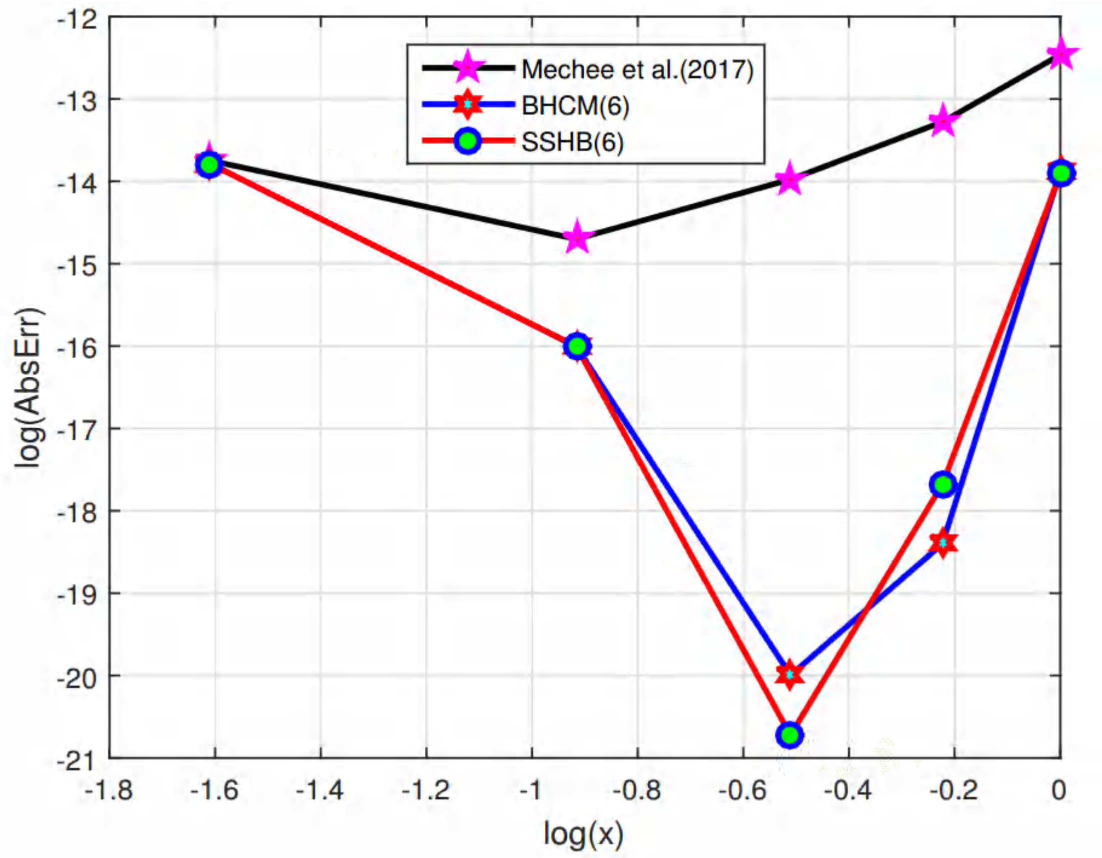

Figure 9: Efficiency curves for thin film flow problem, k=2, h=0.1 in Table 8

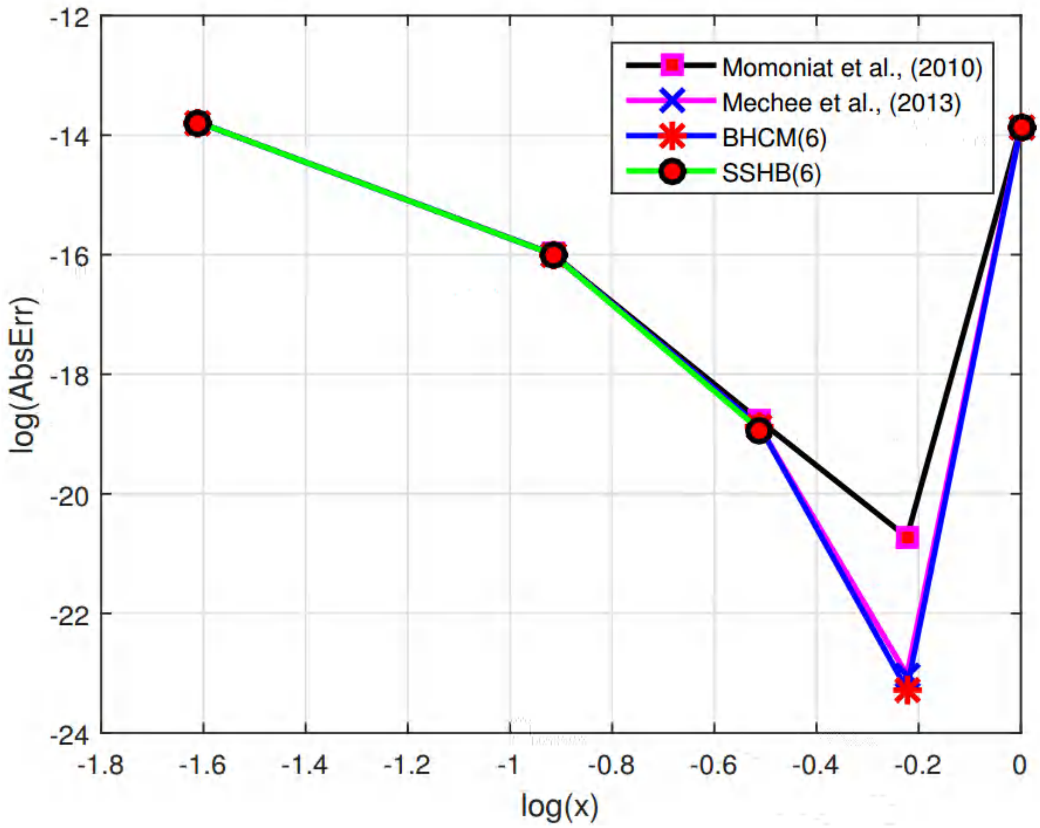

Figure 10: Efficiency curves for thin film flow problem, k=2, h=0.01 in Table 9

Method Application to Thin Film Flow Problem

In order to further test the viability of our derived formula, we applied it to solve the well-known physical problem in fluid dynamics. However, the motion of fluid on a plane surface in which the flow is in the direction of motion along the plane has been extensively discussed by Tuck and Schwartz (1990). Similarly, Momoniat and Mahomed (2010) considered successive reduction of order of thin film flow problem below to a first order ODE, which is then solved by a fourth- order Runge –Kutta method. Mechee, Senu, Ismail, Nikouravan & Siri (2013) have also solved thin film flow problem by using the fifth-order Runge-Kutta method. Also, this same thin film flow problem has been solved by Yap et al., (2014) and is majorly compared with our derived formula below. However, their methods gave reduced computational burden, as well as improved solutions. Therefore, this thin film problem was formulated in the form of a special third order ODEs:

$$y''' = y^{-k} \qquad (50)$$

with the initial conditions: $y(0) = y'(0) = y''(0) = 1$ for $k = 2, 3, h = 0.1$ and $h = 0.01$. The numerical results are presented in Tables 8, 9, 10 and 11 respectively.

Table 8: Numerical results comparing our method with Mechee et al., (2013) and Yap et al., (2014) for Thin Film Flow Problem (50) with h = 0.1 and k = 2

x	Exact	SSHB(6) Approx.	Error in Mechee et al., (2013)	Error in BHCM(6)	Error in SSHB(6)
0.2	1.221211030	1.221210005	1.07x10^{-6}	1.03x 10^{-6}	1.03x 10^{-6}
0.4	1.488834893	1.488834781	4.13x10^{-7}	1.12x 10^{-7}	1.12x 10^{-7}
0.6	1.807361404	1.807361405	8.51 x 10^{-7}	2.07x10^{-9}	1.00x 10^{-9}
0.8	2.179819234	2.179819255	1.71x10^{-6}	1.02x 10^{-8}	2.10x 10^{-8}
1.0	2.608275822	2.608274912	3.86x 10^{-6}	9.35x 10^{-7}	9.10x 10^{-7}

Table 9: Numerical results comparing our method with Momoniat et al., (2010), Mechee et al. (2013) and Yap et al., (2014) for Thin film Flow Problem (50) with h = 0.01 and k = 2

x	Error in Momoniat, (2010)	Error in Mechee, (2013)	Error in BHCM(6), Yap et al., (2014)	Error in SSHB(6)
0.2	1.03x 10^{-6}	1.03x 10^{-6}	1.03x 10^{-6}	1.03x 10^{-6}
0.4	1.14x 10^{-7}	1.13x 10^{-7}	1.13x 10^{-7}	1.13x 10^{-7}
0.6	7.00x 10^{-9}	6.30x 10^{-9}	6.32x 10^{-9}	6.00x 10^{-9}
0.8	1.00x 10^{-9}	1.00x 10^{-10}	7.83x 10^{-11}	0.00 x10+^{00}
1.0	9.55x 10^{-7}	9.54x 10^{-7}	9.54x 10^{-7}	9.54x 10^{-7}

Table 10: Numerical results comparing our method with Mechee et al., (2013) and Yap et al., (2014) for Thin film Flow Problem (50) with h = 0.1 and k = 3

x	Mechee et al., Approx., (2013)	BHCM(6) Approx.	SSHB(6) Approx.
0.0	1.000000000	1.000000000000000000	1.000000000000000000
0.2	1.2211550887	1.2211551426800236	1.221155142607257800
0.4	1.48881049238	1.48881052873784077	1.488105288187583100
0.6	1.8042615558	1.8042625625912998	1.804262566929343300
0.8	2.1715208324	2.1715228333017014	2.171522848573748600
1.0	2.5909549758	2.5909583248983960	2.590958361354658300

Table 11: Numerical results comparing our method with Momoniat et al., (2010), Mechee et al., (2013) and Yap et al., (2014) for Thin film Flow Problem (50) with h = 0.01 and k = 3

x	Approx. in Momoniat et al., (2010)	Approx. in Mechee et al., (2013)	Approx. in BHCM(6)	Approx. in SSHB(6)
0.0	1.000000000	1.000000000	1.000000000000000000	1.000000000000000000
0.2	1.221155142	1.2211551423	1.2211551423957325	1.221155142395771400
0.4	1.488105284	1.4881052838	1.4881052842194118	1.488105284220057700
0.6	1.804262548	1.8042625471	1.8042625481474530	1.804262548150244800
0.8	2.171522797	2.1715227960	2.1715227981283490	2.171522798135284000
1.0	2.590958258	2.5909582556	2.5909582591167280	2.590958259130050000

RESULTS AND DISCUSSIONS

This research paper has considered five numerical examples, excluding special physical problem from thin film flow. While Table 1 shows improved accuracy with its efficiency curves shown in Figure 2. Tables 2 - 4 indicate better approximate solutions that are very close to the exact solutions. This gave the new method better performance in terms of accuracy, as the efficiency curves in Figures 3-5 show small scale errors in our method. In general, varying step-sizes improves the stability of hybrid block methods as this is evident in applying our method to Test Problem 4.2 in Table 3. Similarly, convergency is evident in Table 5 when compared with methods of the same order. Figure 6 clearly depicts very small scale errors as numerical results are close to their exact solutions. Also, Table 6 clearly shows the efficiency of our formula over methods of the same and higher order 7 with improved performance in our method and with the efficiency curves shown in Figure 7. Table 7 has been compared with Kayode et al., (2017) of the same order. Results indicate that our method gave improved accuracy, as can be seen from the efficiency curves in Figure 8. Tables 8 and 9 show slight difference in absolute errors from applying our method on thin film flow problem with k = 2, 3 and h = 0.1,0.01 respectively. Similarly, with k = 3 and h = 0.1, the numerical results of our method agree with BHCM(6) in Table 10 to ten decimal places. Also, in Table 11, the numerical results in our method comply to a thirteen decimal places when compared with BHCM(6) in Yap et al., (2014). This clearly shows the viability of our method over BHCM(6).

CONCLUSION AND FUTURE RESEARCH

This paper has derived and implemented single-step hybrid block method with four off-grid points considered. Six (6) numerical experiments have been considered, including thin film flow problem in engineering, and results from experiment showed that the new method (SSHB), which is of a uniform order 6 gave improved approximations than some of the existing numerical formulae compared in this research. Therefore, we recommend that our derived formula becomes an alternative hybrid block method for the numerical solution of the specific third order ODEs considered. However, future research should consider the formulation of hybrid block methods with adequate odd off-grid points with fixed step discretization and with increased order of accuracy. Their strength should also be confirmed on practical problems.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest as regarding the publication of this research paper.

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