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DERIVATION OF THE DYNAMICAL EQUATIONS OF MOTION OF THE R3BP WITH VARIABLE MASSES AND DISK

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ABSTRACT
This paper explores the dynamical equations of the restricted three-body problem with variable masses of the primaries which are enclosed by a disk, when the masses of the primary and the disk vary with time in accordance with the unified Mestschersky law and motion of the primaries is determined by the Gylden-Mestschersky equation. It is seen that the equations of motion differ from those of the restricted three-body problem with variable masses due to the disk mass.

Keywords: R3BP, Equations of Motion, Variable Masses; Disk

INTRODUCTION
The restricted three-body problem (R3BP) describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. For no general solution in the CR3BP is available, particular solutions are sought to obtain insight into the problem, these particular solutions referred to as the equilibrium points (EP) are five for the classical R3BP; two triangular and three collinear. Euler (1767) determined a set of three collinear EPs and Lagrange (1772) found the triangular EPs. The R3BP has been treated from many angles by various authors and researchers, some of who have considered the effect of some parameters relating to the shape and nature of orbits of the primaries and of the third body. Others have included perturbing forces of Coriolis, centrifugal, radiation pressure and the effect of the gravitational potential from the disk; while some have looked at the formulation whereby the motion of the third body is confined inside one of the primaries. The investigations of the R3BP with mass variations of one, two or all the masses and studies of the periodic orbits have also received attention.
The classical R3BP assumes that the masses of celestial bodies are constant. However, the phenomenon of isotropic radiation or absorption in stars led scientists to formulate the restricted problem of three bodies with variable mass. During evolution, the masses of celestial bodies change, especially in a double star system were masses change rather intensively. As an example, we could mention the motion of rockets, black holes' formation, motion of a satellite around a radiating star surrounded by a cloud and varying its mass due to particles of the cloud, and comets loosing part or all of their masses as a result of roaming around the Sun (or other stars) due to their interaction with the solar wind which blows off particles from their surfaces. The problem of the motion of astronomical objects with variable mass has many interesting applications in stellar, galactic, and planetary dynamics.
The problem of two bodies with variable masses came into science practically following the work of Gylden (1884), who for the relative motion of one mass point m_2 about the other mass point m_1 under the action of mutual gravitational force represented the sum of the masses of these points as varying with time by a certain law $m_1 + m_2 = \mu(t)$. Later, Dufour (1886) examined the astronomical phenomena of variable mass relating the secular variation of lunar acceleration with the increase of the Earth's mass due to the impact of meteorites. Soon afterwards, Mestschersky (1893, 1902) showed that the Gylden problem is a particular case of the problem of two bodies with variable masses under the condition that the laws of variation of the masses vary isotropically. This problem is referred to as the Gylden-Mestschersky problem (GMP).
The first investigation of the existence of the EPs for variable masses in the absence of reactive forces was performed by Orlov (1939), in which the plane problem of three bodies with finite variable masses was considered, and the existence of five analogous particular solutions was established. Sersic (1970, 1973) demonstrated the existence of particular solutions in the R3BP, in which the motion of the primary, variable-mass bodies occur along a straight line passing through the center of mass of the system. Gelf'gat (1973) examined the R3BP of variable mass in which the primary bodies move within the framework of the GMP and assumed that the isotropic mass variation of the masses of the primaries occurs in accordance with unified Mestschersky law (UML). He established the existence of three collinear and two triangular EPs analogous to the classical type $L_i\left(i=1,2...5\right)$. The equations of motion of the CR3BP with variable mass under the assumption that the mass of the infinitesimal body vary with respect to time was established by Shrivastava and Ishwar (1983), while Singh and Ishwar (1984, 1985) investigated the effect of small perturbation in the Coriolis and centrifugal forces on the location and stability of EPs in the R3BP with variable mass under the assumption that the third infinitesimal mass is variable and the primaries are spherical with constant masses.
Dyakov and Reznikov (1986) studied the motion in the vicinity of triangular EPs when the mass ratio of the components is a variable. They specified by investigating Trojan orbits in a system having components with variable mass ratio within the framework of the CR3BP, and found that when the mass variation over the period of the libration is slight, an increase in the relative mass μ of the smaller component above a certain limit (lower than the critical value

$\mu = 0.0385$ and depends only on the initial deviation from the EPs) result in the cutoff of librations. They investigated the stability and established numerically the regions of relative stability of the motion around the points for different mass ratios. Further they used the results obtained to evaluate the hypothesis that the Trojans originated as former satellites of Jupiter.

The coplanar EPs L_6 and L_7 for variable mass were first found by Bekov (1988) after he worked on the same problem of Gelf'gat (1973). These EPs are located outside the plane of revolution of the primary bodies, with isotropic mass variation of the masses of the primaries. Later, on the particular solution of same formulation as Gelf'gat (1973) was investigated by Luk'yanov (1989). He found four new coplanar solutions denoted by L_8 , L_9 , L_{10} and L_{11} , and further sought for the possibility of the existence of infinitely remote solutions $L_{\pm\infty}$. The stability of the EPs in the R3BP with variable masses of the primaries with respect to time according to the UML, with the consideration that the motion of the variable mass primary body is within the framework of the GMP, was studied by Luk'yanov (1990). He showed that the EPs (collinear, triangular and coplanar) for any parameter are stable with respect to the coordinate introduced by Mestschersky (1952). The particular solutions in the restricted collinear three-body problem with variable masses were considered by Bekov (1991). In this case, it was also assumed that the motion of the two primary bodies is in accordance with the GMP and the time variation of the masses of the primaries is determined by the UML. He found the collinear solutions $L_{1,2,3}$ and the spatial solution L_0 (a Lagrangian ring) for different time dependency of the masses and established the domain of existence of these solutions.

The EPs and Hill surfaces in the same formulation as Gelf'gat (1973) and Bekov (1988) were studied by Bekov (1992) using a Gelf'gat transformation in the autonomization of the equations of motion with variable coefficients, to a system of equations with constant coefficients. He investigated the arising and disappearance of collinear $L_{1,2,3}$, triangular $L_{4,5}$, coplanar $L_{6,7}$, ring L_0 and infinitely distant $L_{\pm\infty}$ solutions and the Hill surfaces. The possibility of applying the results obtained to non-stationary double stellar systems was discussed. Bekov (1993) studied the periodic solutions of the GMP. An important role in the dynamic evolution of real gravitating systems is their non-stationarity, connected with mass variation of the system and the additional influence of the variable light pressure from the system's components. Singh and Leke (2010, 2012,2013a,b,c,d) investigated the motion and stability of equilibrium points of the restricted problem under different characterizations when the primaries vary their masses isotropically with time in accordance with the UML with the inclusion that their motion is described by the GMP.

The effect of the isotropic variation of the mass of the star in a planetary system and the possible ejection of a planet from the system was studied by Veras *et al*. (2011). Letelier and Da Silva (2011) studied the particular solutions of the R3BP with variable masses. In the study, particular solutions to the R3BP where the bodies are allowed to either lose or gain mass to or from a static atmosphere, were found. In the case that all the masses are proportional to the same function of time, they found analogous solution to the five stationary solutions of the usual R3BP of constant masses: the three collinear and the two triangular solutions, however, the relative distance of the bodies change with time at the same rate. Further, they observed that under some restrictions, there are also coplanar, infinitely remote and ring solutions.The triangular EPs in photogravitational R3BP with variable mass, in which both the attracting bodies are radiating as well and the infinitesimal body vary its mass with time according to Jeans' law, was studied by Zhang *et al.* (2012).They applied the space-time transformation of Mestschersky (1949) and obtained the differential equations of motion of the problem. They obtained the triangular equilibrium points and found that the triangular points are unstable in the linear sense when the problem with constant mass evolves into the problem with decreasing mass. Tyokyaa and Atsue (2020) examined the positions and linear stability of libration points in the CR3BP under radiation and oblatenes of the more massive primary with constant masses. Singh and Leke (2014) discussed the periodic orbits around triangular points of the restricted three-body problem with variable masses. Ziyad (2018) examined the effect of Poynting-Robertson drag on the circular R3BP with variable masses while Ansari et. al (2019) investigated the effect of variation of charge in the circular R3BP with variable masses.

Some studies of our planetary systems have revealed some disks of dust particles, which are regarded as young analogues of the Kuiper Belt in our Solar system (Greaves et al., 1998). These disks play important roles in the origin of planets' orbital elements. Since the belt of planetesimals often exists within a planetary system and provides the possible mechanism for orbital circularization, it is important to understand the solutions of dynamical systems which show planet-belt interactions. In stellar systems, this phenomenon is also valid. Out of an observed 69 A3-F8 main sequence binary star systems, nearly 60 percent showed dust disks surrounding binary stars. Circumbinary disk that may indicate processes of planet formation have been found around several stars, and are in fact common around binaries with separations less than 3 AU (Trilling *et al.*2007). One notable example is in the HD 98800 system, which comprises two pairs of binary stars separated by around 34 AU (Fig **1**). The binary subsystem HD 98800 B, which consists of two stars of 0.70 and 0.58 solar masses in a highly eccentric orbit with semimajor axis 0.983 AU, is surrounded by a complex dust disc that is being warped by the gravitational effects of the mutually-inclined and eccentric stellar orbits (Akeson *et al*. 2007; Verrier and Evans2008). The other binary subsystem, HD 98800 A, is not associated with significant amounts of dust (Prato*et al*. 2001).

The importance of the problem in astronomy has been addressed by Jiang and Yeh (2004, 2006), where it was shown that the presence of disk resulted in additional equilibrium points of the system. Other works that took into account the gravitational potential from the belt/disk under different assumptions include Singh and Taura (2013, 2014, 2015, 2017) and Jiang and Yeh (2014).

Figure 1: An artist's impression of the binary star system HD 98800 B surrounded by a disk. (Credit: NASA Spitzer Telescope)

An interesting example of mass loss is the real physical scenario of those transiting exoplanets (Fig **2**) whose atmospheres are escaping because of the severe levels of energetic radiations, coming from their very close parent stars, hitting them.

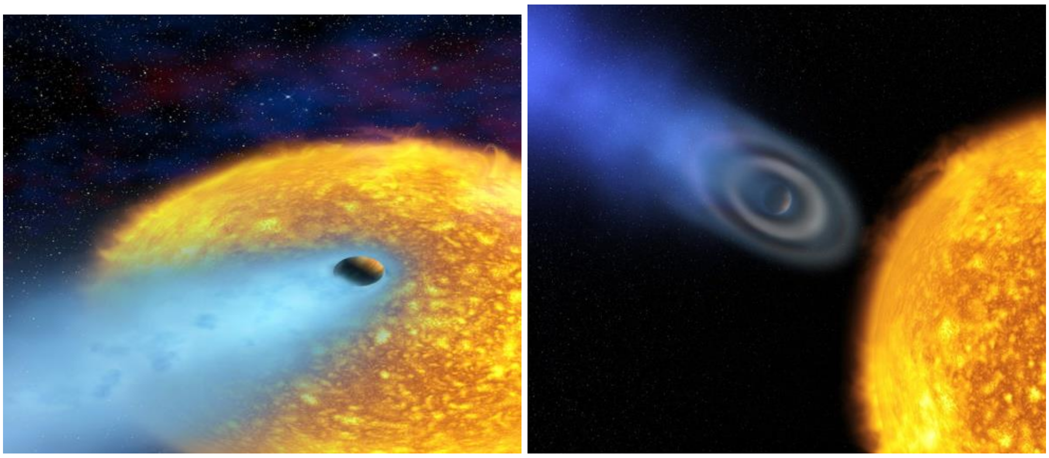

Figure 2: An artistic impression of mass loss of an extrasolar planet **HD 209458b** (**Credit**: European Astronomical observatory)

This image shows a dramatic scorched extrasolar planet HD 209458b in its orbit around a yellow Sun-like star. The image shows the atmosphere of HD 209458b (shown in blue) evaporating off into space. Much of this planet may eventually disappear, leaving only the core, because the amount of hydrogen gas escaping from it is estimated to be at least 10,000 tonnes per second.

Hence, one could consider the formulation of the R3BP with variable mass as follows viz.

i. When masses of both primaries vary with time and the mass of the third body is kept constant (Gelf'gat 1973; Bekov 1988; Luk'yanov 1990 and Singh and Leke 2010).
ii. When the mass of the third body is assumed to vary with time and the masses of the primaries is kept constant (Singh and Ishwar 1984, 1985; Zhang *et al.* 2012).
iii. When the three masses vary with time (Letelier and Da Silva2011).

Our interest is in the first case when masses of both primaries vary with time and the mass of the third body is kept constant. However, we shall also assume that the mass of the disk varies with time.Hence, in the present paper, our aim is to establish the equations of motion of the R3BP with variable masses of the primaries when there is a disk in the configuration. Section one contains the introduction, while section two deals with the dynamical exploration of the problem. The discussion and conclusions are drawn in sections three and four, respectively.

METHODOLOGY

The two-body problem and the integral of area

The two-body problem (2BP) is the starting point for nearly all reference books in the field of astrodynamics. The basic problem describes the motion of two point-masses in mutual gravitational attraction. Newton's law of gravitation leads to closed form solutions to the motions of the bodies with respect to the center of mass. These solutions can be used to analyze orbital properties without the need for cumbersome numerical propagation.

When the formula for gravitation force is applied to the two bodies, the equation relating each body's position with respect to the center of mass of the system may be written:

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{1}$$

where $\vec{v}$ is the velocity, μ is the product of the gravitational constant and the sum of the masses.

To obtain the integral of area, we take cross product of equation (1) with the vector $\vec{r}$, to get

$$\vec{r} \times \frac{d\vec{v}}{dt} = -\frac{\mu}{r^3}\vec{r} \times \vec{r}$$

Since $\vec{r} \times \vec{r} = 0$, we have

$$\vec{r} \times \frac{d\vec{v}}{dt} = 0 \tag{2}$$

Also, $\dfrac{d}{dt}\left(\vec{r} \times \vec{v}\right) = \dfrac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \dfrac{d\vec{v}}{dt}$

From (2), we get

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = \frac{d\vec{r}}{dt} \times \vec{v} \tag{3}$$

But $\dfrac{d\vec{r}}{dt} = \vec{v}$ and since $\vec{v} \times \vec{v} = 0$,

Equation (3) becomes

$$\frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = 0 \tag{4}$$

Integrating (4), we have

$$\vec{r} \times \vec{v} = \vec{C} \tag{5}$$

where $\vec{C}$ is a constant vector called the integral of area, so $\vec{r} \times \vec{v}$ is a constant vector and that $\vec{C}$ is perpendicular to both $\vec{r}$ and $\vec{v}$, which in turns means that $\vec{C}$ is perpendicular to the plane of motion. If in polar coordinates r and θ are taken in this plane, then the velocity component along and perpendicular to the radius vector joining mass m_1 and m_2 are $\dot{r}$ and $r\dot{\theta}$, so that

$$\vec{v} = \dot{r}\vec{i} + r\dot{\theta}\vec{j} \tag{6}$$

Here, $\vec{i}$ and $\vec{j}$ are unit vectors along and perpendicular to the radius vector, hence by equations (5) and (6), we have

$$r\vec{i} \times \left(\dot{r}\vec{i} + r\dot{\theta}\vec{j}\right) = \vec{C}.$$

Taking product, yields

$$r^2\dot{\theta}\vec{k} = \vec{C} \tag{7}$$

The constant vector $\vec{C}$ can be expressed as

$$\vec{C} = C\vec{k} \tag{8}$$

where $\vec{k}$ is a unit vector perpendicular to the plane of the orbit.

Equation (7) now takes the form

$$r^2\dot{\theta} = C \tag{9}$$

Equation (9) is the area integral of the system, $\dot{\theta}$ is the angular velocity of revolution of the bodies of masses m_1 and m_2, and C is the constant of integration

Equation of motion of two-body with variable masses

By the problem of two bodies with variable masses, by analogy with the classical problem of two bodies with constant masses, one understands the problem of motion of two primary bodies, the masses m_1 and m_2 of which vary with time under certain laws and between which only the gravitational force acts. It is usually assumed that the separation of particles from (or their attachment to) the points take place in accordance with Mestschersky's hypothesis, i.e., a contact interaction occurs between the points of variable mass and the separating (or attaching) particles; it is assumed that the masses of the points vary continuously.

The absolute motion of the points is described by the Mestschersky equation for a point of variable mass,

$$\vec{F} = m\dot{\vec{v}} + \left(\vec{v} - \vec{u}\right)\dot{m} \tag{10}$$

where $\vec{F}$ is the sum of all the forces acting on the body and $\vec{v}$ is its velocity, both measured in an inertial coordinate system. Also, $\vec{u}$ is the velocity of the center of mass of the absorbed mass immediately before its union with the body (or of the ejected mass immediately after its ejection). The overdot denotes derivation with respect to the time variable.

Gylden represented the relative motion (equation 1) of mass m_2 about mass m_1 under the action of mutual gravitational force, as the sum of the masses of these points as varying with time by a certain law

$$m_1 + m_2 = \mu\left(t\right) \tag{11}$$

and wrote the differential equation of the problem in the form

$$\ddot{\vec{r}} + \frac{\mu\left(t\right)}{r^3}\vec{r} = 0 \tag{12}$$

Mestschersky showed that the Gylden problem (12) is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same.

There are two special cases of equation (10) to be considered. The first one is when the mass is ejected with the same velocity of the body at any moment $\left(\vec{v} = \vec{u}\right)$, that is, mass ejection does not produce reactive forces. This case can be used to study the motion of a body ejecting mass isotropically (or radiating energy), since the total reactive momentum would be zero.

If $\vec{v} = \vec{u}$, then equation (10) reduces to the form

$$\vec{F} = m\dot{\vec{v}} \tag{13}$$

In this case the relative motion of the problem of two bodies with variable masses is described by the equation

$$\ddot{\vec{r}} = -G\frac{\left(m_1 + m_2\right)}{r^3}\vec{r} \tag{14}$$

Equation (14) is analogous to the equation (1) of the classical problem of two bodies with constant masses, with the difference that now; the sum of the masses is a certain function of time. Equality of the velocities of ejected mass and the body at any moment means that isotropic variation of masses (in Mestschersky term) occurs. Equation (14) is rightfully called the Gylden-Mestschersky problem (GMP).

The second case is when mass variation takes place in the presence of reactive forces. In this case the particles are at rest in an inertial coordinate system, that is $\vec{u} = 0$ and equation (10) becomes

$$\vec{F} = m\dot{\vec{v}} + \dot{m}\vec{v} = \frac{d}{dt}\left(m\vec{v}\right) \tag{15}$$

This case can be used to study the orbit of a star moving through a static atmosphere, whose particles attach or detach to the star as it moves.

In this case the relative motion of the problem of two bodies with variable masses exchanging mass with a static atmosphere surrounding them, is described by the equation

$$\frac{d}{dt}\left(\mu_*\dot{\vec{r}}\right) = -G\frac{\mu_*\left(m_1+m_2\right)\vec{r}}{r^3} \quad (16)$$

where $\mu_* = \frac{m_1 m_2}{\left(m_1+m_2\right)}$

The Mestschersky transformation and unified Mestschersky law

Mestschersky (1902) showed that the Gylden problem is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same, while the relative velocities of the particles separating from them (or attaching to them) equals zero everywhere. He found two laws of variation of the sum of the masses of the points

$$m_1+m_2=\mu(t)=\frac{\mu_0}{a+\alpha t} \quad (17)$$

$$m_1+m_2=\mu(t)=\frac{\mu_0}{\sqrt{\alpha+\beta t+\gamma t^2}} \quad (18)$$

where μ_0, a, α, β and γ are constants.

In the work of Mestschersky (1902), he reduced the GMP through the introduction of new variables and “time” to the equations of the classical problem of two bodies with constant masses by a transformation, which was thereafter known as the Mestschersky (1902) transformation and is given as

$$x=\xi R(t),\ \ y=\eta R(t),\ z=\varsigma R(t),\ \ \frac{dt}{d\tau}=R^2(t)$$

$$r_i=\rho_i R(t),\ \ (i=1,2),\ r=\rho_{12}R(t) \quad (19)$$

where $R(t)=\sqrt{\alpha t^2+2\beta t+\gamma}$; ξ,η,ζ,τ are the new variables and ρ_{12} is constant.

Later, Mestschersky (1952) came up with a law which considers the masses and their sum to vary in the same proportion in such a way that

$$\mu(t)=\frac{\mu_0}{R(t)},\ \ \mu_1(t)=\frac{\mu_{10}}{R(t)},\ \mu_2(t)=\frac{\mu_{20}}{R(t)} \quad (20)$$

where $\mu_1(t)=Gm_1(t)$, $\mu_2(t)=Gm_2(t)$, $\mu(t)=\mu_1(t)+\mu_2(t)$, μ_{10} and μ_{20} are constants.

The law (20) is called the unified Mestschersky (1952) law (UML)and it assures that the centre of the mass of the system moves inertially.

Particular solutions of the Gylden-Mestschersky problem.

We let $\mu(t)=G\left(m_1+m_2\right)$ in equation (14) to get

$$\ddot{\vec{r}}+\frac{\mu}{r^2}\frac{\vec{r}}{r}=0 \quad (21)$$

Now, differentiation of equation (9) with respect to time t and multiplying by $\frac{1}{r}$, yields

$$2\dot{r}\dot{\theta}+r\ddot{\theta}=0 \quad (22)$$

But acceleration in polar coordinates is given by

$$\dot{\vec{v}}=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{i}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\vec{j} \quad (23)$$

We substitute equation (22) in (23), to get

$$\dot{\vec{v}}=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{i} \quad (24)$$

Using equation (24) in (21) yields

$$\ddot{r}-r\dot{\theta}^2+\frac{\mu}{r^2}=0 \quad (25)$$

Now from equation (9), we have

$$\dot{\theta}^2=\frac{C^2}{r^4} \quad (26)$$

Substituting equation (26) in (25), we get

$$\ddot{r}-\frac{C^2}{r^3}+\frac{\mu}{r^2}=0, \quad (27)$$

where $C=r^2\dot{\theta}$ or $r^2\omega=C$,

r is the distance between the bodies, θ is the angle between the straight line passing through m_1 and m_2 and a certain fixed straight line in the plane of motion. $\dot{\theta}=\omega(t)$ is angular velocity of revolution of the bodies and C is the constant of area integral.

Now,from the Mestschersky transformation (19), we have

$$r=\rho_{12}R(t), \quad (28)$$

Differentiating equation (28), twice yields

$$\ddot{r}=\rho_{12}\frac{\left(\alpha\gamma-\beta^2\right)}{R^3(t)} \quad (29)$$

From second equation of system (27), we have

$$C^2=r^4\omega^2 \quad (30)$$

Substituting equation (28), (29) and (30) in the first equation of system (27); multiplying throughout by $\frac{1}{\rho_{12}}$ and simplifying results in the equation

$$\omega(t)=\frac{1}{R^2(t)}\left(\alpha\gamma-\beta^2+\frac{\mu_0}{\rho_{12}^3}\right)^{\frac{1}{2}} \quad (31)$$

Now since $\alpha,\beta,\gamma,\mu_0$ and ρ_{12} are constants, we let

$$\left(\alpha\gamma-\beta^2+\frac{\mu_0}{\rho_{12}^3}\right)^{\frac{1}{2}}=\omega_0,$$

So that equation (31) becomes

$$\omega(t)=\frac{\omega_0}{R^2(t)} \quad (32)$$

Again, substituting equation (28) and (29) in the first equation of system (27) and reducing throughout by $R^3(t)$, yields

$$\rho_{12}(\alpha\gamma-\beta^2)-\frac{C^2}{\rho_{12}^3}+\frac{\mu_0}{\rho_{12}^2}=0 \quad (33)$$

Applying equation (30) in (33), we get

$$(\alpha\gamma-\beta^2)-\omega_0^2+\frac{\mu_0}{\rho_{12}^3}=0 \quad (34)$$

Also, substituting equation (27) and (32) in equation (30), and simplifying, gives

$$C=\rho_{12}^2\omega_0 \quad (35)$$

Using equation (35) in (34), we have

$$\frac{(\beta^2-\alpha\gamma+\omega_0^2)C}{\omega_0}=\frac{\mu_0}{\rho_{12}} \quad (36)$$

But $\mu=\dfrac{\mu_0}{R(t)}$,

Substituting for μ_0 in equation (36) and multiplying throughout by ρ_{12}^2, yields

$$\frac{(\beta^2-\alpha\gamma+\omega_0^2)\rho_{12}^2 C}{\omega_0}=\mu\rho_{12}R(t)$$

With the help of equations (35) and (28), we get the equation

$$\frac{(\beta^2-\alpha\gamma+\omega_0^2)C^2}{\omega_0^2}=r\mu \quad (37)$$

Now, if we suppose that

$$\kappa=\frac{\beta^2-\alpha\gamma+\omega_0^2}{\omega_0^2} \quad (38)$$

Therefore, we can express equation (37) in the form

$$r\mu=\kappa C^2 \quad (39)$$

Equation (39) is a particular integral of the Gylden-Mestschersky problem and κ is a constant defined by equation (38). When $\beta^2-\alpha\gamma=0$, we get $\kappa=1$ and this corresponds to the case when the masses are constant. When $\beta^2-\alpha\gamma>0$, this means that $\kappa>1$ and when $\beta^2-\alpha\gamma<0$, this implies that $\kappa<1$. Since kappa cannot be zero, the range is such that $0<\kappa<\infty$.

Equations of motion of the restricted three-body problem with isotropic mass variations of the primariesand disk

Let us consider a rotating frame of reference $Oxyz$, where O is the origin and suppose that m_1 and m_2 are the masses of the primary bodies and m_3 is the mass of the third body. Let the radius vector from m_3 to m_1 be r_1, m_3 to m_2 be r_2 and the distance between the two primaries be r and let ω be the angular velocity.

Now, the kinetic energy in the rotating frame of reference $0xyz$ is given by

$$T=\frac{1}{2}m_3(\dot{x}^2+\dot{y}^2+\dot{z}^2)+m_3\omega(x\dot{y}-y\dot{x})+\frac{1}{2}m_3(x^2+y^2)\omega^2 \quad (40)$$

Now, the potential energy has the form

$$V=-Gm_3\left[\frac{m_1(t)}{r_1}+\frac{m_2(t)}{r_2}+\frac{M(t)}{\sqrt{r^2+\left(a+\sqrt{b^2+z^2}\right)^2}}\right] \quad (41)$$

where $r_1^2=(x-x_1)^2+y^2+z^2$, $r_2^2=(x-x_2)^2+y^2+z^2$ and G is the gravitational constant while $M(t)$ is the mass of the disk.

The Hamiltonian H is given as

$$H=\frac{1}{2m_3}\left(p_x^2+p_y^2+p_z^2\right)+\omega\left(yp_x-xp_y\right)+V \quad (42)$$

The Hamiltonian canonical equations are given by

$$\dot{x}=\frac{\partial H}{\partial p_x},\quad \dot{y}=\frac{\partial H}{\partial p_y},\quad \dot{z}=\frac{\partial H}{\partial p_z},\ \dot{p}_x=-\frac{\partial H}{\partial x},\ \dot{p}_y=-\frac{\partial H}{\partial y},\quad \dot{p}_z=-\frac{\partial H}{\partial z} \quad (43)$$

where

$$\rho_x=m_3(\dot{x}-\omega y),\quad \rho_y=m_3(\dot{y}+\omega x),\ \rho_z=m_3\dot{z} \quad (44)$$

are the generalized components of momentum,

Now, differentiating (44) w.r.t x and comparing with (43), we get

$$\ddot{x}-2\omega\dot{y}=\omega^2x+\dot{\omega}y-\frac{1}{m_3}\frac{\partial V}{\partial x}$$

$$\ddot{y}+2\omega\dot{x}=\omega^2y-\dot{\omega}x-\frac{1}{m_3}\frac{\partial V}{\partial y}$$

$$m_3\ddot{z} = -\frac{1}{m_3}\frac{\partial V}{\partial z}$$

Using equations (41) produces

$$\ddot{x} - \dot{\omega}y - 2\omega\dot{y} = \omega^2 x - \frac{Gm_1(x-x_1)}{r_1^3} - \frac{Gm_2(x-x_2)}{r_2^3} - \frac{GMx}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}}$$

$$\ddot{y} + \dot{\omega}x + 2\omega\dot{x} = \omega^2 y - \frac{Gm_1 y}{r_1^3} - \frac{Gm_2 y}{r_2^3} - \frac{GMy}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}} \tag{45}$$

$$\ddot{z} = -\frac{Gm_1 z}{r_1^3} - \frac{Gm_2 z}{r_2^3} - \frac{GMz}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}}\left(1 + \frac{a+\sqrt{z^2+b^2}}{\sqrt{z^2+b^2}}\right)$$

Now, we let

$Gm_1(t) = \mu_1(t)$, $Gm_2(t) = \mu_2(t)$

So that equations (45) take the form:

$$\ddot{x} - \dot{\omega}y - 2\omega\dot{y} = \omega^2 x - \frac{\mu_1(x-x_1)}{r_1^3} - \frac{\mu_2(x-x_2)}{r_2^3} - \frac{GMx}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}}$$

$$\ddot{y} + \dot{\omega}x + 2\omega\dot{x} = \omega^2 y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{GMy}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}} \tag{46}$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} - \frac{GMz}{\left(r^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right)^{3/2}}\left(1 + \frac{a+\sqrt{z^2+b^2}}{\sqrt{z^2+b^2}}\right)$$

where $r_1^2 = (x-x_1)^2 + y^2 + z^2$, $r_2^2 = (x-x_2)^2 + y^2 + z^2$

$$\omega^2(t) = \frac{\mu(t)}{\kappa}\left[\frac{1}{r^3} + \frac{2GMr_c}{\left\{r_c^2 + \left(a+\sqrt{z^2+b^2}\right)^2\right\}^{3/2}}\right]$$

r_1 and r_2 are time dependent distances of the third body from the primaries positioned at $(x_1, 0, 0)$ and $(x_2, 0, 0)$, $\mu(t)$ is the product of the gravitational constant and the sum of the masses; $r = \sqrt{x^2+y^2+z^2}$ and r_c is the radial distance of the infinitesimal mass in the R3BP with variable mass. The masses vary in accordance with the UML while motion of the primaries is described by the GMP.

These equations describe the motion of the third body in the gravitational field of the primariesin the Barycentric coordinate system $0xyz$, rotating with an angular velocity $\omega(t)$ about the $z-$ axis perpendicular to the plane of motion of the primaries, while the $x-$ axis always passes through these points under the consideration that both primaries have variable masses and there is a disk to interact with in the configuration.

Now, since the coordinate system is Barycentric, from the property of the center of mass we have

$\mu_1 x_1 + \mu_2 x_2 = 0$

where

$$x_1 = -\frac{\mu_2 r}{\mu_1+\mu_2}, \quad x_2 = \frac{\mu_1 r}{\mu_1+\mu_2} \tag{47}$$

The expressions (47) connect the Barycentric coordinates x_1 and x_2 with the mutual distance r.

DISCUSSION

The equations of motion (46) of the R3BP with variable masses under the influence from a disk has been derived under the condition that the motion of the primaries takes place in accordance with the GMP and their masses vary according to the UML. The mass of the disk has also been assumed to vary in the same way as the masses of the primaries. Therefore, with the help of the potential (41) and the Hamiltonian canonical equations, we have deduced the equations of motion of the R3BP with variable masses and disk. These equations are different from those of Bekov (1988, 1991, 1993) and those of Luk'yanov (1989,1990), Singh and Leke (2010, 2012, 2013a,b,c,d) due to the inclusion of the mass of the disk in the set up. If we put $(t) = 0$, our equations (46) will fully coincide with those of the previous studies of Bekov (1988, 1991,1993) and those of Luk'yanov (1989,1990).

CONCLUSION

This paper explores the dynamical equations of the R3BP with variable masses and disk, when the masses of the primaries and that of the disk vary with time in accordance with the unified Mestschersky Law and the motion of the primaries are governed by the Gylden-Mestschersky problem. The equations derived are affected by the mass of the disk and are different from those of previous studies.

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