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ON THE SEMIGROUP OF DIFUNCTIONAL BINARY RELATIONS

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ABSTRACT

In this paper, we have examine some properties of elements of the semigroup $(\mathcal{D}_X, \diamond)$, where D_X, is the set of all binary relations $\alpha \subseteq X \times X$ satisfying $(x,u),(x,v),(y,u) \in \alpha \Longrightarrow (y,v) \in \alpha$, $(\forall x,y,u,v \in X)$, and $\diamond$ is a binary operation on D_X defined by $(\forall \alpha,\beta \in \mathcal{D}_X)\ (x,y) \in \alpha \diamond \beta \Leftrightarrow x\alpha = y\beta^{-1} \neq \emptyset$, with $x\alpha$ denoting set of images of x under α, and $y\beta^{-1}$ denoting set of pre-images of y under β. In particular, we showed that in the semigroup $(\mathcal{D}_X, \diamond)$ there is no distinction between the concepts of reflexive and symmetric relations. We also presented a characterization of idempotent elements in $(\mathcal{D}_X, \diamond)$ in term of equivalence relations.
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INTRODUCTION

For a non-empty set X, the semigroups $\mathcal{P}_X$, of all partial transformations of X, $\mathcal{T}_X$, of all full transformations of X, $\mathcal{I}_X$, of all partial one-to-one transformations of X, and the group $\mathcal{S}_X$, of all bijections of X, are subsemigroups, under composition, of the larger semigroup $\mathcal{B}_X$ of all binary relations on the set X. It is therefore natural to try to extend concepts from any of these classes of semigroups to the semigroup $\mathcal{B}_X$ or an interesting subsemigroup of $\mathcal{B}_X$. Vernitski, (2007), in trying to find a natural example of inverse semigroup, defined an alternative binary operation with respect to elements of $\mathcal{B}_X$, and identified an interesting subset $\mathcal{D}_X$ of $\mathcal{B}_X$ whose elements are called difunctional relations. This subset is an inverse semigroup, under, which contains the symmetric inverse semigroup $\mathcal{I}_X$ as a subsemigroup. Thus, the inverse semigroup $(\mathcal{D}_X, \diamond)$ too has the universal property of containing isomorphic copies of all inverse semigroups. Difunctional binary relations, which were introduced in (Riguet, 1984), appear in different areas of Discrete Mathematics, for example in (Jaoua et. at., 2004) difunctional relations were used in databases. In this paper, we examine some combinatorial properties of the inverse semigroup $(\mathcal{D}_X, \diamond)$ of all difunctional binary relations.

Preliminaries

Let X be a non-empty set and define an operation $\diamond$ on $\mathcal{B}_X$, the set of all binary relations on X, by the following: For any $\alpha, \beta \in \mathcal{B}_X$,

$$(x,y) \in \alpha \diamond \beta \quad \text{if and only if} \quad x\alpha = y\beta^{-1} \neq \emptyset.$$

Then clearly, this is a binary operation on $\mathcal{B}_X$ and so, $(\mathcal{B}_X, \diamond)$ is a groupoid. This was first defined by Vernitski, (2007). The operation is not associative in $\mathcal{B}_X$.

Though the operation is not associative in $\mathcal{B}_X$, it still possesses some interesting properties which necessitate search for a subset of $\mathcal{B}_X$ in which is associative. First we recall the definition of difunctional relations.

Definition 2.1 Let X be a non-empty set. A binary relation $\alpha \in \mathcal{B}_X$ is called a *difunctional* if for all $x,y,u,v \in X$,
$(x,u),(x,v),(y,u) \in \alpha \Rightarrow (y,v) \in \alpha.$

$$\begin{tikzcd} & u & \\ x \arrow[ur] \arrow[dr] & & y \arrow[ul] \arrow[dl, dashed] \\ & v & \end{tikzcd}$$

For example the relation $\alpha = \{(1,1),(1,3),(2,1),(2,3),(4,2),(5,2)\}$ can easily be checked as being a difunctional relation on $X = \{1,2,3,4,5,6\}$.
The first connection between the operation and difunctional relations is the following observation.

Lemma 2.1 (Vernitski, 2007): For any $\alpha, \beta \in \mathcal{B}_X$, the product $\alpha \diamond \beta$ is a difunctional relation.
Proof. Let $\alpha, \beta \in \mathcal{B}_X$ and suppose $x, y, , u, v \in X$ are such that

$$(x,u),(x,v),(y,u) \in \alpha \diamond \beta$$

then, by definition of the operation $\diamond$, it follows that $y\alpha = u\beta^{-1} = x\alpha = v\beta^{-1} \neq \emptyset$ and hence $(y,v) \in \alpha \diamond \beta$, showing that $\alpha \diamond \beta$ is a difunctional relation.

It immediately follows from this lemma that the set D_X, of all difunctional binary relations on X, is closed under the operation , that is $(\mathcal{D}_X, \diamond)$ is a subgroupoid of $(\mathcal{B}_X, \diamond)$.
It is already known that $\mathcal{D}_X$ is not closed under the usual composition of binary relations. This is easily checked using simple relations, say $\alpha = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$, $\beta = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$ in $\mathcal{D}_X$, and noting that the composition

$$\alpha \circ \beta = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)\}$$

is not in $\mathcal{D}_X$ since $(2,2),(2,1),(3,2) \in \alpha \circ \beta$ but $(3,1) \notin \alpha \circ \beta$. Now, since $\mathcal{D}_X$ is closed under , the next natural question to ask is whether is associative with respect to elements of $\mathcal{D}_X$, even though it is not generally associative with respect to elements of B_X. This is the case as has been shown by Vernitski (2007). We record the following observations about elements of $\mathcal{D}_X$.

Lemma 2.2 (Vernitski, 2007): Let X be a non-empty set, $x,y \in X$ and $\alpha \in \mathcal{D}_X$. Then
(i) $x\alpha \cap y\alpha \neq \emptyset \Rightarrow x\alpha = y\alpha$.
(ii) $x\alpha^{-1} \cap y\alpha^{-1} \neq \emptyset \Rightarrow x\alpha^{-1} = y\alpha^{-1}$.

Lemma 2.3 (Vernitski, 2007): Let X be a non-empty set, $x, y \in X$ and $\alpha \in \mathcal{D}_X$. We have $(x,y) \in \alpha$ if and only if the following three conditions are satisfied:
(i) $x\alpha \neq \emptyset$;
(ii) $y\alpha^{-1} \neq \emptyset$;
(iii) $x\alpha \times y\alpha^{-1} \subseteq \alpha \subseteq x\alpha \times y\alpha^{-1} \cup X \setminus x\alpha \times X \setminus y\alpha^{-1}$.

The combine effect of these lemmas is that a binary relation $\alpha \in B_X$ is difunctional if and only if there exist partitions $\{A_1,A_2,...\}$ and $\{B_1,B_2,...\}$ of dom(α) and im(α) respectively such that α is of the form
$\alpha = A_1 \times B_1 \cup A_2 \times B_2 \cup \cdots$.
Thus a difunctional relation $\alpha \in D_X$ can be written as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots \\ B_1 & B_2 & \cdots \end{pmatrix}.$$

Now, we record the lemma which proves the associatively of $\diamond$ in $\mathcal{D}_X$.

Lemma 2.4 (Vernitski, 2007): For any $\alpha,\beta,\gamma \in \mathcal{D}_X$, $(\alpha \diamond \beta) \diamond \gamma = \alpha \diamond (\beta \diamond \gamma)$.
This together with Lemma 2.1 proves the next theorem.

Theorem 2.5 (Vernitski, 2007): $(\mathcal{D}_X, \diamond)$ is a semigroup.

The semigroup $(\mathcal{D}_X, \diamond)$ will be referred to as *semigroup of difunctional binary relations*. This semigroup enjoys the following properties.

Lemma 2.6 (Vernitski, 2007): Let $\alpha \in D_X$. Then
(i) $\alpha^{-1} \in \mathcal{D}_X$ and $\alpha \diamond \alpha^{-1} \diamond \alpha = \alpha$, that is ($\mathcal{D}_X$,$\diamond$) is a regular semigroup,
(ii) If $\alpha \diamond \alpha = \alpha$ then $\alpha^{-1} = \alpha$,
(iii) If $\delta, \xi \in \mathcal{D}_X$ are idempotents, then $\delta \diamond \xi \subseteq \xi \diamond \delta$, and hence $\delta \diamond \xi = \xi \diamond \delta$.

The first and last parts of this lemma immediately imply the following conclusion.

Theorem 2.7 (Vernitski, 2007): The semigroup $(\mathcal{D}_X, \diamond)$ is an inverse semigroup.

RESULTS ON DIFUNCTIONAL BINARY RELATIONS

As in (Vernitski, 2007), for difunctional binary relation $\alpha = \cup_{i=1}^{r} A_i \times B_i = \begin{pmatrix} A_1 \cdots A_r \\ B_1 \cdots B_r \end{pmatrix}$ in $\mathcal{D}_n$, we write rank(α) = r, dom(α) = $A_1 \cup\cdots\cup A_r$, codom(α) = $B_1 \cup\cdots\cup B_r$, ker(α) = $\{A_1,...,A_r\}$, coker(α) = $\{B_1,...,B_r\}$, def(α) = $|X \setminus \text{dom}(\alpha)|$, codef($\alpha$) = $|X \setminus \text{codom}(\alpha)|$, and call these parameters the *rank*, *domain*, *codomain*, *kernel*, *cokernel*, *defect* and *codefect* of α respectively.

Recall that, for non-negative integers n and k, the *Stirling number of the second kind S*(n,k) is the number of partitions of a set of size n into k (non-empty) subsets. The Bell number $B(n)$ is the total number of partitioned of a set of size n into any number of (non-empty) subsets, that is

$$B(n) = \sum_{k=1}^{n} S(n,k).$$

Note that the Stirling number of the second kind satisfy the following recursion and boundary conditions:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \quad (1)$$

and $S(n,1) = S(n,n) = 1$. Also, $S(n,k) = 0$ for all $k > n$.

In the next result, we show that the concepts of reflexive and symmetric relations are the same on $\mathcal{D}_X$.

Theorem 3.1: Let X be non-empty set and $\alpha \in \mathcal{D}_X$. Then, the following are equivalent:
(i) α is reflexive on $dom(\alpha)$;
(ii) α is symmetric on $dom(\alpha)$.

Proof. Suppose $x,y \in dom(\alpha)$ and that α is reflexive. Then $(x,x),(y,y) \in \alpha$. If $(x,y) \in \alpha$, then, by definition of difunctional relation, $(y,x) \in \alpha$. Thus, α is symmetric.
Conversely, if α is symmetric, then $\alpha = \alpha^{-1}$. This implies $x\alpha = x\alpha^{-1}$ for all $x \in dom(\alpha)$, and since $\alpha \in \mathcal{D}_X$, $x\alpha^{-1} \times x\alpha \subseteq \alpha$. Thus, by Lemma 2.3, $(x,x) \in \alpha$ showing that α is reflexive.
It follows from Lemma 2.6 and Proposition 3.1 that the only reflexive (symmetric) relations in $\mathcal{D}_X$ are the idempotent relations in $(\mathcal{D}_X, \diamond)$. These are elements $\xi \in \mathcal{D}_X$ that can be written in the form

$$\xi = \cup_{i=1}^{\infty}(A_i \times A_i) = \begin{pmatrix} A_1 & A_2 & \cdots \\ A_1 & A_2 & \cdots \end{pmatrix}$$

for some partition $\{A_1, A_2, \dots\}$ of $dom(\xi)$. The following proposition is then immediate.

Theorem 3.2: An element of $(\mathcal{D}_X, \diamond)$ is an idempotent if and only if it is an equivalence relation on its domain.

Proof. Suppose $\xi = \cup_{i=1}^{\infty}(A_i \times A_i)$ is an idempotent in $(\mathcal{D}_X, \diamond)$. Then, by Proposition 3.1, ξ is both reflexive and symmetric on its domain. Now if $(x,y),(y,z) \in \xi$, then $x,y,z \in A_i$ for some i. But then $(x,z) \in \xi$, so that ξ is transitive. Thus, ξ is an equivalence relation.
Conversely, if $\xi \in \mathcal{D}_X$ is an equivalence relation on its domain, then ξ is both reflexive and symmetric. Thus, by our observation preceding this proposition, ξ must be an idempotent.
We already know that the set $E(\mathcal{D}_X)$ of idempotents in $(\mathcal{D}_X, \diamond)$ is a semilattice (that is a commutative semigroup of idempotents). This is not the case when we are dealing with transitive relations in $\mathcal{D}_X$. The product of two transitive relations in $(\mathcal{D}_X, \diamond)$ is not necessarily a transitive relation.

For instance the relations

$$\alpha = \begin{pmatrix} \{1,2\} & \{4,5\} \\ \{1,3\} & \{4,6\} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \{1,3\} & \{4,6\} \\ \{1,3,5\} & \{2,6\} \end{pmatrix}$$

Are transitive in D_X with $X = \{1,2,3,4,5,6\}$, but their product

$$\alpha\beta = \begin{pmatrix} \{1,2\} & \{4,5\} \\ \{1,3,5\} & \{2,6\} \end{pmatrix}$$ is not.

Theorem 3.3: A relation $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots \\ B_1 & B_2 & \cdots \end{pmatrix} \in \mathcal{D}_X$ is transitive if and only if, for all $i \neq j$ in X, $A_i \cap B_j = \emptyset$.

Proof. Let $\alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ B_1 & \cdots & B_r \end{pmatrix}$ be a transitive elation in D_X and suppose contrary that $A_i \cap B_j \neq \emptyset$ for some $i \neq j$. Let $x \in A_i \cap B_j$ and $x, y, z \in X$ be such that $(x,y),(y,z) \in \alpha$. Then $y, z \in B_i$, $y \in A_i$ and, by transitivity, $(x,z) \in \alpha$. Now since $x \in B_j$ there must exists $w \in A_j$ such that $(w,x) \in \alpha$ and so, by transitivity, $(w,y) \in \alpha$. Thus, $y \in B_j$ so that $y \in B_i \subseteq B_j$. But this is a contradiction, since $i \neq j$ and $\{B_1, \ldots, B_r\}$ is a partition of $im(\alpha)$.

Conversely, suppose $A_i \cap B_j = \emptyset$ for all $i \neq j$ in X and let $(x,y),(y,z) \in \alpha$. Then, $y \in B_i$ and $z \in B_j$ for some i and j. Thus, $y \in A_i$ so that $y \in A_i \cap B_j$, form which it follows, by hypothesis, that $i = j$. Thus $z \in Bj = Bi$ implies $(x,z) \in \alpha$, showing that α is transitive.

CONCLUSION

This paper studied the semigroup of a special class of binary relations on a non-empty set. This semigroup is called the semigroup of difunctional binary operations. The structure of its elements is examined and a characterization of idempotents elements in the semigroup is given in term of equivalence relations.

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