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TRANSMUTED NEW WEIGHTED EXPONENTIAL DISTRIBUTION: ITS DISTRIBUTIONAL PROPERTIES AND APPLICATIONS TO DATASETS FROM RAINFALL AND BREAST CANCER STUDIES

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ABSTRACT

This research paper a new distribution known as the Transmuted New Weighted Exponential Distribution (TNWED) was developed. The probability density function (pdf) and cumulative distribution function (cdf) of the developed distribution were determined using the transmutation map (TM). This is an extension of the well-known exponential distribution. The new distribution's properties have been thoroughly studied. The estimation of the distribution parameters was done using the method of maximum likelihood estimation. The performance of the TNWED has been evaluated using real life datasets and the results show that the TNWED fits the datasets more better compared to the fits of the other two distributions (New Weighted Exponential distribution and Exponential distribution) considered in this study.

Keywords: Distribution, Estimation, Likelihood, Transmuted**,** Weighted-Exponential

INTRODUCTION

The time between series of events in a poison process is described by the well-known single parameter exponential distribution, which only has a constant failure rate (Lawless, 2003). It has also been employed extensively in many different contexts. An expansion of the exponential distribution based on the Lehmann type I alternative is the two-parameter generalized-exponential (GE) distribution developed by Gupta and Kundu (1999). The skew-normal distribution was developed by Azzalini (1985) using a novel technique for adding a skewness parameter to the normal distribution based on a weighted function. Many skew symmetric distributions, such as the Nadarajah skew-logistic distribution, have been produced using Azzalini's theory applied to other symmetric distributions. In order to develop a new class of weighted exponential distributions, which are generalisations of the exponential distribution and resemble the Weibull, gamma, and exponentiated exponential distributions, Gupta and Kundu (2009) followed a methodology similar to Azzalini's for adding a skewness parameter to the exponential distribution. The study showed that, the weighted exponential distribution can be used to analyze positively skewed data, like the distributions mentioned above, and data coming from hidden truncated models.

Bashir and Naqvi (2016) employed Azzalini's Methodology to develop a new weighted exponential distribution, in comparison to the weighted exponential distribution created by Gupta and Kundu (2009), is more straightforward and easier to handle mathematically.

A random variable X is said to follow a New Weighted Exponential distribution by Bashir and Naqvi (2016) with parameters α *and* β if its probability density function (*pdf*) is given as

$$g(x;\alpha,\beta)=(\alpha-\beta)e^{-(\alpha-\beta)x} \quad (1)$$

With corresponding cumulative distribution function (*cdf*) as

$$G(x;\alpha,\beta)=1-e^{-(\alpha-\beta)x} \quad (2)$$

For $x>0,\alpha>0, 0<\beta<1$ where α *and* β are the shape and scale parameters respectively.

Some extensions of the Exponential distribution and transmuted distributions include the odd generalized exponential-inverse lomax by Falgore *et al.* (2018), the Lomax-exponential by Ieren and Kuhe (2018), the transmuted odd generalized exponential-exponential by Abdullahi et al. (2018), generalized transmuted-inverted exponential by Usman et al. (2019), the odd generalized exponential-exponential by Maiti and Pramanik (2015), exponential new weighted Weibull by Umar et al. (2018), transmuted generalized linear exponential by Elbatel *et al.* (2013), transmuted exponential Lomax by Abdullahi and Ieren (2018), Odd Generalized Exponential-New Weighted Exponential by Abdullahi and Abba (2017) and transmuted Rayleigh by Merovci (2013). Khan *et al*. (2017) presented a three-parameter transmuted generalized exponential distribution and test the potentiality of the distribution using survival data. The transmuted linear exponential distribution was introduced by Tian *et al.* (2014) which is a generalization of linear exponential distribution.

The *cdf* and *pdf* of the Transmuted New weighted exponential Distribution (TNWED) are obtained using the steps developed by Shaw and Buckley (2007). A random variable *X* is said to have a transmuted distribution function if its *pdf* and *cdf* are respectively given by:

$$f(x)=g(x)[1+\lambda-2\lambda G(x)] \quad (3)$$

and

$$F(x)=(1+\lambda)G(x)-\lambda[G(x)]^2 \quad (4)$$

where; $x>0$, and $-1\leq\lambda\leq 1$ is the transmuted parameter, *G(x)* is the *cdf* of any continuous distribution while *f(x)* and *g(x)* are the associated *pdf* of *F(x)* and *G(x)*, respectively.

Using equations (1) and (2) in (3) and (4) and simplifying, we obtain the *cdf* and *pdf* of the transmuted New Weighted Exponential distribution as follows (equations 3 and 4 being the link function for the transmutation parameter):

$$F(x)=1+e^{-(\alpha-\beta)x}\left[\lambda-1-\lambda e^{-(\alpha-\beta)x}\right] \quad (5)$$

and

$$f(x)=(\alpha-\beta)e^{-(\alpha-\beta)x}\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right] \quad (6)$$

respectively. Where, $x>0,\alpha>0, 0<\beta<1, -1\leq\lambda\leq 1$, α *and* β are the shape and scale parameters respectively and while λ is called the transmuted parameter.

The *pdf* and *cdf* of the *TNWED* using some parameter values are displayed in figures 1 and 2 as follows.

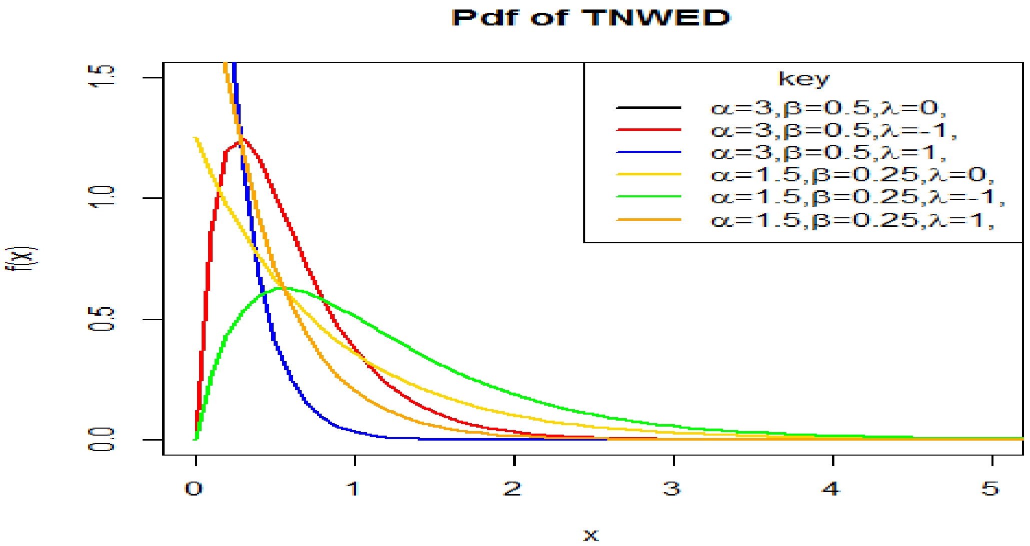

Figure 1: The graph of *pdf* of the *TNWED* at some chosen parameter values of α, β and λ respectively.

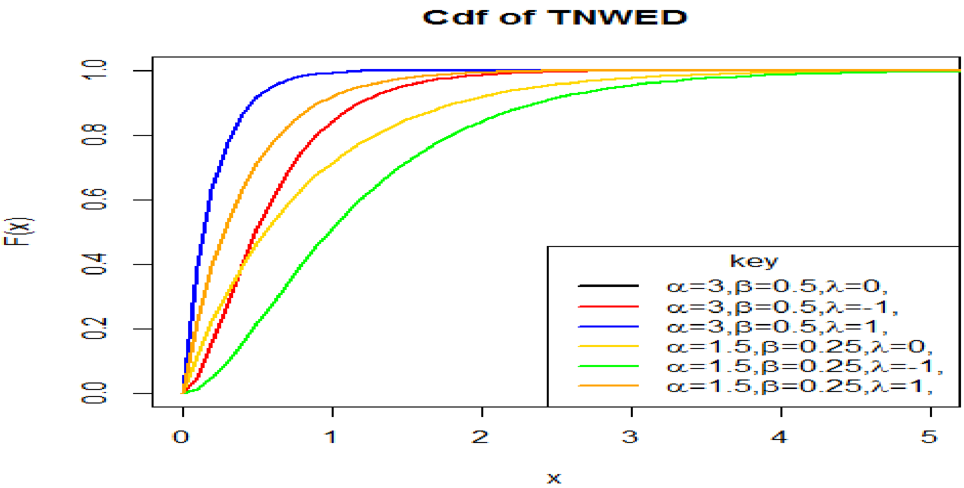

Figure 2: The graph of *cdf* of the *TNWED* at some chosen parameter values, α, β and λ respectively.

The plot for the *pdf* reveals that the *TNWED* is positively skewed while that of the *cdf* show that it is also a valid distribution since limits of 0 and 1 are proven as *X* approaches zero and positive infinity respectively.

Properties

In section 2, we define and go through some of the characteristics of the TNWED distribution.

Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_{-\infty}^{\infty} x^n f(x)\,dx = \int_{0}^{\infty} x^n f(x)\,dx \qquad (7)$$

the n^{th} moment of X is obtained using integration by substitution and is given as:

$$\mu_n^{'} = \frac{(1-\lambda)\Gamma(n+1)}{(\alpha-\beta)^n} + \frac{\lambda\Gamma(n+1)}{(2\alpha-2\beta)^n} \qquad (8)$$

The Mean

The mean of the *TNWED* can be obtained from the n^{th} moment of the distribution when $n=1$ as follows:

$$\mu_1^{'} = \frac{2-\lambda}{(2\alpha-2\beta)} \qquad (9)$$

Also the 2nd moment of the *TNWED* is obtained from the n^{th} moment of the distribution when $n=2$ as

$$\mu_2^{'} = \frac{8-6\lambda}{(2\alpha-2\beta)^2} \qquad (10)$$

The Variance

The n^{th} central moment or moment about the mean of X, say μ_n, can be obtained as

$$\mu_n = E\left(X-\mu_1^{'}\right)^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \mu_1^{'i}\mu_{n-i}^{'} \qquad (11)$$

The variance of X for *TNWED* is obtained from the central moment when $n=2$, that is,

$$Var(X) = E(X^2) - \{E(X)\}^2 \qquad (12)$$

$$Var(X) = \mu_2' - \{\mu_1'\}^2 \qquad (13)$$

$$Var = \frac{8-6\lambda}{(2\alpha-2\beta)^2} - \left(\frac{2-\lambda}{(2\alpha-2\beta)}\right)^2$$

Where μ_1' and μ_2' are the mean and second moment of the *TNWED.*

Moment Generating Function

The *mgf* of a random variable *X* can be defined as,

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)dx \qquad (14)$$

$$M_x(t) = E\left(e^{tx}\right) = \sum_{n=0}^{\infty} \frac{t^p}{p!}\mu_p' = \sum_{p=0}^{\infty} \frac{t^p}{p!}\left\{\frac{(1-\lambda)\Gamma(p+1)}{(\alpha-\beta)^p} + \frac{\lambda\Gamma(p+1)}{(2\alpha-2\beta)^p}\right\} \qquad (15)$$

Reliability analysis of the *TNWED*.

Survival Function:. Mathematically, the survival function is given as:

$$S(x) = 1 - F(x) \qquad (16)$$

Applying the *cdf* of the *TNWED* in (5), the survival function for the *TNWED* is obtained as:

$$S(x) = e^{-(\alpha-\beta)x}\left[1 - \lambda + \lambda e^{-(\alpha-\beta)x}\right] \qquad (17)$$

The following is a plot for the survival function of the *TNWED* using different parameter values α, β and λ respectively as shown in Figure **3** below;

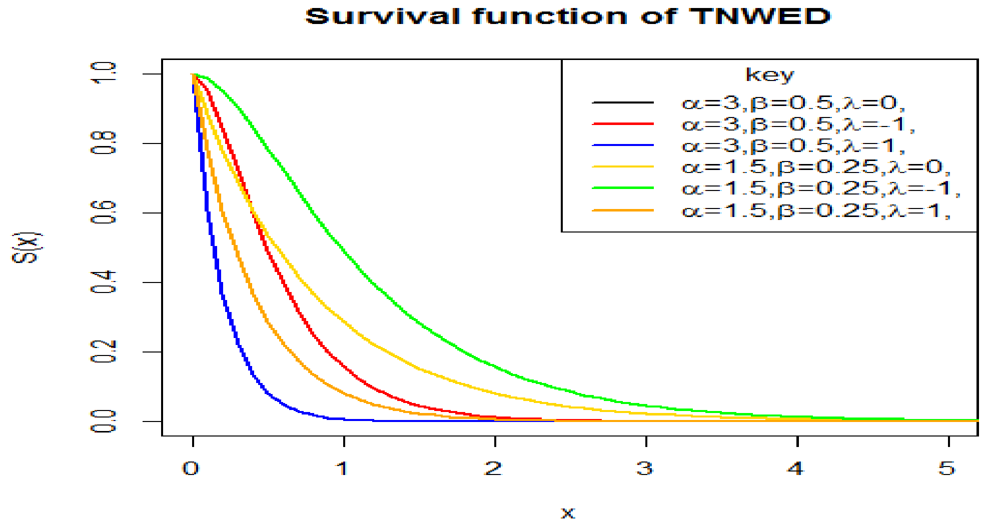

Figure 3: Survival function of the *TNWED* at some chosen parameter values, α, β and λ respectively.

The graph in Figure **3** shows that the probability of the survival is higher at initial time or early age and it decreases as time increases which implies that the distribution will be useful for modeling real life situations.

Hazard Function: The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)} \qquad (18)$$

Meanwhile, the expression for the hazard rate of the *TNWED* is given by

$$h(x) = \frac{(\alpha-\beta)\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right]}{\left[\lambda-1-\lambda e^{-(\alpha-\beta)x}\right]} \qquad (19)$$

where $\alpha > 0, 0 < \beta < 1$ and $-1 \leq \lambda \leq 1$.

The following are some possible curves for the hazard rate at some values of the model parameters

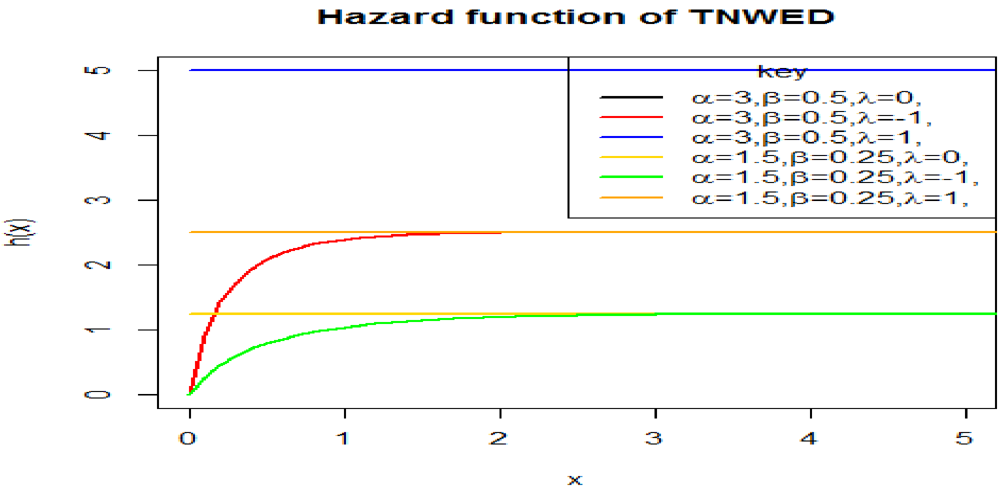

Figure 4: Hazard function of the *TNWED* at some parameter values, α, β and λ respectively.

We can see from Figure 4 that the value of the hazard function can be in different forms such as increasing and constant failure rate. Thus the *TNWED* may be appropriate for modeling time dependent events whose risk or failure rate increases and those with constant risk from the early stage to the end.

Quantile Function

Taking *F(x)* to be the *cdf* of the *TNWED* and inverting it will give us the quantile function as follows:

$$F(x) = 1 + e^{-(\alpha-\beta)x}\left[\lambda - 1 - \lambda e^{-(\alpha-\beta)x}\right] = u \qquad (20)$$

Simplifying equation (20) above, we obtain:

$$Q(u) = X_q = \left\{-\frac{ln\left(\frac{\lambda+u-2}{\lambda}\right)}{2(\alpha-\beta)}\right\} \quad (21)$$

The function derived in (21) is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question.

When u=0.5 in (21) we obtain the median as (22)

$$Q(0.5) = X_q = \left\{-\frac{ln\left(\lambda+\frac{1}{2}-\frac{2}{\lambda}\right)}{2(\alpha-\beta)}\right\} \quad (22)$$

Order Statistics

Let $X_{(1)}$ denote the smallest of $X_1, X_2, \ldots, X_n$, $X_{(2)}$ denote the second smallest of $X_1, X_2, \ldots, X_n$, and similarly $X_{(i)}$ denote the i^{th} smallest of $X_1, X_2, \ldots, X_n$.Then the random variables $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, called the order statistics of the sample $X_1, X_2, \ldots, X_n$, thus the pdf of the i^{th} order statistic, $X_{(i)}$, is given by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}[1-F(x)]^{n-i} \quad (23)$$

Where $f(x)$ and $F(x)$ are the *pdf* and *cdf* of the *TNWED* respectively.

Using (5) and (6), the *pdf* of the i^{th} order statistics$X_{i:n}$, can be expressed from (23) as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\left\{(\alpha-\beta)e^{-(\alpha-\beta)x}\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right]\left[1+e^{-(\alpha-\beta)x}(\lambda-1-\lambda e^{-(\alpha-\beta)x})\right]^{i-1}\left[e^{-(\alpha-\beta)x}(1-\lambda+\lambda e^{-(\alpha-\beta)x})\right]^{n-i}\right\} \quad (24)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *TNWED* are given by;

$$f_{1:n}(x) = n\left\{(\alpha-\beta)e^{-(\alpha-\beta)x}\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right]\left[e^{-(\alpha-\beta)x}(1-\lambda+\lambda e^{-(\alpha-\beta)x})\right]^{n-1}\right\} \quad (25)$$

$$f_{n:n}(x) = n\left\{(\alpha-\beta)e^{-(\alpha-\beta)x}\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right]\left[1+e^{-(\alpha-\beta)x}(\lambda-1-\lambda e^{-(\alpha-\beta)x})\right]^{n-1}\right\} \quad (26)$$

respectively.

Estimation of Parameters of TNWED using Maximum Likelihood Method

Let $X_{1, \ldots,}X_n$ be a sample of size 'n' independently and identically distributed random variables from the *TNWED* with unknown parameters α, β, λ defined previously. The *pdf* of the *TNWED* is given as

$$f(x) = (\alpha-\beta)e^{-(\alpha-\beta)x}\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x}\right]$$

The likelihood function is given by;

$$L(X_1, X_2, \ldots, X_n/\alpha, \beta, \theta, \lambda) = (\alpha-\beta)^n e^{-(\alpha-\beta)\sum_{i=1}^n x_i} \sum_{i=1}^n \left[1-\lambda+2\lambda e^{-(\alpha-\beta)x_i}\right] \quad (27)$$

Equation (27) can be rewritten as

$$L(X_1, X_2, \ldots, X_n/\alpha, \beta, \theta, \lambda) = (\alpha-\beta)^n\, exp\left(-(\alpha-\beta)\sum_{i=1}^n x_i\right)\sum_{i=1}^n [1-\lambda+2\lambda\, exp(-(\alpha-\beta)x_i)]$$

Let the log-likelihood function,$l = log\, L\,(X_1, X_2, \ldots, X_n/\alpha, \beta, \lambda)$ therefore

$$l = n\, log(\alpha-\beta) - (\alpha-\beta)\sum_{i=1}^n x_i + \sum_{i=1}^n log\left[1-\lambda+2\lambda e^{-(\alpha-\beta)x_i}\right] \quad (28)$$

Differentiating l partially with respect to α, β, λ respectively gives;

$$\frac{\partial l}{\alpha} = \frac{n}{(\alpha-\beta)} - \sum_{i=1}^n x_i - \sum_{i=1}^n log\left[\frac{2x_i\lambda e^{-(\alpha-\beta)x_i}}{(1-\lambda+2\lambda e^{-(\alpha-\beta)x_i})}\right] \quad (29)$$

$$\frac{\partial l}{\beta} = -\frac{n}{(\alpha-\beta)} + \sum_{i=1}^n x_i + \sum_{i=1}^n log\left[\frac{2x_i\lambda e^{-(\alpha-\beta)x_i}}{(1-\lambda+2\lambda e^{-(\alpha-\beta)x_i})}\right] \quad (30)$$

$$\frac{\partial l}{\lambda} = \sum_{i=1}^n \left[\frac{2\lambda e^{-(\alpha-\beta)x_i-1}}{(1-\lambda+2\lambda e^{-(\alpha-\beta)x_i})}\right] \quad (31)$$

Equating equations (29), (30) and (31) to zero, we obtain the most likely parameter estimations by solving for the solution of the nonlinear system of equations. When data sets are provided, the solution can only be determined numerically with the use of appropriate statistical tools, such as Python, R, SAS, etc.

Application to a real life datasets

Two datasets, a summary of descriptive statistics, and applications to a few chosen extensions of the exponential distribution are presented in this section. We have contrasted the TNWED's performance with that of the New Weighted Exponential distribution and the conventional Exponential distribution (ED). We have evaluated this performance of the above listed models using the *AIC* (Akaike Information Criterion). It is considered that the model with the smallest value of AIC will be chosen as the best model to fit the data.

Data set I: The first data set (Chen et al., 2010) corresponds to fifty-two ordered annual maximum antecedent rainfall measurements in mm from Maple Ridge in British Columbia, Canada. The data is summarized as follows:

Table 1: Summary Statistics for Dataset I

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	52	264.9	528.1	596.8	672.0	595	895.1	16158	-0.1895	0.3454

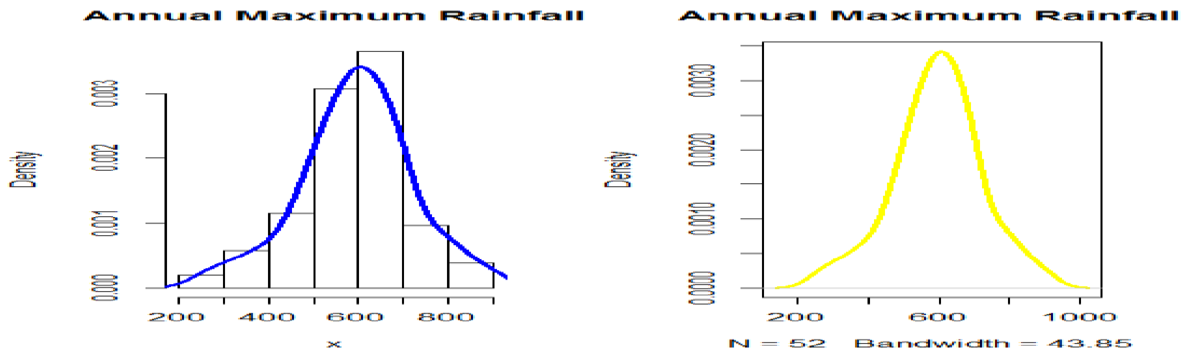

Figure 5: A Histogram and density plot for Dataset I

From the descriptive statistics in Table 1 as well as the histogram and density plot shown above in Figure 5 we observed that the data set is approximately normal and therefore not suitable for skewed distributions such as the proposed model.

Table 2: The performance of the selected models using the *AIC* value of the models evaluated at the *MLEs* based on dataset I.

Distributions	Parameter estimates (standard errors)	-ll=(-log-likelihood value)	*AIC*	Ranks
TNWED	α =0.3707 (0.4714) β =0.3682 (0.4710) λ = -0.7930 (0.5419)	-370.9731	747.9462	**1**
NWED	α =0.06685 (0.27784) β =0.06516 (0.27751)	-384.2055	772.4109	**3**
ED	λ =0.00168 (0.00023)	-384.2053	770.4106	**2**

From Table 2 and comparing the AIC value for each model reveals that the TNWED performs better than the NWED and ED. These is due to the decision rule which says that the distribution or model with smaller value of the test statistics (AIC) will be taken as the most adequate or efficient model and comparing the value of the AIC for each model show that the TNWED has better performance compared to the NWED and ED.

The following figure displayed the histogram and estimated densities and cdfs of the fitted models for dataset I.

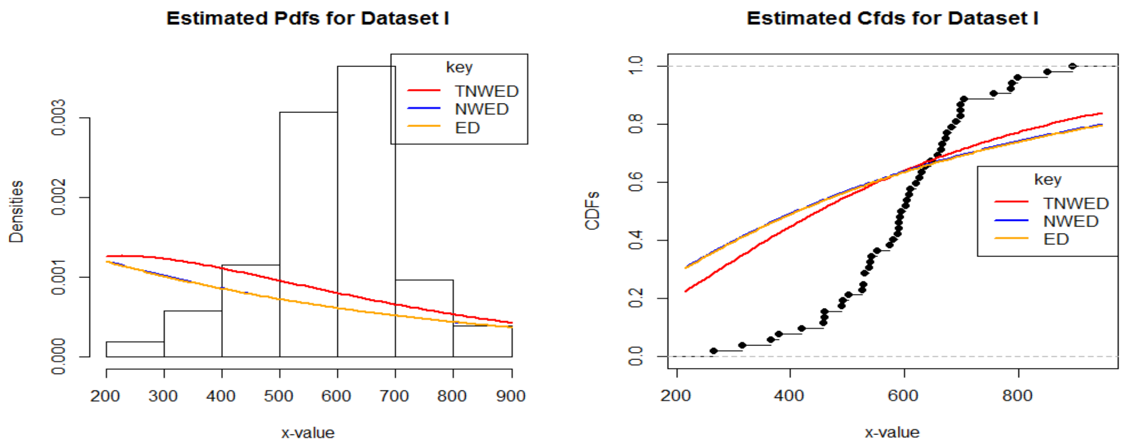

Figure 6: Histogram and plots of the estimated densities (pdfs) and cdf's of the TNWED, NWED and ED fitted to dataset I.

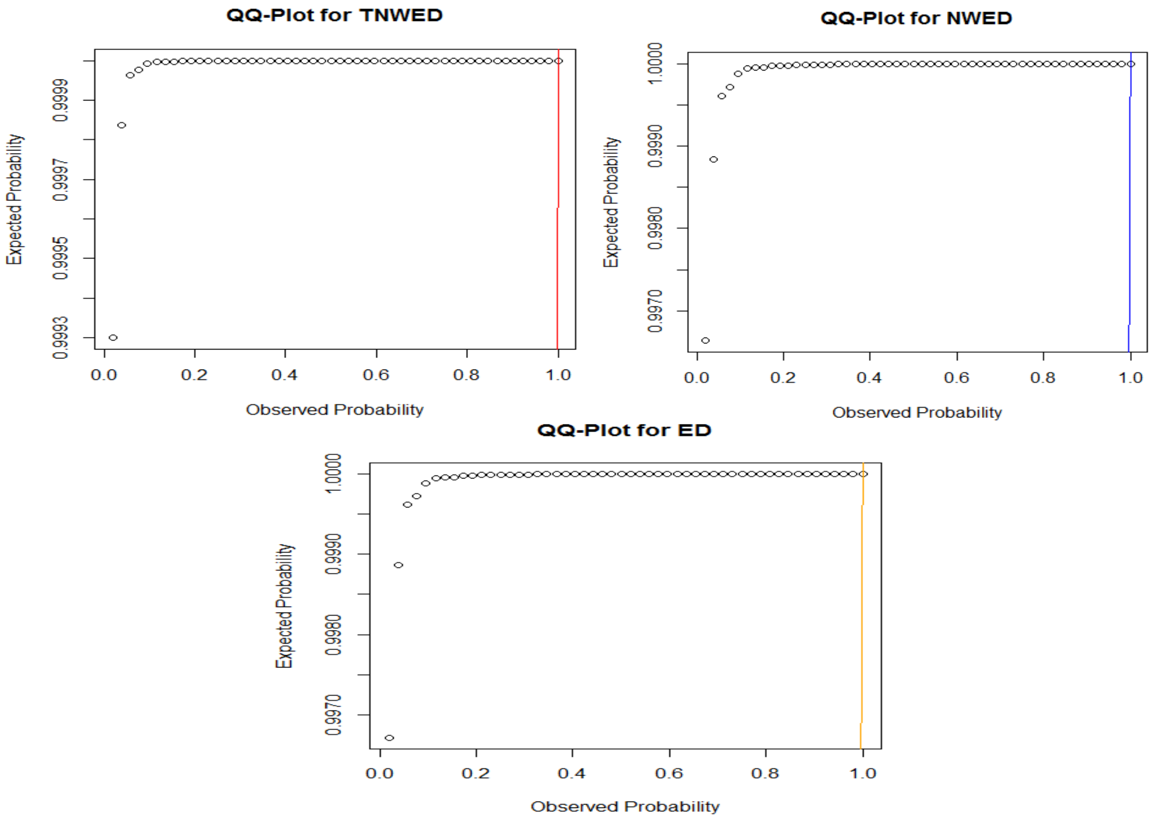

Figure 7: Probability plots for the fit of the TNWED, NWED and ED based on dataset I.

It is obvious from the probability density plots in Figure 6 that the TNWED, despite having a better fit to dataset I than NWED and ED, is a poor model for the dataset. The reason is that the first dataset (dataset I) is normally distributed while the three fitted distributions are positively skewed and cannot claim to fit the dataset. This instance also identifies the limitation associated with using these information criteria (AIC, BIC, CAIC, HQIC) since they may not always provide adequate information for decision making in some research works. Moreover, the Q-Q plots also confirm our reasons that the three considered distributions are not in any way good for the dataset because they are skewed to the right while the dataset is approximately normal.

Data set II: This second data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929-1938 from Lee (1992), Ramos *et al*. (2013) and Oguntunde *et al.* (2017). The observations are as follows: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0,47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0. Its summary is given as follows:

Table 3: Summary Statistics for the dataset II

parameter	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	121	0.30	17.5	40.00	60.0	46.33	154.00	1244.464	1.04318	0.40214

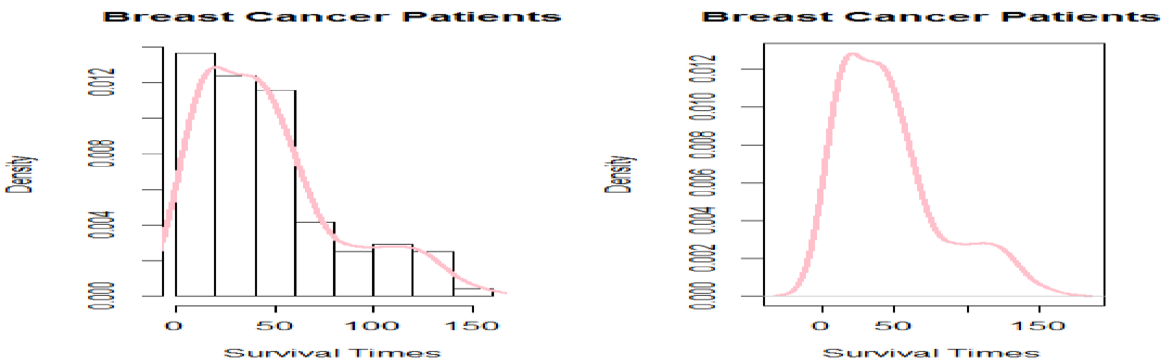

Figure 8: A Histogram and density plot for Data set II

From the descriptive statistics in Table 3 and the histogram as well as the density plot shown in Figures 8 above, it was clearly demonstrated that dataset II was observed to be positively skewed, which means is not normally distributed and therefore suitable for distributions that are skewed to the right and have variety of shapes just like the TNWED.

Table 4: The performance of the selected models using the *AIC* value of the models evaluated at the *MLEs* based on dataset II

Distributions	Parameter estimates (standard errors)	-ll=(-log-likelihood value)	*AIC*	Ranks
TNWED	α =0.4285 (1.3282) β =0.3989 (1.3280) λ =-0.7806 (0.1404)	-578.9889	1163.978	**1**
NWED	α =0.4031 (NA) β =0.3816 (NA)	-585.1278	1174.256	**3**
ED	λ =0.02159 (0.00196)	-585.1277	1172.255	**2**

Now, comparing the value of the *AIC* for all the distributions, it is very clear that the *TNWED* which is heavily skewed to the right due to our extension by means of transmutation is favorable compares to the original NWED and the traditional exponential distribution (ED) in terms of performance. The decision rule, which states that the distribution or model with a smaller value of the test statistics (AIC) should be considered as the most suited or efficient model, is again the reason for this result. This conclusion also acknowledges that generalizing or extending any continuous distribution always results in a compound distribution that fits the data better than the classical distribution.

To further prove the above decision, the following figure displayed the histogram and estimated densities and cdf's of the fitted models for dataset II under study.

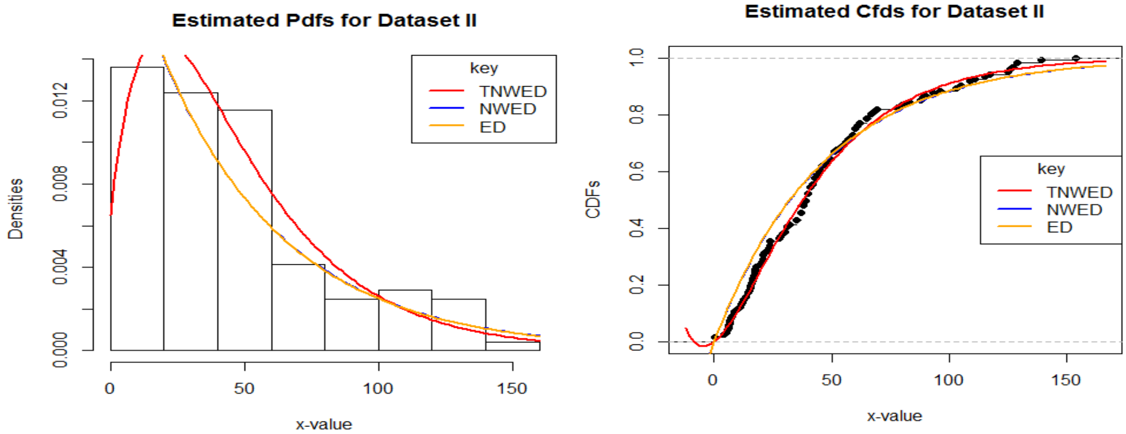

Figure 9: Histogram and plots of the estimated densities (pdfs) and cdfs of the TNWED, NWED and ED fitted to dataset II.

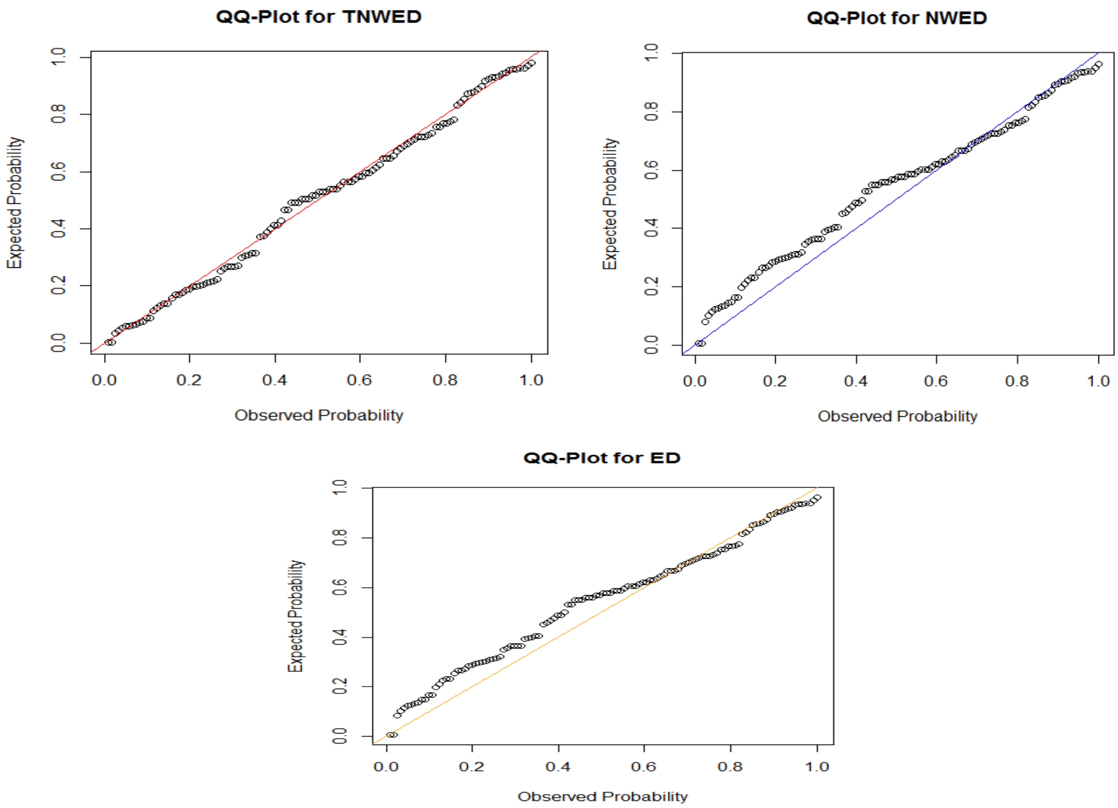

Figure 10: Probability plots for the fit of the TNWED, NWED and ED based on dataset II

From the estimated density plots and *cdf's* in Figure 9 it can be clearly seen that the TNWED fits the second dataset (dataset II) better than the other two distributions (NWED and ED) and is therefore regarded as the best for the data. Furthermore, a graphical comparison of the PP-Plots corresponding to these fits confirms our decision that TNWED is better as demonstrated in Figure 10. The plots in Figure 9 and Figure 10 above reveal that there is no much difference between the performance of the NWED and that of the ED unlike what is found with the TNWED.

CONCLUSION

This article developed a new distribution named a Transmuted New Weighted Exponential distribution (*TNWED)*. The distributional characteristics of the developed TNWED have been thoroughly researched and some have been deduced. It has also been done to derive certain expressions for its moments, moment generating function, survival function, hazard function, and ordered statistics. Depending on the parameter settings, several distribution plots showed that the distribution's shape is skewed to the right or left. Maximum likelihood estimation methodology has been used to estimate the model parameters. The implications of the plots for the survival function suggest that age-dependent events or variables, whose survival declines with passage of time or where survival rate declines with age, could be modelled using the transmuted new weighted exponential distribution. The performance of the TNWED has been evaluated using two real life datasets from previous researches and the results show that our proposed distribution fits the datasets more better compared to the fits of the other two distributions (New Weighted Exponential distribution and Exponential distribution) considered in this study.

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